# Convergence Theorem for a Family of New Modified Halley's Method in Banach Space 

Rongfei Lin, ${ }^{1}$ Yueqing Zhao, ${ }^{1}$ Qingbiao Wu, ${ }^{2}$ and Jueliang $\mathbf{H u}^{3}$<br>${ }^{1}$ Department of Mathematics, Taizhou University, Linhai, Zhejiang 317000, China<br>${ }^{2}$ Department of Mathematics, Zhejiang University, Hangzhou, Zhejiang 310027, China<br>${ }^{3}$ Department of Mathematics, Zhejiang Sci-Tech University, Hangzhou, Zhejiang 310018, China

Correspondence should be addressed to Qingbiao Wu; qbwu@zju.edu.cn
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#### Abstract

We establish convergence theorems of Newton-Kantorovich type for a family of new modified Halley's method in Banach space to solve nonlinear operator equations. We present the corresponding error estimate. To show the application of our theorems, two numerical examples are given.


## 1. Introduction

Solving the nonlinear operator equation is an important issue in the engineering and technology field. Finding solutions of the nonlinear operator equation in Banach spaces also is a very general subject which is widely used in both theoretical and applied areas of mathematics for many years. Many problems may be formulated in terms of finding zeros. These roots cannot be expressed in closed form generally. Thus, in order to solve nonlinear equations, we have to use approximate methods (see [1]). In this study, we consider establishing the Newton-Kantorovich convergence theorems for a family of new modified Halley's method with third-order Banach space by using majorizing function which is used to solve the nonlinear operator equation. Consider

$$
\begin{equation*}
F(x)=0 \tag{1}
\end{equation*}
$$

where $F$ is defined as an open convex $\Omega$ of a Banach space $X$ with values in a Banach space $Y$.

There are kinds of methods to find a solution of (1). Iterative methods are often used to solve this problem (see [1]). If we use the famous Newton's method, we can do as follows:

$$
\begin{equation*}
x_{n+1}=x_{n}-F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right), \quad(n \geq 0)\left(x_{0} \in \Omega\right) \tag{2}
\end{equation*}
$$

Under the reasonable hypothesis, Newton's method is second-order convergence.

Since Kantorovich presented the famous convergence result (see [2]), many Newton-Kantorovich type convergence theorems were gotten (see [3-11]). To improve the convergence order, many modified methods have been presented (see [12-15]). The famous Halley's method is third-order convergence which was widely discussed (see [16-23]). The famous Halley's method is defined as follows:

$$
\begin{align*}
x_{n+1}= & x_{n}-\left[I+\frac{1}{2} L_{F}\left(x_{n}\right)\left(I-\frac{1}{2} L_{F}\left(x_{n}\right)\right)^{-1}\right]  \tag{3}\\
& \times F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right), \quad n=0,1, \ldots
\end{align*}
$$

where

$$
\begin{equation*}
L_{F}(x)=F^{\prime}(x)^{-1} F^{\prime \prime}(x) F^{\prime}(x)^{-1} F(x), \quad x \in \Omega \tag{4}
\end{equation*}
$$

In the year of 2001, Gutiérrez and Hernández [24] proposed the Super-Halley method, which is an acceleration of Newton's method with third order. Consider

$$
\begin{equation*}
x_{n+1}=x_{n}-\left[I+\frac{1}{2} L_{F}\left(x_{n}\right)\left(I-L_{F}\left(x_{n}\right)\right)^{-1}\right] F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right) . \tag{5}
\end{equation*}
$$

Using the majorant principle, the authors also established a semilocal convergence theorem for the Super-Halley method under weaker conditions, which is defined as follows:

$$
\begin{equation*}
\left\|F^{\prime}\left(x_{0}\right)^{-1}\left(F^{\prime \prime}(x)-F^{\prime \prime}(y)\right)\right\| \leq k\|x-y\| . \tag{6}
\end{equation*}
$$

Extending the family of scalar iterative processes considered by Hernandez and Salanova in [25], Gutiérrez and Hernández [26] presented a one-parameter family of iterative processes

$$
\begin{align*}
x_{\alpha, n+1}= & x_{\alpha, n}-\left[I+\frac{1}{2} L_{F}\left(x_{\alpha, n}\right)\left(I-\alpha L_{F}\left(x_{\alpha, n}\right)\right)^{-1}\right]  \tag{7}\\
& \times F^{\prime}\left(x_{\alpha, n}\right)^{-1} F x_{\alpha, n} .
\end{align*}
$$

This family includes, as particular cases, Chebyshev's method ( $\alpha=0$ ), Halley's method ( $\alpha=1 / 2$ ), and convex acceleration of Newton's method ( $\alpha=1$ ). Under Kantorovich-type assumptions, the authors obtained results on existence and uniqueness of solution of (1).

The calculation of the second derivative of the function $F$ is needed in the above iterative methods. For avoiding the calculation $F^{\prime \prime}\left(x_{n}\right)$ and keeping higher order of convergence at the same time, some authors have studied the convergence of the iteration (3) by using difference quotient of the first derivative to replace the second derivative.

Based on Halley's method in which the second derivative is replaced with a finite difference between first derivatives, that is,

$$
\begin{equation*}
F^{\prime \prime}\left(x_{n}\right) \simeq \frac{F^{\prime}\left(y_{n}\right)-F^{\prime}\left(x_{n}\right)}{y_{n}-x_{n}} \tag{8}
\end{equation*}
$$

a class of iterative methods with free second derivative is obtained [27]. Consider

$$
\begin{equation*}
x_{n+1}=x_{n}-\frac{2 \theta F\left(x_{n}\right)}{(2 \theta-1) F^{\prime}\left(x_{n}\right)+F^{\prime}\left(y_{n}\right)} \tag{9}
\end{equation*}
$$

where $\theta \in(0,1]$ and $y_{n}=x_{n}-\theta F\left(x_{n}\right) / F^{\prime}\left(x_{n}\right)$.
For $\theta=1$, a third-order method is obtained [28]:

$$
\begin{equation*}
x_{n+1}=x_{n}-\frac{2 F\left(x_{n}\right)}{F^{\prime}\left(x_{n}\right)+F^{\prime}\left(x_{n}-F\left(x_{n}\right) / F^{\prime}\left(x_{n}\right)\right)} \tag{10}
\end{equation*}
$$

For $\theta=1 / 2$, another third-order method is obtained [29, 30]:

$$
\begin{equation*}
x_{n+1}=x_{n}-\frac{F\left(x_{n}\right)}{F^{\prime}\left(x_{n}-F\left(x_{n}\right) /\left(2 F^{\prime}\left(x_{n}\right)\right)\right)} \tag{11}
\end{equation*}
$$

These methods do not require the second derivative although they can converge cubically. Thus, the approach to remove the second derivative is important and interesting for deriving more new high-order iterative methods free from second derivative from third-order iterative methods with second derivative.

