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Research Article

Discussions on Recent Results for α-ψ-Contractive Mappings

N. Hussain,[1] M. A. Kutbi,[1] S. Khaleghizadeh,[2] and P. Salimi[3]

[1] *Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*
[2] *Department of Mathematics, Payame Noor University, P.O. Box 19395-3697, Tehran, Iran*
[3] *Young Researchers and Elite Club, Rasht Branch, Islamic Azad University, Rasht, Iran*

Correspondence should be addressed to P. Salimi; salimipeyman@gmail.com

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We establish certain fixed point results for α-η-generalized convex contractions, α-η-weakly Zamfirescu mappings, and α-η-Ćirić strong almost contractions. As an application, we derive some Suzuki type fixed point theorems and certain new fixed point theorems in metric spaces endowed with a graph and a partial order. Moreover, we discuss some illustrative examples to highlight the realized improvements.

1. Introduction

Banach contraction principle states that every contraction mapping defined on a complete metric space X has a unique fixed point and that point can be obtained as a limit of repeated iteration of the mapping at any point of X. This fundamental fixed point theorem has laid the foundation of metric fixed point theory which is very important due to its applications in different fields such as image processing, physics, computer science [1], economics, and telecommunication (see for more details [2–11]).

Istrăţescu [12] introduced and studied the notion of convex contractions. Recently Miandaragh et al. [13] proved certain results for generalized convex contractions on complete metric spaces. Salimi et al. [14] modified the concept of α-admissible mappings introduced and studied by Samet et al. [15], Karapınar and Samet [16], and Salimi and Karapınar [17]. We establish certain fixed point results for α-η-generalized convex contractions, α-η-weakly Zamfirescu mappings, and α-η-Ćirić strong almost contractions. As an application, we shall derive corresponding results in metric spaces endowed with a graph and a partial order.

2. Discussion on α-η-ψ-Contractive Mappings

We shall denote by Ψ the family of nondecreasing functions $\psi : [0,+\infty) \to [0,+\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is the nth iterate of ψ. Clearly, if $\psi \in \Psi$, then $\psi(t) < t$ for all $t > 0$.

Samet et al. [15] introduced following concept.

Definition 1. Let (X, d) be a metric space, let $T : X \to X$ be a self-mapping, and let $\alpha : X \times X \to [0,\infty)$ be a function. One says that T is an α-ψ-contractive mapping if

$$\alpha(x, y)\, d(Tx, Ty) \le \psi(d(x, y)) \tag{1}$$

holds for all $x, y \in X$, where $\psi \in \Psi$.

By taking $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = rt$, where $0 \le r < 1$, α-ψ-contractive mapping reduces to Banach contraction mapping.

We suggest the following notion as generalization of α-ψ-contractive mappings.

Definition 2. Let (X, d) be a metric space, let $T : X \to X$ be a self-mapping, and let $\alpha, \eta : X \times X \to [0,\infty)$ be two functions. One says that T is an α-η-ψ-contractive mapping if for all $x, y \in X$ with $\eta(x, Tx) \le \alpha(x, y)$ we have

$$d(Tx, Ty) \le \psi(d(x, y)) \tag{2}$$

for some $\psi \in \Psi$.

Example 3. Let $X = [0,\infty)$ be endowed with usual metric and let $T : X \to X$ be defined by $Tx = rx$, where $0 < r < 1$.

Also, let $\alpha, \eta : X \times X \to [0, \infty)$ be two functions such that only $\alpha(x_0, y_0) \geq 1/r$ for some $x_0, y_0 \in X$ with $x_0 \neq y_0$. Then, T is not an α-ψ-contractive mapping while it is a Banach contraction and α-η-ψ-contractive mapping. In fact,

$$\begin{aligned}\alpha(x_0, y_0) d(Tx_0, Ty_0) &\geq \frac{1}{r}|rx_0 - ry_0| \\ &= |x_0 - y_0| > \psi(|x_0 - y_0|) \\ &= \psi(d(x_0, y_0))\end{aligned} \tag{3}$$

while $d(Tx, Ty) \leq rd(x, y)$ holds for all $x, y \in X$ where $\psi(t) = rt$.

Example 4. Let $X = [0, 1]$ be endowed with usual metric and let $T : X \to X$ be defined by $Tx = (1/4)x^2$. Also, let $\alpha, \eta : X \times X \to [0, \infty)$ be two functions such that only $\alpha(0, 1) = 8$. Then, T is not an α-ψ-contractive mapping while it is a Banach contraction and α-η-ψ-contractive mapping. In fact,

$$\alpha(0, 1) d(T0, T1) = 2 > 1 > \psi(1) = \psi(d(0, 1)) \tag{4}$$

while $d(Tx, Ty) \leq (1/2)d(x, y)$ holds for all $x, y \in X$ where $\psi(t) = (1/2)t$.

Similarly, one may develop other examples of self-mappings that are not α-ψ-contractive mappings while they are Banach contraction and α-η-ψ-contractive mappings.

Remark 5. It is worth to notice that there is no Banach contraction mapping which is not α-η-ψ-contractive. Indeed, let T be a Banach contraction mapping on X with contraction constant k such that T is not an α-η-ψ-contractive mapping. Then for all $\psi \in \Psi$, there exists $x_0, y_0 \in X$ such that $\alpha(x_0, y_0) \geq \eta(x_0, Tx_0)$ and $d(Tx_0, Ty_0) > \psi(d(x_0, y_0))$. But $\psi(t) = kt$ produces a contradiction to the fact that T is a Banach contraction mapping.

More recently, Miandaragh et al. [13] introduced the following notions.

Definition 6. Let (X, d) be a metric space and let $T : X \to X$ be a self-mapping. One says T is a generalized convex contraction if there exist $a, b \geq 0$ with $a+b < 1$ and a function $\alpha : X \times X \to [0, \infty)$ such that

$$\alpha(x, y) d(T^2x, T^2y) \leq ad(Tx, Ty) + bd(x, y) \tag{5}$$

holds for all $x, y \in X$.

Definition 7. Let (X, d) be a metric space and let $T : X \to X$ be a self-mapping. One says T is a generalized convex contraction of order 2 if there exist $a_1, a_2, b_1, b_2 \geq 0$ with $a_1 + a_2 + b_1 + b_2 < 1$ and a function $\alpha : X \times X \to [0, \infty)$ such that

$$\begin{aligned}\alpha(x, y) d(T^2x, T^2y) \leq{}& a_1 d(x, Tx) + a_2 d(Tx, T^2x) \\ &+ b_1 d(y, Ty) + b_2 d(Ty, T^2y)\end{aligned} \tag{6}$$

holds for all $x, y \in X$.

On the basis of the above facts, we suggest the notions of generalized convex contraction and generalized convex contraction of order 2 as follows.

Definition 8. Let (X, d) be a metric space, let $T : X \to X$ be a self-mapping, and let $\alpha, \eta : X \times X \to [0, \infty)$ be two functions. Then T is said to be an α-η-generalized convex contraction if

$$\begin{aligned}x, y \in X, \quad \eta(x, Tx) \leq \alpha(x, y) \Longrightarrow{}& d(T^2x, T^2y) \\ &\leq ad(Tx, Ty) + bd(x, y),\end{aligned} \tag{7}$$

where $a, b \geq 0$ with $a + b < 1$.

Definition 9. Let (X, d) be a metric space, let $T : X \to X$ be a self-mapping, and let $\alpha, \eta : X \times X \to [0, \infty)$ be two functions. Then T is said to be an α-η-generalized convex contraction of order 2 if

$$\begin{aligned}x, y \in X, \quad \eta(x, Tx) \leq \alpha(x, y) \Longrightarrow{}& d(T^2x, T^2y) \\ &\leq a_1 d(x, Tx) + a_2 d(Tx, T^2x) \\ &\quad + b_1 d(y, Ty) + b_2 d(Ty, T^2y),\end{aligned} \tag{8}$$

where, $a_1, a_2, b_1, b_2 \geq 0$ and $a_1 + a_2 + b_1 + b_2 < 1$.

Example 10. Let $X = [0, \infty)$ be endowed with usual metric and let $T : X \to X$ be defined by $Tx = rx$, where $0 < r < 1$. Also, let $\alpha, \eta : X \times X \to [0, \infty)$ be two functions such that $\alpha(x_0, y_0) \geq 1/r^2$ for some $x_0, y_0 \in X$ with $x_0 \neq y_0$. Then, T is not a generalized convex contraction while it is a convex contraction and α-η-generalized convex contraction. Indeed,

$$\begin{aligned}\alpha(x_0, y_0) d(T^2x_0, T^2y_0) &= \frac{1}{r^2}|r^2x_0 - r^2y_0| \\ &= |x_0 - y_0| \\ &> (a + b)|x_0 - y_0| \\ &\geq a|rx_0 - ry_0| + b|x_0 - y_0| \\ &= ad(Tx_0, Ty_0) + bd(x_0, y_0)\end{aligned} \tag{9}$$

for all $a, b \in \mathbb{R}_+$ with $a + b < 1$. That is, T is not a generalized convex contraction mapping. But if we choose $a = r^2/2$ and $b = r/2$ then,

$$\begin{aligned}d(T^2x, T^2y) &= r^2|x - y| \\ &= \left(\frac{r^2}{2} + \frac{r^2}{2}\right)|x - y| \\ &= \frac{r^2}{2}|x - y| + \frac{r^2}{2}|x - y| \\ &\leq \frac{r^2}{2}|x - y| + \frac{r}{2}|x - y| \\ &= ad(Tx, Ty) + bd(x, y)\end{aligned} \tag{10}$$

holds for all $x, y \in X$. That is, T is a convex contraction and α-η-generalized convex contraction mapping.

Example 11. Let $X = [0, \infty)$ be endowed with metric

$$d(x, y) = \begin{cases} \max\{x, y\}, & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases} \tag{11}$$

Let $T : X \to X$ be defined by $Tx = (1/4)x$ and let $\alpha, \eta : X \times X \to [0, \infty)$ be two functions such that $\alpha(0, 1) = 16$. Then T is not a generalized convex contraction of order 2 while it is a convex contraction of order 2 and α-η-generalized convex contraction of order 2 mapping. Indeed, if we choose $x = 0$ and $y = 1$ then,

$$\begin{aligned} &\alpha(0,1)\, d\left(T^2 0, T^2 1\right) \\ &\quad = 1 > b_1 + \frac{b_2}{4} = a_1 d(0, T0) \\ &\qquad + a_2 d\left(T0, T^2 0\right) + b_1 d(1, T1) + b_2 d\left(T1, T^2 1\right) \end{aligned} \tag{12}$$

holds for all $a_1, a_2, b_1, b_2 \geq 0$ with $a_1 + a_2 + b_1 + b_2 < 1$. That is, T is not a generalized convex contraction of order 2. But, if we choose $a_1 = b_1 = 1/32$ and $a_2 = b_2 = 1/8$ then,

$$\begin{aligned} d\left(T^2 x, T^2 y\right) &= \frac{1}{16}\max\{x, y\} \leq \frac{1}{16}x + \frac{1}{16}y \\ &= \frac{1}{32}x + \frac{1}{32}x + \frac{1}{32}y + \frac{1}{32}y \\ &= \frac{1}{32}\max\left\{x, \frac{1}{4}x\right\} + \frac{1}{8}\max\left\{\frac{1}{4}x, \frac{1}{16}x\right\} \\ &\quad + \frac{1}{32}\max\left\{y, \frac{1}{4}y\right\} + \frac{1}{8}\max\left\{\frac{1}{4}y, \frac{1}{16}y\right\} \\ &\leq a_1 d(x, Tx) + a_2 d\left(Tx, T^2 x\right) \\ &\quad + b_1 d(y, Ty) + b_2 d\left(Ty, T^2 y\right) \end{aligned} \tag{13}$$

holds for all $x, y \in X$ with $x \neq y$. Moreover, if $x = y$, then

$$\begin{aligned} d\left(T^2 x, T^2 y\right) = 0 &\leq a_1 d(x, Tx) + a_2 d\left(Tx, T^2 x\right) \\ &\quad + b_1 d(y, Ty) + b_2 d\left(Ty, T^2 y\right) \end{aligned} \tag{14}$$

and so,

$$\begin{aligned} d\left(T^2 x, T^2 y\right) &\leq a_1 d(x, Tx) + a_2 d\left(Tx, T^2 x\right) \\ &\quad + b_1 d(y, Ty) + b_2 d\left(Ty, T^2 y\right) \end{aligned} \tag{15}$$

holds for all $x, y \in X$. That is, T is a convex contraction of order 2 and α-η-generalized convex contraction of order 2 mapping.

Remark 12. We cannot find a self-mapping T and functions $\alpha, \eta : X \times X \to [0, \infty)$ such that T is a convex contraction mapping (or convex contraction of order 2) which is not a α-η-generalized convex contraction (or α-η-generalized convex contraction of order 2).

3. Fixed Point Results for Modified Convex Contractions

Let $\epsilon > 0$ be given. A point x in a metric space (X, d) is called an ϵ-fixed point of the self-map T on X whenever $d(x, Tx) < \epsilon$. We say that T has an approximate fixed point (or T has the approximate fixed point property) whenever T has an ϵ-fixed point for all $\epsilon > 0$; see [18, 19].