By directly replacing $L_{F}\left(x_{n}\right)$ with a new approximation

$$
\begin{equation*}
L_{F}\left(x_{n}\right)=\frac{1}{2} \frac{F^{\prime \prime}\left(x_{n}\right) F\left(x_{n}\right)}{F^{\prime}\left(x_{n}\right)^{2}} \simeq \frac{F\left(y_{n}\right)+(\theta-1) F\left(x_{n}\right)}{\theta^{2} F\left(x_{n}\right)} \tag{12}
\end{equation*}
$$

a class of modifications of Halley's method free from second derivative [31] is obtained; that is,

$$
\begin{equation*}
x_{n+1}=x_{n}-\frac{\theta^{2} F\left(x_{n}\right)}{\left(\theta^{2}-\theta+1\right) F\left(x_{n}\right)-F\left(y_{n}\right)} \frac{F\left(x_{n}\right)}{F^{\prime}\left(x_{n}\right)}, \tag{13}
\end{equation*}
$$

where $\theta \in R, \theta \neq 0$, and $y_{n}=x_{n}-\theta F\left(x_{n}\right) / F^{\prime}\left(x_{n}\right)$. This modified Halley's method is cubically convergent for any nonzero real number $\theta$.

Now, we consider a new finite difference approximation of $F^{\prime \prime}(x)$ :

$$
\begin{equation*}
p F^{\prime \prime}\left(x_{n}\right)\left(y_{n}-x_{n}\right) \simeq F^{\prime}\left(x_{n}+p\left(y_{n}-x_{n}\right)\right)-F^{\prime}\left(x_{n}\right), \tag{14}
\end{equation*}
$$

where

$$
\begin{equation*}
y_{n}=x_{n}-F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right), \quad \lambda>0, \tag{15}
\end{equation*}
$$

and from Halley's method (3), we obtain a family of new modified Halley's method with parameters $p$ and $\alpha$ :

$$
\begin{gather*}
y_{n}=x_{n}-F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right), \\
H\left(x_{n}, y_{n}\right)=\frac{1}{p} F^{\prime}\left(x_{n}\right)^{-1}\left[F^{\prime}\left(x_{n}+p\left(y_{n}-x_{n}\right)\right)-F^{\prime}\left(x_{n}\right)\right], \\
\alpha \in[0,1], \quad p \in(0,1], \\
x_{n+1}=y_{n}-\frac{1}{2} H\left(x_{n}, y_{n}\right)\left[I-\alpha H\left(x_{n}, y_{n}\right)\right]\left(y_{n}-x_{n}\right) . \tag{16}
\end{gather*}
$$

This includes the modified Chebyshev iteration (see [8, 9]) for $p=1 / 2, \alpha=0$ and the Jarratt iteration (see [32, 33]) for $p=2 / 3, \alpha=1$. In this paper, we discuss the convergence of modified Halley's methods for solving nonlinear operator equations in Banach spaces and establish convergence theorems of Newton-Kantorovich's type. The corresponding error estimate is also given. Finally, two examples are provided to show the application of our theorem.

## 2. Convergence Theorem

In the section, we establish a Newton-Kantorovich type convergence theorem and present the error estimate. Denote $g(t)=\left((1 / 2) K t^{2}\right)-(t / \beta)+(\eta / \beta)$, where $K, \beta, \eta$ are positive real numbers. Write $h=K \beta \eta, t^{*}=((1-\sqrt{1-2 h}) / h) \eta$, and $t^{* *}=((1+\sqrt{1-2 h}) / h) \eta$, where $t^{*}, t^{* *}$ are the roots of the equation $g(t)=0$. Let

$$
\begin{gather*}
s_{n}=t_{n}-g^{\prime}\left(t_{n}\right)^{-1} g\left(t_{n}\right), \quad t_{0}=0, \\
H_{g}\left(t_{n}, s_{n}\right)=\frac{1}{p} g^{\prime}\left(t_{n}\right)^{-1}\left[g^{\prime}\left(t_{n}+p\left(s_{n}-t_{n}\right)\right)-g^{\prime}\left(t_{n}\right)\right], \\
\alpha \in[0,1], \quad p \in(0,1], \\
t_{n+1}=s_{n}-\frac{1}{2} H_{g}\left(t_{n}, s_{n}\right)\left[1-\alpha H_{g}\left(t_{n}, s_{n}\right)\right]\left(s_{n}-t_{n}\right) . \tag{17}
\end{gather*}
$$

Firstly, we get some lemmas.

Lemma 1. Assume $\left\{t_{n}\right\}$ and $\left\{s_{n}\right\}$ are the sequences generated by (17). If $h \leq 1 / 2$, then the sequences $\left\{t_{n}\right\}$ and $\left\{s_{n}\right\}$ are monotonically increasing and converge to $t^{*}$. Moreover, one has

$$
\begin{gather*}
0 \leq t_{n}<s_{n}<t_{n+1}<s_{n+1}<\cdots<t^{*} \\
t^{*}-t_{n+1}=\left(t^{*}-t_{n}\right)^{3}\left[\left(t^{*}-t_{n}\right)^{3}+4\left(t^{*}-t_{n}\right)^{2}\left(t^{* *}-t_{n}\right)\right. \\
+5\left(t^{*}-t_{n}\right)\left(t^{* *}-t_{n}\right)^{2} \\
\\
\left.+(2-2 \alpha)\left(t^{* *}-t_{n}\right)^{3}\right] \\
\times\left(\left[\left(t^{*}-t_{n}\right)+\left(t^{* *}-t_{n}\right)\right]^{5}\right)^{-1} \\
t^{* *}-t_{n+1}=\left(t^{* *}-t_{n}\right)^{3}\left[\left(t^{* *}-t_{n}\right)^{3}+4\left(t^{* *}-t_{n}\right)^{2}\left(t^{*}-t_{n}\right)\right. \\
\\
\quad+5\left(t^{* *}-t_{n}\right)\left(t^{*}-t_{n}\right)^{2}  \tag{18}\\
\\
\left.\quad+(2-2 \alpha)\left(t^{*}-t_{n}\right)^{3}\right] \\
\times\left(\left[\left(t^{*}-t_{n}\right)+\left(t^{* *}-t_{n}\right)\right]^{5}\right)^{-1}
\end{gather*}
$$