Definition 13 (see [14]). Let T be a self-mapping on X and let $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. One says that T is an α-admissible mapping with respect to η if

$$x, y \in X, \quad \alpha(x, y) \geq \eta(x, y) \Longrightarrow \alpha(Tx, Ty) \geq \eta(Tx, Ty). \tag{16}$$

Note that if we take $\eta(x, y) = 1$, then T is called α-admissible mapping.

We shall need the following result.

Lemma 14 (see [18]). *Let (X, d) be a metric space and let T be an asymptotic regular self-map on X; that is, $d(T^n x, T^{n+1} x) \to 0$ as $n \to \infty$ for all $x \in X$. Then T has the approximate fixed point property.*

Theorem 15. *Let (X, d) be a complete metric space and let T be a modified generalized convex contraction on X. If T is an α-admissible mapping with respect to η and $\alpha(x, Tx) \geq \eta(x, Tx)$ for all $x \in X$, then T has an approximate fixed point.*

Proof. Let $\alpha(w, Tw) \geq \eta(w, Tw)$ for all $w \in X$. Since T is an α-admissible mapping with respect to η, then we deduce that $\alpha(Tw, T^2 w) \geq \eta(Tw, T^2 w)$ for all $w \in X$. By continuing this process, we get $\alpha(T^n w, T^{n+1} w) \geq \eta(T^n w, T^{n+1} w)$ for all $n \in \mathbb{N} \cup \{0\}$ and for all $w \in X$. By taking $\vartheta = d(T^2 w, Tw) + d(Tw, w)$ and $r = a + b$ we have $d(Tw, T^2 w) \leq \vartheta$. Let $x = w$ and $y = Tw$; then by (7),

$$d\left(T^3 w, T^2 w\right) \leq ad\left(T^2 w, Tw\right) + bd(w, Tw) \leq r\vartheta. \tag{17}$$

By continuing this process we get

$$d\left(T^m w, T^{m+1} w\right) \leq 2r^l \vartheta, \tag{18}$$

where $m = 2l$ or $m = 2l + 1$. This implies that $d(T^m w, T^{m+1} w) \to 0$ for all $w \in X$. By applying Lemma 14, T has an approximate fixed point. □

Let T be a self-mapping and let $\alpha, \eta : X \times X \to [0, \infty)$ be two functions. We say that X has the $H^\star$-property whenever for all $x, y \in \operatorname{Fix}(T)$ with $\alpha(x, y) < \eta(x, Tx)$, and there exists $z \in X$ such that $\alpha(x, z) \geq \eta(x, z)$ and $\alpha(y, z) \geq \eta(y, z)$. Also for all $x, y \in X$ we have, $\eta(x, Tx) \leq \eta(x, y)$.

Theorem 16. *Let (X, d) be a complete metric space and let T be a modified generalized convex contraction on X. Also suppose that T is continuous and α-admissible mapping with respect to η. If there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$, then T has a fixed point. Moreover, T has a unique fixed point when X has $H^\star$-property.*

Proof. Define a sequence $\{x_n\}$ in X by $x_n = T^n x_0$ for all $n \in \mathbb{N}$. Since T is an α-admissible mapping with respect to η and $\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq \eta(x_0, Tx_0) = \eta(x_0, x_1)$, we deduce that $\alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geq \eta(x_1, x_2) = \eta(Tx_0, Tx_1)$. By continuing this process, we get $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) = \eta(x_n, Tx_n)$ for all $n \in \mathbb{N} \cup \{0\}$. Since T is a modified generalized convex contraction, so from (7) we get

$$d(x_n, x_{n+1}) \leq ad(x_{n-1}, x_n) + bd(x_{n-2}, x_{n-1}). \tag{19}$$

By taking $\vartheta = d(x_2, x_1) + d(x_1, x_0)$ and $r = a + b$ we have

$$d(x_m, x_{m+1}) \leq r^l \vartheta, \tag{20}$$

where $m = 2l$ or $m = 2l + 1$. Let $m = 2l$. Then for $n = 2p$ with $p > 2$, $l \geq 1$, and $m < n$ we deduce

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) \\ &\quad + d(x_{m+2}, x_{m+3}) + \cdots + d(x_{n-1}, x_n) \\ &= d(x_2 l, x_{2l+1}) + d(x_{2l+1}, x_{2l+2}) \\ &\quad + d(x_{2l+2}, x_{2l+3}) + \cdots + d(x_{2p-1}, x_{2p}) \\ &\leq r^l \vartheta + r^l \vartheta + r^{l+1} \vartheta + \cdots + r^{p-1} \vartheta \\ &= 2r^l \vartheta + 2r^{l+1} \vartheta + \cdots + 2r^{p-1} \vartheta \leq \frac{2r^l}{1-r} \vartheta. \end{aligned} \tag{21}$$

Similarly, for $m = 2l$ and $n = 2p + 1$ with $p \geq 1$, $l \geq 1$, and $m < n$ we get

$$d(x_m, x_n) \leq \frac{2r^l}{1-r} \vartheta. \tag{22}$$

Now, assume that $m = 2l + 1$. Then for $n = 2p$ with $p \geq 2$, $l \geq 1$, and $m < n$ we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) \\ &\quad + d(x_{m+2}, x_{m+3}) + \cdots + d(x_{n-1}, x_n) \\ &= d(x_{2l}, x_{2l+1}) + d(x_{2l+1}, x_{2l+2}) \\ &\quad + d(x_{2l+2}, x_{2l+3}) + \cdots + d(x_{2p-1}, x_{2p}) \\ &\leq r^l \vartheta + r^{l+l} \vartheta + r^{l+1} \vartheta + \cdots + r^p \vartheta \\ &\leq 2r^l \vartheta + 2r^{l+1} \vartheta + \cdots + 2r^p \vartheta \leq \frac{2r^l}{1-r} \vartheta. \end{aligned} \tag{23}$$

Similarly, for $m = 2l + 1$ and $n = 2p + 1$ with $p \geq 1$, $l \geq 1$, and $m < n$ we deduce

$$d(x_m, x_n) \leq \frac{2r^l}{1-r} \vartheta. \tag{24}$$

Hence, for all $m, n \in \mathbb{N}$ with $m < n$ we have

$$d(x_m, x_n) \leq \frac{2r^l}{1-r} \vartheta. \tag{25}$$

Taking limit as $l \to \infty$ in the above inequality we get $d(x_m, x_n) \to 0$. That is, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, then there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Since T is continuous, then $z = Tz$.

Let $x, y \in \mathrm{Fix}(T)$, where $x \neq y$. For prove of uniqueness we consider the following cases.

Case 1. Let $\alpha(x, y) \geq \eta(x, Tx)$. Since T is a modified generalized convex contraction, then we have

$$\begin{aligned} d(x, y) &= d(T^2 x, T^2 y) \leq ad(Tx, Ty) + bd(x, y) \\ &= ad(x, y) + bd(x, y) \\ &= (a + b) d(x, y) < d(x, y) \end{aligned} \tag{26}$$

which is a contradiction.

Case 2. Let $\alpha(x, y) < \eta(x, Tx)$. Since X has H^*-property, then there exists $z \in X$ such that $\alpha(x, z) \geq \eta(x, z)$ and $\alpha(y, z) \geq \eta(y, z)$. Now, since T is an α-admissible mapping with respect to η, then we can deduce $\alpha(x, T^n z) \geq \eta(x, T^n z) \geq \eta(x, Tx)$ and $\alpha(y, T^n z) \geq \eta(y, T^n z) \geq \eta(y, Ty)$. First we assume that $\alpha(x, T^n z) \geq \eta(x, Tx)$. So by hypothesis we get

$$d(x, T^{n+2} z) \leq ad(x, T^{n+1} z) + bd(x, T^n z). \tag{27}$$

By taking $\vartheta = d(x, Tz) + d(x, z)$ and $r = a + b$ we have

$$d(x, T^m x) \leq r^l \vartheta, \tag{28}$$

where $m = 2l$ or $m = 2l + 1$. Therefore, $T^m z \to x$. Similarly, we can show that $T^m z \to y$. That is, $x = y$ which is a contradiction. Therefore, T has a unique fixed point. □

Theorem 17. *Let (X, d) be a metric space and let T be a modified generalized convex contraction of order 2 on X. If T is an α-admissible mapping with respect to η and $\alpha(x, Tx) \geq \eta(x, Tx)$ for all $x \in X$, then T has an approximate fixed point.*

Proof. As in proof of Theorem 15 we can conclude that $\alpha(T^n w, T^{n+1} w) \geq \eta(T^n w, T^{n+1} w)$ for all $n \in \mathbb{N}$ and all $w \in X$. Put $r = a_1 + a_2 + b_1$, $\beta = 1 - b_2$, and $\vartheta = d(T^2 w, Tw) + d(Tw, w)$. From (8) with $x = w$ and $y = Tw$ we have

$$\begin{aligned} d(T^2 w, T^3 w) &\leq a_1 d(w, Tw) + a_2 d(Tw, T^2 w) \\ &\quad + b_1 d(Tw, T^2 w) + b_2 d(T^2 w, T^3 w) \\ &\leq r\vartheta + b_2 d(T^2 w, T^3 w) \end{aligned} \tag{29}$$

which implies that $(1 - b_2) d(T^2 w, T^3 w) \leq (a_1 + a_2 + b_1)\vartheta$. That is, $d(T^2 w, T^3 w) \leq (r/\beta)\vartheta$. Again from (8) with $x = Tw$ and $y = T^2 w$ we get

$$\begin{aligned} d(T^3 w, T^4 w) &\leq a_1 d(Tw, T^2 w) + a_2 d(T^2 w, T^3 w) \\ &\quad + b_1 d(T^2 w, T^3 w) + b_2 d(T^3 w, T^4 w) \\ &\leq r\vartheta + b_2 d(T^3 w, T^4 w) \end{aligned} \tag{30}$$

which implies that $d(T^3w, T^4w) \leq (r/\beta)\vartheta$. Similarly, $d(T^4w, T^5w) \leq (r/\beta)^2\vartheta$ and $d(T^5w, T^6w) \leq (r/\beta)^2\vartheta$. By continuing this process, we get $d(T^mw, T^{m+1}w) \leq (r/\beta)^l\vartheta$ for all $w \in X$ when $m = 2l$ or $m = 2l + 1$. This implies that $d(T^mw, T^{m+1}w) \to 0$ for all $w \in X$. By Lemma 14 T has an approximate fixed point. □

Theorem 18. *Let (X, d) be a complete metric space and let T be a modified generalized convex contraction of order 2 on X. Also suppose that T is an α-admissible with respect to η and continuous mapping. If there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$, then T has a fixed point. Moreover, T has a unique fixed point when X has $H^\star$-property.*

Proof. Define a sequence $\{x_n\}$ in X by $x_n = T^nx_0$ for all $n \in \mathbb{N}$. Put $r = a_1 + a_2 + b_1$ and $\beta = 1 - b_2$ and $\vartheta = d(T^2x_0, Tx_0) + d(Tx_0, x_0)$. From (8) with $x = x_0$ and $y = Tx_0$ we have

$$\begin{aligned} d\left(T^2x_0, T^3x_0\right) &\leq a_1 d\left(x_0, Tx_0\right) + a_2 d\left(Tx_0, T^2x_0\right) \\ &\quad + b_1 d\left(Tx_0, T^2x_0\right) + b_2 d\left(T^2x_0, T^3x_0\right) \\ &\leq r\vartheta + b_2 d\left(T^2x_0, T^3x_0\right) \end{aligned} \tag{31}$$

which implies that $(1-b_2)d(T^2x_0, T^3x_0) \leq (a_1+a_2+b_1)\vartheta$. That is, $d(T^2x_0, T^3x_0) \leq (r/\beta)\vartheta$. Again from (8) with $x = Tx_0$ and $y = T^2x_0$ we get

$$\begin{aligned} d\left(T^3x_0, T^4x_0\right) &\leq a_1 d\left(Tx_0, T^2x_0\right) + a_2 d\left(T^2x_0, T^3x_0\right) \\ &\quad + b_1 d\left(T^2x_0, T^3x_0\right) + b_2 d\left(T^3x_0, T^4x_0\right) \\ &\leq r\vartheta + b_2 d\left(T^3x_0, T^4x_0\right) \end{aligned} \tag{32}$$

which implies that $d(T^3x_0, T^4x_0) \leq (r/\beta)\vartheta$. Similarly, $d(T^4x_0, T^5x_0) \leq (r/\beta)^2\vartheta$ and $d(T^5x_0, T^6x_0) \leq (r/\beta)^2\vartheta$. By continuing this process, we get $d(T^mx_0, T^{m+1}x_0) \leq (r/\beta)^l\vartheta$ when $m = 2l$ or $m = 2l + 1$. Let $m = 2l$. Then for $n = 2p$ with $p > 2$, $l \geq 1$, and $m < n$ we deduce

$$\begin{aligned} & d\left(x_m, x_n\right) \\ & \leq d\left(x_m, x_{m+1}\right) + d\left(x_{m+1}, x_{m+2}\right) \\ & \quad + d\left(x_{m+2}, x_{m+3}\right) + \cdots + d\left(x_{n-1}, x_n\right) \\ & = d\left(x_{2l}, x_{2l+1}\right) + d\left(x_{2l+1}, x_{2l+2}\right) \\ & \quad + d\left(x_{2l+2}, x_{2l+3}\right) + \cdots + d\left(x_{2p-1}, x_{2p}\right) \\ & \leq \left(\frac{r}{\eta}\right)^l\vartheta + \left(\frac{r}{\eta}\right)^l\vartheta + \left(\frac{r}{\eta}\right)^{l+1}\vartheta + \cdots + \left(\frac{r}{\eta}\right)^{p-1}\vartheta \\ & = 2\left(\frac{r}{\eta}\right)^l\vartheta + 2\left(\frac{r}{\eta}\right)^{l+1}\vartheta + \cdots + 2\left(\frac{r}{\eta}\right)^{p-1}\vartheta \leq \frac{2(r/\eta)^l}{1-(r/\eta)}\vartheta. \end{aligned} \tag{33}$$