Proof. By (17) we can get

$$
\begin{gather*}
g(t)=\frac{K}{2}\left(t^{*}-t\right)\left(t^{* *}-t\right), \\
g^{\prime}(t)=-\frac{K}{2}\left[\left(t^{*}-t\right)+\left(t^{* *}-t\right)\right], \\
s_{n}-t_{n}=-g^{\prime}\left(t_{n}\right)^{-1} g\left(t_{n}\right)=\frac{\left(t^{*}-t_{n}\right)\left(t^{* *}-t_{n}\right)}{\left(t^{*}-t_{n}\right)+\left(t^{* *}-t_{n}\right)}, \\
H_{g}\left(t_{n}, s_{n}\right)=g^{\prime}\left(t_{n}\right)^{-1} K\left(s_{n}-t_{n}\right)=\frac{-2\left(t^{*}-t_{n}\right)\left(t^{* *}-t_{n}\right)}{\left[\left(t^{*}-t_{n}\right)+\left(t^{* *}-t_{n}\right)\right]^{2}}, \\
t_{n+1}-s_{n}=-\frac{1}{2} H_{g}\left(t_{n}, s_{n}\right)\left[1-\alpha H_{g}\left(t_{n}, s_{n}\right)\right]\left(s_{n}-t_{n}\right) . \tag{19}
\end{gather*}
$$

By a simple calculation and mathematical induction, it is easy to prove Lemma 1.

Lemma 2. Let $F(x)$ be a nonlinear operator defined on a convex domain $\Omega$ of a Banach space $X$ with values in a Banach space Y. Assume that F has second-order continuous Frechet derivatives on $\Omega$. If $\left\{x_{n}\right\},\left\{y_{n}\right\}$ are the sequences generated by (16), then

$$
\begin{aligned}
F & \left(x_{n+1}\right) \\
= & \int_{0}^{1} F^{\prime \prime}\left(y_{n}+t\left(x_{n+1}-y_{n}\right)\right)(1-t) d t\left(x_{n+1}-y_{n}\right)^{2} \\
& +\int_{0}^{1}\left[F^{\prime \prime}\left(x_{n}+t\left(y_{n}-x_{n}\right)\right)(1-t)\right. \\
& \left.\quad-\frac{1}{2} F^{\prime \prime}\left(x_{n}+p t\left(y_{n}-x_{n}\right)\right)\right] d t\left(y_{n}-x_{n}\right)^{2}
\end{aligned}
$$

$$
\begin{align*}
& -\frac{1-\alpha}{2} \int_{0}^{1} F^{\prime \prime}\left(x_{n}+p t\left(y_{n}-x_{n}\right)\right) d t \\
& \times\left(y_{n}-x_{n}\right) H\left(x_{n}, y_{n}\right)\left(y_{n}-x_{n}\right) \\
& -\frac{\alpha}{2} \int_{0}^{1}\left[F^{\prime \prime}\left(x_{n}+t\left(y_{n}-x_{n}\right)\right)\right. \\
& \left.\quad-F^{\prime \prime}\left(x_{n}+p t\left(y_{n}-x_{n}\right)\right)\right] d t \\
& \times\left(y_{n}-x_{n}\right) H\left(x_{n}, y_{n}\right)\left(y_{n}-x_{n}\right) \\
& +\frac{\alpha}{2} \int_{0}^{1} F^{\prime \prime}\left(x_{n}+t\left(y_{n}-x_{n}\right)\right) d t \\
& \times\left(y_{n}-x_{n}\right) H\left(x_{n}, y_{n}\right) H\left(x_{n}, y_{n}\right)\left(y_{n}-x_{n}\right) . \tag{20}
\end{align*}
$$

Proof. Consider

$$
\begin{aligned}
& F\left(x_{n+1}\right) \\
& =F\left(x_{n+1}\right)-F\left(y_{n}\right)-F^{\prime}\left(y_{n}\right)\left(x_{n+1}-y_{n}\right) \\
& +F\left(y_{n}\right)+F^{\prime}\left(y_{n}\right)\left(x_{n+1}-y_{n}\right) \\
& =\int_{0}^{1} F^{\prime \prime}\left(y_{n}+t\left(x_{n+1}-y_{n}\right)\right)(1-t) d t\left(x_{n+1}-y_{n}\right)^{2} \\
& +F\left(y_{n}\right)+F^{\prime}\left(y_{n}\right)\left(x_{n+1}-y_{n}\right), \\
& F^{\prime}\left(x_{n}\right) H\left(x_{n}, y_{n}\right) \\
& =\frac{1}{p}\left[F^{\prime}\left(x_{n}+p\left(y_{n}-x_{n}\right)\right)-F^{\prime}\left(x_{n}\right)\right] \\
& =\int_{0}^{1} F^{\prime \prime}\left(x_{n}+p t\left(y_{n}-x_{n}\right)\right) d t\left(y_{n}-x_{n}\right) \text {, } \\
& F\left(y_{n}\right)+F^{\prime}\left(y_{n}\right)\left(x_{n+1}-y_{n}\right) \\
& =F\left(y_{n}\right)-F\left(x_{n}\right)-F^{\prime}\left(x_{n}\right)\left(y_{n}-x_{n}\right) \\
& -\frac{1}{2} F^{\prime}\left(y_{n}\right) H\left(x_{n}, y_{n}\right)\left[I-\alpha H\left(x_{n}, y_{n}\right)\right]\left(y_{n}-x_{n}\right) \\
& =\int_{0}^{1} F^{\prime \prime}\left(x_{n}+t\left(y_{n}-x_{n}\right)\right)(1-t) d t\left(y_{n}-x_{n}\right)^{2} \\
& -\frac{1}{2}\left[F^{\prime}\left(y_{n}\right)-F^{\prime}\left(x_{n}\right)\right] H\left(x_{n}, y_{n}\right)\left[I-\alpha H\left(x_{n}, y_{n}\right)\right] \\
& \times\left(y_{n}-x_{n}\right) \\
& -\frac{1}{2} F^{\prime}\left(x_{n}\right) H\left(x_{n}, y_{n}\right)\left[I-\alpha H\left(x_{n}, y_{n}\right)\right]\left(y_{n}-x_{n}\right) \\
& =\int_{0}^{1} F^{\prime \prime}\left(x_{n}+t\left(y_{n}-x_{n}\right)\right)(1-t) d t\left(y_{n}-x_{n}\right)^{2} \\
& -\frac{1}{2} \int_{0}^{1} F^{\prime \prime}\left(x_{n}+p t\left(y_{n}-x_{n}\right)\right) d t\left(y_{n}-x_{n}\right)^{2} \\
& +\frac{\alpha}{2} \int_{0}^{1} F^{\prime \prime}\left(x_{n}+p t\left(y_{n}-x_{n}\right)\right) d t\left(y_{n}-x_{n}\right)
\end{aligned}
$$

$$
\begin{align*}
& \times H\left(x_{n}, y_{n}\right)\left(y_{n}-x_{n}\right) \\
& -\frac{1}{2} \int_{0}^{1} F^{\prime \prime}\left(x_{n}+t\left(y_{n}-x_{n}\right)\right) d t\left(y_{n}-x_{n}\right) \\
& \times H\left(x_{n}, y_{n}\right)\left(y_{n}-x_{n}\right) \\
& +\frac{\alpha}{2} \int_{0}^{1} F^{\prime \prime}\left(x_{n}+t\left(y_{n}-x_{n}\right)\right) d t\left(y_{n}-x_{n}\right) H\left(x_{n}, y_{n}\right) \\
& \times H\left(x_{n}, y_{n}\right)\left(y_{n}-x_{n}\right) \tag{21}
\end{align*}
$$

Hence,

$$
\begin{align*}
& F\left(x_{n+1}\right) \\
&= \int_{0}^{1} F^{\prime \prime}\left(y_{n}+t\left(x_{n+1}-y_{n}\right)\right)(1-t) d t\left(x_{n+1}-y_{n}\right)^{2} \\
&+\int_{0}^{1}\left[F^{\prime \prime}\left(x_{n}+t\left(y_{n}-x_{n}\right)\right)(1-t)\right. \\
&\left.\quad-\frac{1}{2} F^{\prime \prime}\left(x_{n}+p t\left(y_{n}-x_{n}\right)\right)\right] d t\left(y_{n}-x_{n}\right)^{2} \\
&-\frac{1-\alpha}{2} \int_{0}^{1} F^{\prime \prime}\left(x_{n}+t\left(y_{n}-x_{n}\right)\right) d t \\
& \times\left(y_{n}-x_{n}\right) H\left(x_{n}, y_{n}\right)\left(y_{n}-x_{n}\right) \\
&-\frac{\alpha}{2} \int_{0}^{1}\left[F^{\prime \prime}\left(x_{n}+t\left(y_{n}-x_{n}\right)\right)\right. \\
& \quad\left.\quad-F^{\prime \prime}\left(x_{n}+p t\left(y_{n}-x_{n}\right)\right)\right] d t \\
&+\frac{\alpha}{2} \int_{0}^{1} F^{\prime \prime}\left(x_{n}\right) H\left(x_{n}, y_{n}\right)\left(y_{n}-x_{n}\right) \\
&\left.\times\left(y_{n}-x_{n}\right) H\left(x_{n}, y_{n}\right)\right) d t \\
&=\left(x_{n}, y_{n}\right)\left(y_{n}-x_{n}\right) . \tag{22}
\end{align*}
$$

This completes the proof.
Theorem 3. Let $X$ and $Y$ be Banach space and let $\Omega \subset X$ be an open convex domain. Assume $F: \Omega \subset X \rightarrow Y$ has secondorder continuous Frechet derivatives. For an initial value $x_{0} \in$ $\Omega$ and fixed parameters $\alpha \in[0,1), p \in(0,1]$, if $F^{\prime}\left(x_{0}\right)^{-1}$ exists and the conditions

$$
\begin{gathered}
\left\|F^{\prime}\left(x_{0}\right)^{-1}\right\| \leq \beta, \quad\left\|y_{0}-x_{0}\right\|=\left\|F^{\prime}\left(x_{0}\right)^{-1} F\left(x_{0}\right)\right\| \leq \eta \\
\left\|F^{\prime \prime}(x)\right\| \leq M, \quad\left\|F^{\prime \prime}(x)-F^{\prime \prime}(y)\right\| \leq N\|x-y\| \\
x, y \in \Omega \\
\max \left\{M \sqrt{1+\frac{(2+3 p) N}{6 \beta(1-\alpha) M^{2}}}, M \sqrt[3]{1+\frac{(1-p) N}{2 \beta M^{2}}}\right\} \leq K \\
\overline{S\left(x_{0}, t^{*}\right)} \subset \Omega
\end{gathered}
$$

are satisfied and $h=K \beta \eta \leq 1 / 2$, then the sequence $\left\{x_{n}\right\}_{n \geq 0}$ generated by (16) is well defined and converges to a unique solution $x^{*}$ of (1) in $S\left(x_{0}, t^{* *}\right)$.

Theorem 4. Assume $F$ satisfies conditions in Theorem 3. Denoting $\theta=\left(t^{*} / t^{* *}\right)=(1-\sqrt{1-2 h}) /(1+\sqrt{1-2 h})$, one has the following:
(i) when $h=K \beta \eta<6 \sqrt{2}-8=0.4852 \cdots$,

$$
\begin{equation*}
\left\|x_{n}-x^{*}\right\| \leq t^{*}-t_{n} \leq \frac{\left(1-\theta^{2}\right) \eta}{(1-(1 / \sqrt{2}))[\sqrt{2} \theta]^{3^{n}}}[\sqrt{2} \theta]^{3^{n}-1} \tag{24}
\end{equation*}
$$

(ii) when $\alpha \geq 3 / 8, h \leq 4(9 \sqrt{5}-20)=0.4984 \cdots$,

$$
\begin{align*}
\left\|x_{n}-x^{*}\right\| & \leq t^{*}-t_{n} \\
& \leq \frac{\left(1-\theta^{2}\right) \eta}{(1-(2 / \sqrt{5}))[(\sqrt{5} / 2) \theta]^{3^{n}}}\left[\frac{\sqrt{5}}{2} \theta\right]^{3^{n}-1} . \tag{25}
\end{align*}
$$