Similarly, for $m = 2l$ and $n = 2p + 1$ with $p \geq 1$, $l \geq 1$, and $m < n$ we get

$$d\left(x_m, x_n\right) \leq \frac{2(r/\eta)^l}{1-(r/\eta)}\vartheta. \tag{34}$$

Now, assume that $m = 2l + 1$. Then for $n = 2p$ with $p \geq 2$, $l \geq 1$, and $m < n$ we have

$$\begin{aligned} & d\left(x_m, x_n\right) \\ & \leq d\left(x_m, x_{m+1}\right) + d\left(x_{m+1}, x_{m+2}\right) \\ & \quad + d\left(x_{m+2}, x_{m+3}\right) + \cdots + d\left(x_{n-1}, x_n\right) \\ & = d\left(x_2l, x_{2l+1}\right) + d\left(x_{2l+1}, x_{2l+2}\right) \\ & \quad + d\left(x_{2l+2}, x_{2l+3}\right) + \cdots + d\left(x_{2p-1}, x_{2p}\right) \\ & \leq \left(\frac{r}{\eta}\right)^l\vartheta + \left(\frac{r}{\eta}\right)^{l+l}\vartheta + \left(\frac{r}{\eta}\right)^{l+1}\vartheta + \cdots + \left(\frac{r}{\eta}\right)^{p}\vartheta \\ & \leq 2\left(\frac{r}{\eta}\right)^l\vartheta + 2\left(\frac{r}{\eta}\right)^{l+1}\vartheta + \cdots + 2\left(\frac{r}{\eta}\right)^{p}\vartheta \leq \frac{2(r/\eta)^l}{1-(r/\eta)}\vartheta. \end{aligned} \tag{35}$$

Similarly, for $m = 2l + 1$ and $n = 2p + 1$ with $p \geq 1$, $l \geq 1$, and $m < n$ we deduce

$$d\left(x_m, x_n\right) \leq \frac{2(r/\eta)^l}{1-(r/\eta)}\vartheta. \tag{36}$$

Hence, for all $m, n \in \mathbb{N}$ with $m < n$ we have

$$d\left(x_m, x_n\right) \leq \frac{2(r/\eta)^l}{1-(r/\eta)}\vartheta. \tag{37}$$

Taking limit as $l \to \infty$ in the above inequality we get $d(x_m, x_n) \to 0$. That is, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Now since T is a continuous mapping then T has a fixed point z. If X has the $H^\star$-property, then by using a similar method to that in the proof of Theorem 16, we can prove uniqueness of the fixed point of T. □

4. α-η-Weakly Zamfirescu Mappings

In this section we introduce the notion of α-η-weakly Zamfirescu mapping and establish fixed point results.

Definition 19. Let (X, d) be a metric space and let T be a self-mapping on X. Assume there exists $\gamma : X \times X \to [0, 1]$ with $\theta(a, b) := \sup\{\gamma(x, y) : a \leq d(x, y) \leq b\} < 1$ for all $0 < a \leq b$, such that

$$\begin{aligned} & x, y \in X, \quad \eta\left(x, Tx\right) \\ & \quad \leq \alpha\left(x, y\right) \Longrightarrow d\left(Tx, Ty\right) \\ & \quad \leq \gamma\left(x, y\right) \max\left\{d\left(x, y\right), \frac{d\left(x, Tx\right) + d\left(y, Ty\right)}{2},\right. \\ & \qquad \left.\frac{d\left(x, Ty\right) + d\left(y, Tx\right)}{2}\right\}, \end{aligned} \tag{38}$$

and then T is a modified α-weakly Zamfirescu mapping.

Theorem 20. *Let (X, d) be a metric space and let T be an α-η-weakly Zamfirescu mapping on X. If T is an α-admissible mapping with respect to η and $\alpha(x, Tx) \geq \eta(x, Tx)$ for all $x \in X$, then T has an approximate fixed point.*

Proof. For a given $w \in X$, we define the sequence $\{x_n\}$ by $x_n = T^n w$. As in proof of Theorem 15 we can conclude that $\alpha(T^n w, T^{n+1} w) \geq \eta(T^n w, T^{n+1} w)$ for all $n \in \mathbb{N}$ and all $w \in X$. Now since T is an α-η-weakly Zamfirescu mapping, then

$$\begin{aligned} d(x_n, x_{n+1}) &= d\left(TT^{n-1}w, TT^n w\right) \\ &\leq \gamma\left(T^{n-1}w, T^n w\right) \\ &\quad \times \max\left\{ d\left(T^{n-1}w, T^n w\right), \frac{d\left(T^{n-1}w, TT^{n-1}w\right) + d\left(T^n w, TT^n w\right)}{2}, \frac{d\left(T^{n-1}w, TT^n w\right) + d\left(T^n w, TT^{n-1} w\right)}{2} \right\} \\ &\leq \gamma(x_{n-1}, x_n) \max\left\{ d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}, \frac{d(x_{n+1}, x_{n-1}) + d(x_n, x_n)}{2} \right\} \\ &\leq \gamma(x_{n-1}, x_n) \max\left\{ d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}, \frac{d(x_{n+1}, x_n) + d(x_n, x_{n-1})}{2} \right\} \\ &= \gamma(x_{n-1}, x_n) \max\left\{ d(x_{n-1}, x_n), \frac{d(x_{n+1}, x_n) + d(x_n, x_{n-1})}{2} \right\}. \end{aligned} \tag{39}$$

Now if $\max\{d(x_n, x_{n-1}), (d(x_n, x_{n+1}) + d(x_{n-1}, x_n))/2\} = (d(x_n, x_{n+1}) + d(x_{n-1}, x_n))/2$, then

$$d(x_{n+1}, x_n) \leq \gamma(x_n, x_{n-1}) \left[\frac{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)}{2} \right] \tag{40}$$

which implies

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \frac{\gamma(x_n, x_{n-1})}{2 - \gamma(x_n, x_{n-1})} d(x_{n-1}, x_n) \\ &\leq \gamma(x_n, x_{n-1}) d(x_{n-1}, x_n) \end{aligned} \tag{41}$$

and so $\{x_n\}$ is a nonincreasing sequence and converges to a real number $s = \inf_{n \geq 1} d(x_{n-1}, x_n)$. Assume that $s > 0$. Now since $0 < s \leq d(x_{n+1}, x_n) \leq d(x_1, x_0)$ for all $n \in \mathbb{N} \cup \{0\}$ and $\gamma(x_n, x_{n-1}) \leq \theta$ for all $n \in \mathbb{N} \cup \{0\}$, where $\theta = \theta(d, d(x_1, x_0))$, thus

$$s \leq d(x_{n+1}, x_n) \leq \theta^n d(x_1, x_0) \tag{42}$$

for all $n \in \mathbb{N} \cup \{0\}$. This implies $s = 0$, which is a contradiction. Therefore,