## 3. Proof of Theorem

Proof of Theorem 3. To prove Theorem 3, we first prove that following items are true for all $n \geq 0$ :

$$
\begin{aligned}
& \left(\mathrm{I}_{n}\right) x_{n} \in \overline{S\left(x_{0}, t_{n}\right)} ; \\
& \left(\mathrm{II}_{n}\right)\left\|F^{\prime}\left(x_{n}\right)^{-1}\right\| \leq-g^{\prime}\left(t_{n}\right)^{-1} ; \\
& \left(\mathrm{III}_{n}\right)\left\|y_{n}-x_{n}\right\| \leq s_{n}-t_{n} ; \\
& \left(\mathrm{IV}_{n}\right) y_{n} \in \overline{S\left(x_{0}, s_{n}\right)} ; \\
& \left(\mathrm{V}_{n}\right)\left\|x_{n+1}-y_{n}\right\| \leq t_{n+1}-s_{n} .
\end{aligned}
$$

It is easy to check for the case $n=0$ by the initial conditions. By using mathematical induction, assume that the above statements are true for some fixed $n \geq 0$. Then we have

$$
\begin{align*}
& \left(\mathrm{I}_{n+1}\right): \\
& \qquad \begin{aligned}
\left\|x_{n+1}-x_{0}\right\| & \leq\left\|x_{n+1}-y_{n}\right\|+\left\|y_{n}-x_{0}\right\| \\
& \leq t_{n+1}-s_{n}+s_{n}=t_{n+1}
\end{aligned} \tag{26}
\end{align*}
$$

$\left(\mathrm{II}_{n+1}\right):$

$$
\begin{align*}
\left\|F^{\prime}\left(x_{n+1}\right)-F^{\prime}\left(x_{0}\right)\right\| & \leq M\left\|x_{n+1}-x_{0}\right\| \\
& \leq M t_{n+1}<K t^{*}=K \eta \frac{1-\sqrt{1-2 h}}{h} \\
& \leq \frac{1}{\beta}=\frac{1}{\left\|f^{\prime}\left(x_{0}\right)^{-1}\right\|} . \tag{27}
\end{align*}
$$

By Banach Lemma, we get that $F^{\prime}\left(x_{n+1}\right)^{-1}$ exists and

$$
\begin{aligned}
& \left\|F^{\prime}\left(x_{n+1}\right)^{-1}\right\| \\
& \leq \frac{\left\|F^{\prime}\left(x_{0}\right)^{-1}\right\|}{1-\left\|F^{\prime}\left(x_{0}\right)^{-1}\right\| \cdot\left\|F^{\prime}\left(x_{n+1}\right)-F^{\prime}\left(x_{0}\right)\right\|} \\
& \leq \frac{\beta}{1-\beta M\left\|x_{n+1}-x_{0}\right\|} \\
& \leq \frac{1}{(1 / \beta)-K\left\|x_{n+1}-x_{0}\right\|} \leq \frac{1}{(1 / \beta)-K t_{n+1}} \\
& =-g^{\prime}\left(t_{n+1}\right)^{-1}
\end{aligned}
$$

$\left(\mathrm{III}_{n+1}\right)$ :

$$
\begin{aligned}
& \| \int_{0}^{1}\left[F^{\prime \prime}\left(x_{n}+t\left(y_{n}-x_{n}\right)\right)(1-t)\right. \\
& \left.\quad-\frac{1}{2} F^{\prime \prime}\left(x_{n}+p t\left(y_{n}-x_{n}\right)\right)\right] d t \| \\
& \leq\left\|\int_{0}^{1}\left[F^{\prime \prime}\left(x_{n}+t\left(y_{n}-x_{n}\right)\right)-F^{\prime \prime}\left(x_{n}\right)\right](1-t) d t\right\| \\
& \quad+\frac{1}{2} \| \int_{0}^{1}\left[F^{\prime \prime}\left[\left(x_{n}+p t\left(y_{n}-x_{n}\right)\right)-F^{\prime \prime}\left(x_{n}\right)\right] d t \|\right. \\
& \leq \frac{(2+3 p) N}{12}\left\|y_{n}-x_{n}\right\| \\
& \left\|H\left(x_{n+1}, y_{n+1}\right)\right\| \\
& =\frac{1}{p \| F^{\prime}\left(x_{n+1}\right)^{-1} \int_{0}^{1} F^{\prime \prime}\left(x_{n+1}+p t\left(y_{n+1}-x_{n+1}\right)\right)} \\
& \quad \times p\left(y_{n+1}-x_{n+1}\right) d t \|
\end{aligned}
$$

By Lemma 2 and $0<-g^{\prime}\left(t_{n}\right) \leq 1 / \beta$,

$$
\begin{aligned}
& \left\|F\left(x_{n+1}\right)\right\| \\
& \leq \frac{M}{2}\left\|x_{n+1}-y_{n}\right\|^{2}+\frac{(2+3 p) N}{12} \\
& \quad \times\left\|y_{n}-x_{n}\right\|^{3}+\frac{(1-\alpha) M^{2}}{-2 g^{\prime}\left(t_{n}\right)}\left\|y_{n}-x_{n}\right\|^{3} \\
& \quad+\frac{\alpha M^{3}}{2\left[g^{\prime}\left(t_{n}\right)\right]^{2}}\left\|y_{n}-x_{n}\right\|^{4}+\frac{\alpha M N(1-p)}{-4 g^{\prime}\left(t_{n}\right)}\left\|y_{n}-x_{n}\right\|^{4}
\end{aligned}
$$

$$
\begin{align*}
\leq & \frac{K}{2}\left\|x_{n+1}-y_{n}\right\|^{2}+\left[1+\frac{N(1-p)}{2 \beta M^{2}}\right] \frac{\alpha M^{3}\left\|y_{n}-x_{n}\right\|^{4}}{2\left[g^{\prime}\left(t_{n}\right)\right]^{2}} \\
& +\left[1+\frac{(2+3 p) N}{6 \beta(1-\alpha) M^{2}}\right] \frac{(1-\alpha) M^{2}}{-2 g^{\prime}\left(t_{n}\right)}\left\|y_{n}-x_{n}\right\|^{3} \\
\leq & \frac{K}{2}\left(t_{n+1}-s_{n}\right)^{2}+\frac{\alpha K^{3}}{2} \frac{\left(s_{n}-t_{n}\right)^{4}}{\left[g^{\prime}\left(t_{n}\right)\right]^{2}}-\frac{(1-\alpha) K^{2}}{2 g^{\prime}\left(t_{n}\right)}\left(s_{n}-t_{n}\right)^{3} \\
= & g\left(t_{n+1}\right) . \tag{30}
\end{align*}
$$

Hence, we deduce that

$$
\begin{align*}
\left\|y_{n+1}-x_{n+1}\right\| & \leq\left\|-F^{\prime}\left(x_{n+1}\right)^{-1}\right\|\left\|F\left(x_{n+1}\right)\right\| \\
& \leq-g^{\prime}\left(t_{n+1}\right)^{-1} g\left(t_{n+1}\right)=s_{n+1}-t_{n+1} \tag{31}
\end{align*}
$$

Moreover, we have $\left(\mathrm{IV}_{n+1}\right)$ :

$$
\begin{align*}
\left\|y_{n+1}-x_{0}\right\| & \leq\left\|y_{n+1}-x_{n+1}\right\|+\left\|x_{n+1}-x_{0}\right\| \\
& \leq\left(s_{n+1}-t_{n+1}\right)+t_{n+1}=s_{n+1} \tag{32}
\end{align*}
$$

$\left(\mathrm{V}_{n+1}\right)$ :

$$
\begin{align*}
\| & x_{n+2}-y_{n+1} \| \\
= & \left\|-\frac{1}{2} H\left(x_{n+1}, y_{n+1}\right)\left[I-\alpha H\left(x_{n+1}, y_{n+1}\right)\right]\left(y_{n+1}-x_{n+1}\right)\right\| \\
\leq & \frac{1}{2} \frac{M\left\|y_{n+1}-x_{n+1}\right\|}{-g^{\prime}\left(t_{n+1}\right)}\left[1+\alpha \frac{M\left\|y_{n+1}-x_{n+1}\right\|}{-g^{\prime}\left(t_{n+1}\right)}\right] \\
& \times\left\|y_{n+1}-x_{n+1}\right\| \\
\leq & -\frac{1}{2} \frac{K\left(s_{n+1}-t_{n+1}\right)}{g^{\prime}\left(t_{n+1}\right)}\left[1-\alpha \frac{K\left(s_{n+1}-t_{n+1}\right)}{g^{\prime}\left(t_{n+1}\right)}\right]\left(s_{n+1}-t_{n+1}\right) \\
= & t_{n+2}-s_{n+1} . \tag{33}
\end{align*}
$$

By Lemma 1, if $h \leq 1 / 2$, then the sequence $\left\{x_{n},\right\}_{n \geq 0}$ generated by (16) is well defined, remains in $\overline{S\left(x_{0}, t^{*}\right)}$ for all $n \geq 0$, and converges to a solution $x^{*}$ of (1).

To show uniqueness, let us assume that there exists a second solution $y^{*}$ of (1) in $S\left(x_{0}, t^{* *}\right)$. Then

$$
\begin{align*}
& \left\|F^{\prime}\left(x_{0}\right)^{-1} \int_{0}^{1} F^{\prime}\left(x^{*}+t\left(y^{*}-x^{*}\right)\right) d t-I\right\| \\
& \leq\left\|F^{\prime}\left(x_{0}\right)^{-1}\right\|\left\|\int_{0}^{1} F^{\prime}\left[x^{*}+t\left(y^{*}-x^{*}\right)\right]-F^{\prime}\left(x_{0}\right) d t\right\| \\
& \leq \beta M \int_{0}^{1}\left\|x^{*}+t\left(y^{*}-x^{*}\right)-x_{0}\right\| d t  \tag{34}\\
& \leq \beta M \int_{0}^{1}\left[(1-t)\left\|x^{*}-x_{0}\right\|+t\left\|y^{*}-x_{0}\right\|\right] d t \\
& <\frac{\beta M}{2}\left(t^{*}+t^{* *}\right) \leq 1 .
\end{align*}
$$

By Banach Lemma, we can obtain that the inverse of the linear operator $\int_{0}^{1} F^{\prime}\left[x^{*}+t\left(y^{*}-x^{*}\right)\right] d t$ exists and

$$
\begin{equation*}
F\left(y^{*}\right)-F\left(x^{*}\right)=\int_{0}^{1} F^{\prime}\left[x^{*}+t\left(y^{*}-x^{*}\right)\right] d t\left(y^{*}-x^{*}\right) \tag{35}
\end{equation*}
$$

We conclude that $x^{*}=y^{*}$. The proof of Theorem 3 is completed.

Proof of Theorem 4. By Lemma 1, we have

$$
\begin{align*}
& t^{*}-t_{n+1} \\
& \begin{aligned}
&=\left(t^{*}-t_{n}\right)^{3} {\left[\left(t^{*}-t_{n}\right)^{3}+4\left(t^{*}-t_{n}\right)^{2}\left(t^{* *}-t_{n}\right)\right.} \\
&+5\left(t^{*}-t_{n}\right)\left(t^{* *}-t_{n}\right)^{2}+(2-2 \alpha) \\
&\left.\times\left(t^{* *}-t_{n}\right)^{3}\right] \\
& \times\left(\left[\left(t^{*}-t\right)+\left(t^{* *}-t\right)\right]^{5}\right)^{-1} \\
&<\left(t^{*}-t_{n}\right)^{3}\left[5\left(t^{*}-t_{n}\right)^{2}\left(t^{* *}-t_{n}\right)\right. \\
&\left.+5\left(t^{*}-t_{n}\right)\left(t^{* *}-t_{n}\right)^{2}+2\left(t^{* *}-t_{n}\right)^{3}\right] \\
& \times\left(\left[\left(t^{*}-t\right)+\left(t^{* *}-t\right)\right]^{5}\right)^{-1} .
\end{aligned}
\end{align*}
$$