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = \lim_{n \to \infty} d\left(T^n w, T^{n+1} w\right) = 0 \tag{43}$$

for a given $w \in X$. By Lemma 14 T has an approximate fixed point. □

Theorem 21. *Let (X, d) be a complete metric space and let T be an α-η-weakly Zamfirescu mapping on X. Also suppose that T is an α-admissible mapping with respect to η and continuous mapping. If there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$, then T has a fixed point.*

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. Define a sequence $\{x_n\}$ as in Theorem 15. By the similar proof as in proof of Theorem 20 we deduce

$$d(x_{n+1}, x_n) \leq \gamma(x_n, x_{n-1}) d(x_{n-1}, x_n) \tag{44}$$

for all $n \in \mathbb{N} \cup \{0\}$. As in proof of Theorem 28 [20], we deduce that $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Now since T is an continuous mapping, so $Tz = z$. □

Example 22. Let $X = [0, \infty)$ be endowed with usual metric. Define $T : X \to X$ and $\alpha, \eta : X \times X \to [0, \infty)$ by

$$Tx = \begin{cases} \frac{1}{5}x, & \text{if } x \in [0,1] \\ \frac{\left(3x^2 + x^{x+1}\right)(4 - x)}{60} + \frac{x - 1}{x^2 + 1} & \text{if } x \in (1,4] \\ \frac{3(20 - x)}{16(x^2 + 1)} + \frac{100}{16}(x - 4) & \text{if } x \in (4,20) \\ 5x, & \text{if } x \in [20, \infty), \end{cases}$$

$$\alpha(x, y) = \begin{cases} 15, & \text{if } x, y \in [0,1] \\ \frac{1}{2}, & \text{otherwise}, \end{cases} \qquad \eta(x, y) = 1. \tag{45}$$

Let $x = 0$ and $y = 1$ and let $\gamma : X \times X \to [0,1]$ be a given function. Then,

$$\begin{aligned}\alpha(0,1)\, d(T0, T1) &= 15 \times \frac{1}{5} = 3 > \gamma(0,1) \\ &\times \max\left\{d(0,1), \frac{d(0,T0)+d(1,T1)}{2}, \right. \\ &\left. \frac{d(0,T1)+d(1,T0)}{2}\right\}.\end{aligned} \tag{46}$$

That is, T is not an α-weakly Zamfirescu mapping. Therefore, Theorem 3.3 of [13] can not be applied for this example.

Further, if $x = 0$ and $y = 20$, then

$$\begin{aligned}d(T0, T20) &= 100 > 60 \geq 60\gamma(0,20) \\ &= \gamma(0,20) \max\left\{d(0,20), \frac{d(0,T0)+d(20,T20)}{2}, \right. \\ &\left. \frac{d(0,T20)+d(20,T0)}{2}\right\}.\end{aligned} \tag{47}$$

That is, T is not a weakly Zamfirescu mapping.

But if $\alpha(x,y) \geq 1$, then $x, y \in [0,1]$. Therefore,

$$\begin{aligned}d(Tx, Ty) &= \frac{1}{5} d(x,y) \leq \frac{1}{4} \max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \right. \\ &\left. \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.\end{aligned} \tag{48}$$

Put $\gamma(x,y) = 1/4$ and so

$$\begin{aligned}d(Tx, Ty) &= \frac{1}{5} d(x,y) \leq \gamma(x,y) \\ &\times \max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \right. \\ &\left. \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.\end{aligned} \tag{49}$$

That is, there exists $\gamma : X \times X \to [0,1]$ with $\theta(a,b) := \sup\{\gamma(x,y) : a \leq d(x,y) \leq b\} < 1$ for all $0 < a \leq b$, such that

$$\begin{aligned}d(Tx, Ty) \leq \gamma(x,y) \max&\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \right. \\ &\left. \frac{d(x,Ty)+d(y,Tx)}{2}\right\}\end{aligned} \tag{50}$$

holds for all $x, y \in X$ with $\alpha(x,y) \geq 1$. Then T is an α-η-weakly Zamfirescu mapping. Clearly T has a fixed point by our result.

5. From α-Ćirić Strong Almost Contraction to Suzuki Type Contraction

Definition 23 (see [21]). Let (X,d) be a metric space and let T be a self-mapping on X. Then T is called a Ćirić strong almost contraction, if there exists a constant $r \in [0,1)$ such that

$$d(Tx, Ty) \leq rM(x,y) + Ld(y, Tx) \tag{51}$$

for all $x, y \in X$, where $L \geq 0$ and

$$\begin{aligned}M(x,y) = \max&\left\{d(x,y), d(x,Tx), d(y,Ty), \right. \\ &\left. \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.\end{aligned} \tag{52}$$

Now we generalize the notion of Ćirić strong almost contraction mapping as follows.

Definition 24. Let (X,d) be a metric space and let $\alpha, \eta : X \times X \to [0,\infty)$ be two functions. A mapping $T : X \to X$ is called an α-η-Ćirić strong almost contraction, if there exists a constant $r \in [0,1)$ such that

$$\eta(x, Tx) \leq \alpha(x,y) \Longrightarrow d(Tx, Ty) \leq rM(x,y) + Ld(y, Tx) \tag{53}$$

for all $x, y \in X$, where $L \geq 0$ and

$$\begin{aligned}M(x,y) = \max&\left\{d(x,y), d(x,Tx), d(y,Ty), \right. \\ &\left. \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.\end{aligned} \tag{54}$$

Moreover, if we take $\eta(x,y) = 1$ for all $x, y \in X$, then we say T is a modified α-Ćirić strong almost contraction mapping.

Theorem 25. *Let (X,d) be a complete metric space and T be a continuous α-η-Ćirić strong almost contraction on X. Also suppose that T is an α-admissible mapping with respect to η. If there exists a $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$, then T has a fixed point.*

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. For a given $x_0 \in X$, we define the sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_n$. Now since T is an α-admissible mapping with respect to η, then $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0) = \eta(x_0, x_1)$. By continuing this process we have

$$\eta(x_{n-1}, Tx_{n-1}) = \eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n) \tag{55}$$

for all $n \in \mathbb{N}$. Since T is an α-η-Ćirić strong almost contraction mapping, so we obtain

$$\begin{aligned}d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \leq rM(x_{n-1}, x_n) \\ &+ Ld(x_n, Tx_{n-1}) = rM(x_{n-1}, x_n),\end{aligned} \tag{56}$$

where

$$
\begin{aligned}
M(x_{n-1}, x_n) &= \max\left\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \right.\\
&\qquad \left.\frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{2}\right\}\\
&= \max\left\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \right.\\
&\qquad \left.\frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2}\right\}\\
&\leq \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \right.\\
&\qquad \left.\frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}\right\}\\
&\leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}
\end{aligned}
\tag{57}
$$

which implies

$$d(x_n, x_{n+1}) \leq r \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \tag{58}$$

Now if $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$, then

$$d(x_n, x_{n+1}) \leq r d(x_n, x_{n+1}) < d(x_n, x_{n+1}) \tag{59}$$

which is a contradiction. Hence, $d(x_n, x_{n+1}) \leq r d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. Now it is easy to show that $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, so there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Continuity of T implies that $z = Tz$. □

Theorem 26. *Let (X, d) be a metric space and let T be a self-mapping on X. Also, suppose that $\alpha, \eta : X \times X \to [0, \infty)$ be two functions. Assume that the following assertions holds true:*

(i) *T is an α-admissible mapping with respect to η;*

(ii) *T is an α-η-Ćirić strong almost contraction on X;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;*

(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to \infty$, then either*

$$
\begin{aligned}
&\eta\left(Tx_n, T^2 x_n\right) \leq \alpha(Tx_n, x)\\
&\textit{or}\quad \eta\left(T^2 x_n, T^3 x_n\right) \leq \alpha\left(T^2 x_n, x\right)
\end{aligned}
\tag{60}
$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. Define a sequence $\{x_n\}$ in X by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. Now as in the proof of Theorem 25 we have $\alpha(x_{n+1}, x_n) \geq \eta(x_{n+1}, x_n)$ for all $n \in \mathbb{N}$ and there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Let $d(z, Tz) \neq 0$. From (iv) either

$$
\begin{aligned}
&\eta\left(Tx_{n-1}, T^2 x_{n-1}\right) \leq \alpha(Tx_{n-1}, z)\\
&\text{or}\quad \eta\left(T^2 x_{n-1}, T^3 x_{n-1}\right) \leq \alpha\left(T^2 x_{n-1}, z\right)
\end{aligned}
\tag{61}
$$

holds for all $n \in \mathbb{N}$. Then,

$$\eta(x_n, Tx_n) \leq \alpha(x_n, z) \quad \text{or} \quad \eta(x_{n+1}, Tx_{n+1}) \leq \alpha(x_{n+1}, z) \tag{62}$$

holds for all $n \in \mathbb{N}$. Let $\eta(x_n, Tx_n) \leq \alpha(x_n, z)$ hold for all $n \in \mathbb{N}$. Since T is an α-η-Ćirić strong almost contraction, so we get

$$
\begin{aligned}
d(x_{n+1}, Tz) &= d(Tx_n, Tz) \leq rM(x_n, z)\\
&\quad + Ld(z, Tx_n) = rM(x_n, z) + Ld(z, x_{n+1}),
\end{aligned}
\tag{63}
$$

where

$$
\begin{aligned}
M(x_n, z) &= \max\left\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), \right.\\
&\qquad \left.\frac{d(x_n, Tz) + d(z, Tx_n)}{2}\right\}\\
&= \max\left\{d(x_n, z), d(x_n, x_{n+1}), d(z, Tz), \right.\\
&\qquad \left.\frac{d(x_n, Tz) + d(z, x_{n+1})}{2}\right\}.
\end{aligned}
\tag{64}
$$

Taking limit as $n \to \infty$ in the above inequality we get

$$d(z, Tz) \leq r d(z, Tz) < d(z, Tz) \tag{65}$$

which is a contradiction. Hence, $d(z, Tz) = 0$. That is, $z = Tz$. By the similar method we can show that $z = Tz$ if $\eta(x_{n+1}, Tx_{n+1}) \leq \alpha(x_{n+1}, z)$ holds for all $n \in \mathbb{N}$. □

If in Theorem 26 we take $\eta(x, y) = 1$ for all $x, y \in X$, then we obtain following corollary.

Corollary 27. *Let (X, d) be a metric space and let T be a self-mapping on X. Also, suppose that $\alpha : X \times X \to [0, \infty)$ is a function. Assume that the following assertions holds true:*

(i) *T is an α-admissible mapping;*

(ii) *T is modified α-Ćirić strong almost contraction on X;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ with $x_n \to x$ as $n \to \infty$, then either*

$$\alpha(Tx_n, x) \geq 1 \quad \textit{or} \quad \alpha\left(T^2 x_n, x\right) \geq 1 \tag{66}$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point.

If in Theorem 26 we take $\alpha(x, y) = \eta(x, y) = 1$ for all $x, y \in X$, we obtain following result.

Corollary 28 (Theorem 2.2 of [21]). *Let (X, d) be complete metric space and let T be a Ćirić strong almost contraction on X. Then T has a fixed point.*

Example 29. Let $X = [0, +\infty)$. We endow X with usual metric. Define $T : X \to X, \alpha, \eta : X \times X \to [0, \infty)$ by

$$Tx = \begin{cases} \frac{1}{4}x^2, & \text{if } x \in [0,1] \\ \frac{x^3 + 2x + 1}{\sqrt{x^2+1}}, & \text{if } x \in (1,2] \\ 3x & \text{if } x \in (2,\infty) \end{cases}$$
$$\alpha(x,y) = \begin{cases} \frac{1}{2}, & \text{if } x, y \in [0,1] \\ \frac{1}{8}, & \text{otherwise,} \end{cases}$$
$$\eta(x,y) = \frac{1}{4}. \tag{67}$$

Let $\alpha(x, y) \geq \eta(x, y)$, and then $x, y \in [0, 1]$. On the other hand, $Tw \in [0, 1]$ for all $w \in [0, 1]$. Then, $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$. That is, T is an α-admissible mapping with respect to η. If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to \infty$, then $Tx_n, T^2x_n, T^3x_n \in [0, 1]$ for all $n \in \mathbb{N}$. That is,

$$\eta\left(Tx_n, T^2x_n\right) \leq \alpha\left(Tx_n, x\right),$$
$$\eta\left(T^2x_n, T^3x_n\right) \leq \alpha\left(T^2x_n, x\right) \tag{68}$$

hold for all $n \in \mathbb{N}$. Clearly, $\alpha(0, T0) \geq \eta(0, T0)$. Let $\alpha(x, y) \geq \eta(x, Tx)$. Now, if $x \notin [0, 1]$ or $y \notin [0, 1]$, then $1/8 \geq 1/4$, which is a contradiction. So, $x, y \in [0, 1]$. Therefore,

$$\begin{aligned} d(Tx, Ty) &= \frac{1}{4}\left|x^2 - y^2\right| \\ &= \frac{1}{4}|x-y||x+y| \leq \frac{1}{2}|x-y| \\ &\leq \frac{1}{2}M(x,y) + Ld(y, Tx). \end{aligned} \tag{69}$$

Therefore T is an α-η-Ćirić strong almost contraction. Hence, all conditions of Theorem 26 hold and T has a fixed point. Let $x = 3$ and $y = 9$; then

$$\begin{aligned} d(T3, T9) &= 18 > 18r + L.0 \\ &= r \max\left\{d(3,9), d(3,T3), d(9,T9), \frac{d(3,T9) + d(9,T3)}{2}\right\} + Ld(9,T3). \end{aligned} \tag{70}$$

That is, T is not a Ćirić strong almost contraction. Hence, Corollary 28 (Theorem 2.2 of [21]) cannot be applied for this example.