Similarly,

$$
\begin{align*}
& t^{* *}-t_{n+1} \\
& \begin{aligned}
&=\left(t^{* *}-t_{n}\right)^{3}\left[\left(t^{* *}-t_{n}\right)^{3}+4\left(t^{* *}-t_{n}\right)^{2}\left(t^{*}-t_{n}\right)\right. \\
&\left.+5\left(t^{* *}-t_{n}\right)\left(t^{*}-t_{n}\right)^{2}+(2-2 \alpha)\left(t^{*}-t_{n}\right)^{3}\right] \\
& \times\left(\left[\left(t^{*}-t\right)+\left(t^{* *}-t\right)\right]^{5}\right)^{-1} \\
&>\left(t^{* *}-t_{n}\right)^{3}\left[\left(t^{* *}-t_{n}\right)^{3}+4\left(t^{* *}-t_{n}\right)^{2}\left(t^{*}-t_{n}\right)\right. \\
&\left.+5\left(t^{* *}-t_{n}\right)\left(t^{*}-t_{n}\right)^{2}\right]
\end{aligned} \\
& \quad \times\left(\left[\left(t^{*}-t\right)+\left(t^{* *}-t\right)\right]^{5}\right)^{-1} .
\end{align*}
$$

Hence,

$$
\begin{align*}
\frac{t^{*}-t_{n+1}}{t^{* *}-t_{n+1}} & <2\left[\frac{t^{*}-t_{n}}{t^{* *}-t_{n}}\right]^{3}<2 \cdot 2^{3}\left[\frac{t^{*}-t_{n-1}}{t^{* *}-t_{n-1}}\right]^{3^{2}} \\
& <\cdots<2 \cdot 2^{3} \cdots 2^{3^{n}}\left[\frac{t^{*}-t_{0}}{t^{* *}-t_{0}}\right]^{3^{n+1}}=\sqrt{2}^{3^{n+1}-1} \theta^{3^{n+1}} \tag{38}
\end{align*}
$$

Because $t^{* *}-t_{n}=t^{*}-t_{n}+t^{* *}-t^{*}=t^{*}-t_{n}+\left(\left(1-\theta^{2}\right) \eta\right) / \theta$, we obtain

$$
\begin{equation*}
t^{*}-t_{n} \leq \frac{\left(1-\theta^{2}\right) \eta}{(1-(1 / \sqrt{2}))[\sqrt{2} \theta]^{3^{n}}}[\sqrt{2} \theta]^{3^{n}-1} \tag{39}
\end{equation*}
$$

Table 1: Error computing results of (42).

| $n$ | Newton method | Method (16) with $p=2 / 3, \alpha=1 / 2$ |
| :--- | :---: | :---: |
| 1 | $5.215 \times 10^{-1}$ | $2.2371 \times 10^{-3}$ |
| 2 | $6.610 \times 10^{-2}$ | $1.3667 \times 10^{-9}$ |
| 3 | $14.017 \times 10^{-4}$ | $3.1229 \times 10^{-28}$ |

When $\alpha \geq 3 / 8$,

$$
\begin{align*}
& t^{*}- t_{n+1} \\
&=\left(t^{*}-t_{n}\right)^{3}\left[\left(t^{*}-t_{n}\right)^{3}+4\left(t^{*}-t_{n}\right)^{2}\left(t^{* *}-t_{n}\right)\right. \\
&\left.+5\left(t^{*}-t_{n}\right)\left(t^{* *}-t_{n}\right)^{2}+(2-2 \alpha)\left(t^{* *}-t_{n}\right)^{3}\right] \\
& \times\left(\left[\left(t^{*}-t\right)+\left(t^{* *}-t\right)\right]^{5}\right)^{-1} \\
&<\left(t^{*}-t_{n}\right)^{3}\left[5\left(t^{*}-t_{n}\right)^{2}\left(t^{* *}-t_{n}\right)\right. \\
&\left.+5\left(t^{*}-t_{n}\right)\left(t^{* *}-t_{n}\right)^{2}+\left(\frac{5}{4}\right)\left(t^{* *}-t_{n}\right)^{3}\right] \\
& \times\left(\left[\left(t^{*}-t\right)+\left(t^{* *}-t\right)\right]^{5}\right)^{-1} \\
&= \frac{5}{4}\left(t^{*}-t_{n}\right)^{3}\left[4\left(t^{*}-t_{n}\right)^{2}\left(t^{* *}-t_{n}\right)\right. \\
&\left.\quad+4\left(t^{*}-t_{n}\right)\left(t^{* *}-t_{n}\right)^{2}+\left(t^{* *}-t_{n}\right)^{3}\right] \\
& \times\left(\left[\left(t^{*}-t\right)+\left(t^{* *}-t\right)\right]^{5}\right)^{-1} . \tag{40}
\end{align*}
$$

Hence, we get

$$
\begin{gather*}
\frac{t^{*}-t_{n+1}}{t^{* *}-t_{n+1}}<\frac{5}{4}\left[\frac{t^{*}-t_{n}}{t^{* *}-t_{n}}\right]^{3}<\left(\frac{\sqrt{5}}{2}\right)^{3^{n+1}-1} \theta^{3^{n+1}}, \\
\left\|x_{n}-x^{*}\right\| \leq t^{*}-t_{n} \leq \frac{\left(1-\theta^{2}\right) \eta}{1-(2 / \sqrt{5})[(\sqrt{5} / 2) \theta]^{3^{n}}}\left[\frac{\sqrt{5}}{2} \theta\right]^{3^{n}-1} . \tag{41}
\end{gather*}
$$

The proof of Theorem 4 is completed.

## 4. Applications

Example 1. Consider the case as follows:

$$
\begin{equation*}
x(s)=1+\frac{1}{4} x(s) \int_{0}^{1} \frac{s}{s+t} x(t) d t \tag{42}
\end{equation*}
$$

where the space is $X=C[0,1]$ with norm

$$
\begin{equation*}
\|x\|=\max _{0 \leq s \leq 1}|x(s)| . \tag{43}
\end{equation*}
$$

This equation arises in the theory of the radiative transfer, neutron transport, and kinetic theory of gasses. Let us define the operator $F$ on $X$ by

$$
\begin{equation*}
F(x)=\frac{1}{4} x(s) \int_{0}^{1} \frac{s}{s+t} x(t) d t-x(s)+1 \tag{44}
\end{equation*}
$$

TABLE 2: Error computing results $\left(\left|x_{n}-x^{*}\right|\right)$.

| Step | Newton's method | Halley's method (3) | Method (16) with $p=2 / 3, \alpha=1$ | Method (16) with $p=1 / 2, \alpha=0$ |
| :--- | :---: | :---: | :---: | :---: |
| $n=1$ | $2.56574 \times 10^{-2}$ | $2.13749 \times 10^{-3}$ | $1.58011 \times 10^{-3}$ | $5.74362 \times 10^{-3}$ |
| $n=2$ | $4.8337 \times 10^{-4}$ | $3.04697 \times 10^{-9}$ | $1.19327 \times 10^{-11}$ | $1.98875 \times 10^{-7}$ |
| $n=3$ | $1.75164 \times 10^{-7}$ | $8.84002 \times 10^{-27}$ | $3.90354 \times 10^{-44}$ | $8.35734 \times 10^{-21}$ |
| $n=4$ | $2.30119 \times 10^{-14}$ | $2.15878 \times 10^{-79}$ | $4.47038 \times 10^{-174}$ | $6.20201 \times 10^{-61}$ |

Then, for $x_{0}=1$, we can obtain

$$
\begin{gather*}
M=2 \cdot \frac{1}{4} \max _{0 \leq s \leq 1}\left|\int_{0}^{1} \frac{s}{s+t} d t\right|=\frac{\ln 2}{2}=0.3465, \\
N=0, \quad K=M=0.3465, \quad \beta=\left\|F^{\prime}\left(x_{0}\right)^{-1}\right\|=1.5304, \\
\eta=\left\|F^{\prime}\left(x_{0}\right)^{-1} F^{\prime}\left(x_{0}\right)\right\|=0.2652, \\
h=K \beta \eta=0.1406<0.4852, \quad t^{*}=0.2870, \\
t^{* *}=3.485, \quad \theta=0.08240 . \tag{45}
\end{gather*}
$$

That means the hypotheses of the theorem are satisfied and for $\alpha=1 / 2>3 / 8$ the error bound becomes

$$
\begin{align*}
& \left\|x_{n}-x^{*}\right\| \\
& \leq t^{*}-t_{n} \leq \frac{\left(1-\theta^{2}\right) \eta}{(1-(2 / \sqrt{5}))[(\sqrt{5} / 2) \theta]^{3^{n}}}\left[\frac{\sqrt{5}}{2} \theta\right]^{3^{n}-1}  \tag{46}\\
& =\frac{0.2634}{1-0.8944 \cdot(0.07370)^{3^{n}}}(0.07370)^{3^{n}-1}
\end{align*}
$$

For $n=1,2,3$, we get

$$
\begin{align*}
& \left\|x_{1}-x^{*}\right\| \leq t^{*}-t_{1} \leq 2.2371 \times 10^{-3} \\
& \left\|x_{2}-x^{*}\right\| \leq t^{*}-t_{2} \leq 1.3667 \times 10^{-9}  \tag{47}\\
& \left\|x_{3}-x^{*}\right\| \leq t^{*}-t_{3} \leq 3.1229 \times 10^{-28}
\end{align*}
$$

In practical computation, we use a discretization process. By Gauss-Legendre quadrature formula with 8 nodes:

$$
\begin{equation*}
\int_{0}^{1} \varphi(t) d t \simeq \sum_{j=1}^{8} \omega_{j} \varphi\left(t_{j}\right), \tag{48}
\end{equation*}
$$

we approximate the integral equation (42), where the nodes $t_{j}$ and the weights $\omega_{j}$ are known. Denote $x\left(t_{i}\right)$ by $x_{i}, i=$ $1,2, \ldots 8$, so we can transform (42) into the following system of nonlinear equations:

$$
\begin{equation*}
x_{i}=1+\frac{1}{4} x_{i} \sum_{j=1}^{8} a_{i j} x_{j} \tag{49}
\end{equation*}
$$

where $a_{i j}=w_{j}\left(t_{i} /\left(t_{i}+t_{j}\right)\right)$. Then, we rewrite the above system in the matrix form. Consider

$$
\begin{equation*}
F(X)=X-V-\frac{1}{4} A W, \tag{50}
\end{equation*}
$$

where $X=\left(x_{1}, x_{2}, \ldots, x_{8}\right)^{T}, V=(1,1, \ldots, 1)^{T}, A=$ $\left(a_{i j}\right)_{i, j=1}^{8}$, and $W=\left(x_{1}^{2}, x_{2}^{2}, \ldots, x_{8}^{2}\right)^{T}$. We also get

$$
\begin{equation*}
F^{\prime}(X)=I-\frac{1}{2} A \operatorname{diag}\left\{x_{1}, x_{2}, \ldots, x_{8}\right\} . \tag{51}
\end{equation*}
$$

After 3 iterative steps, we can obtain the numerical solution of (42) by method (16) with $p=2 / 3, \alpha=1 / 2$, and the solution $X_{*}=(1.2751,1.4418,1.5595,1.6486,1.7189$, $1.7761,1.8236,1.8638)^{T}$, and we present the error computing result of (42) in Table 1.

Example 2. Now we employ iterative methods (16) to solve the equation and compare these methods with Newton's method, Halley's method, and modified Halley's methods (16). We define as follows:

$$
\begin{equation*}
f(x)=x^{n}-R, \quad n>2, R \in(0,+\infty) . \tag{52}
\end{equation*}
$$

Denote $x^{*}=\sqrt[n]{R}$, by (16) $x_{n+1}=v\left(x_{n}\right)$, where $v(x)=x-$ $\left[1-(1 / 2) H(x)+(\alpha / 2) H^{2}(x)\right](f(x)) /\left(f^{\prime}(x)\right), H(x)=\left(f^{\prime}(x-\right.$ $\left.\left.p f(x) / f^{\prime}(x)\right)-f^{\prime}(x)\right) /\left(p f^{\prime}(x)\right)$. We have $v^{\prime}\left(x^{*}\right)=v^{\prime \prime}\left(x^{*}\right)=$ $v^{\prime \prime \prime}\left(x^{*}\right)=0$ if

$$
\begin{equation*}
p=\frac{2}{3}, \quad \alpha=1 . \tag{53}
\end{equation*}
$$

So, we get the convergence of the sequence $\left\{x_{n}\right\}$ generated by modified Halley's method (16) with four orders when $p=$ $2 / 3, \alpha=1$.

Now, we compare some of these methods for the calculus of $x^{*}=\sqrt[4]{16}=2$. We analyze the errors $x_{n}-x^{*}$ for these methods with Newton's method, Halley's method, and the family of new modified Halley's methods (16). In these cases, we have taken $n=4, x_{0}=2.2$ (see Table 2).

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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