As an application of the above results, we obtain the following Suzuki type fixed point theorem [22].

Theorem 30. *Let (X, d) be a complete metric space and let T be a self-mapping on X. Assume that there exists $r \in [0, 1)$ such that*

$$\frac{1}{1+r}d(x,Tx) \leq d(x,y)$$
$$\textit{implies} \quad d(Tx,Ty) \leq rM(x,y) + Ld(y,Tx) \tag{71}$$

for all $x, y \in X$, where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2}\right\}. \tag{72}$$

Then T has a fixed point.

Proof. Define $\alpha, \eta : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = d(x,y), \qquad \eta(x,y) = \lambda(r)d(x,y) \tag{73}$$

for all $x, y \in X$, where $0 \leq r < 1$ and $\lambda(r) = 1/(1 + r)$. Now, since $\lambda(r)d(x, y) \leq d(x, y)$ for all $x, y \in X$, then $\eta(x, y) \leq \alpha(x, y)$ for all $x, y \in X$. That is, conditions (i) and (iii) of Theorem 26 hold true. Let $\{x_n\}$ be a sequence with $x_n \to x$ as $n \to \infty$. Assume that $d(Tx_n, T^2x_n) = 0$ for some n. Then, $Tx_n = T^2x_n$. That is Tx_n is a fixed point of T and we have nothing to prove. Hence we assume $Tx_n \neq T^2x_n$ for all $n \in \mathbb{N}$. Since $\lambda(r)d(Tx_n, T^2x_n) \leq d(Tx_n, T^2x_n)$ for all $n \in \mathbb{N}$, then from (82) we get

$$d\left(T^2x_n, T^3x_n\right) \leq rM\left(Tx_n, T^2x_n\right) + Ld\left(T^2x_n, T^2x_n\right), \tag{74}$$

where

$$M\left(Tx_n, T^2x_n\right) = \max\left\{d\left(Tx_n, T^2x_n\right), d\left(T^2x_n, T^3x_n\right), \frac{d\left(Tx_n, T^3x_n\right)}{2}\right\} \tag{75}$$

which implies

$$d\left(T^2x_n, T^3x_n\right) \leq rd\left(Tx_n, T^2x_n\right). \tag{76}$$

Assume that there exists $n_0 \in \mathbb{N}$, such that

$$\eta\left(Tx_{n_0}, T^2x_{n_0}\right) > \alpha\left(Tx_{n_0}, x\right),$$
$$\eta\left(T^2x_{n_0}, T^3x_{n_0}\right) > \alpha\left(T^2x_{n_0}, x\right). \tag{77}$$

Then,

$$\begin{aligned} &\lambda(r)\, d\left(Tx_{n_0}, T^2x_{n_0}\right) > d\left(Tx_{n_0}, x\right), \\ &\lambda(r)\, d\left(T^2x_{n_0}, T^3x_{n_0}\right) > d\left(T^2x_{n_0}, x\right). \end{aligned} \tag{78}$$

So by (76) we have

$$\begin{aligned} &d\left(Tx_{n_0}, T^2x_{n_0}\right) \\ &\quad \le d\left(Tx_{n_0}, x\right) + d\left(T^2x_{n_0}, x\right) \\ &\quad < \lambda(r)\, d\left(Tx_{n_0}, T^2x_{n_0}\right) + \lambda(r)\, d\left(T^2x_{n_0}, T^3x_{n_0}\right) \\ &\quad \le \lambda(r)\, d\left(Tx_{n_0}, T^2x_{n_0}\right) + r\lambda(r)\, d\left(Tx_{n_0}, T^2x_{n_0}\right) \\ &\quad = \lambda(r)(1+r)\, d\left(Tx_{n_0}, T^2x_{n_0}\right) = d\left(Tx_{n_0}, T^2x_{n_0}\right) \end{aligned} \tag{79}$$

which is a contradiction. Hence, either

$$\begin{aligned} &\eta\left(Tx_n, T^2x_n\right) \le \alpha\left(Tx_n, x\right) \\ \text{or}\quad &\eta\left(T^2x_n, T^3x_n\right) \le \alpha\left(T^2x_n, x\right) \end{aligned} \tag{80}$$

holds for all $n \in \mathbb{N}$. That is, condition (iv) of Theorem 26 holds. Let $\eta(x, Tx) \le \alpha(x, y)$. So, $\lambda(r)d(x, Tx) \le d(x, y)$. Then from (82) we get $d(Tx, Ty) \le rM(x, y) + Ld(y, Tx)$. Hence, all conditions of Theorem 26 hold and T has a fixed point. □

Corollary 31 (see [23], Theorem 3.2). *Let (X, d) be a complete metric space and let T be a self-mapping on X. Define a nonincreasing function $\rho : [0, 1) \to (1/2, 1]$ by*

$$\rho(r) = \frac{1}{1+r}. \tag{81}$$

Assume that there exists $r \in [0, 1)$ such that

$$\rho(r)\, d(x, Tx) \le d(x, y) \ \textit{ implies } d(Tx, Ty) \le r d(x, y) \tag{82}$$

for all $x, y \in X$. Then T has a unique fixed point.

6. Fixed Point Results on Metric Spaces Endowed with Graph

Consistent with [1, 24], let (X, d) be a metric space, and Δ denotes the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops; that is, $E(G) \supseteq \Delta$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph (see [24]) by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length N ($N \in \mathbb{N}$) is a sequence $\{x_i\}_{i=0}^{N}$ of $N + 1$ vertices such that $x_0 = x, x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for $i = 1, \ldots, N$. A graph G is connected if there is a path between any two vertices. G is weakly connected if $\widetilde{G}$ is connected (see for details [23–25]).

Definition 32 (see [24]). A mapping $T : X \to X$ is called G-continuous, if given $x \in X$ and sequence $\{x_n\}$:

$$\begin{gathered} x_n \longrightarrow x \quad \text{as } n \longrightarrow \infty, \\ \left(x_n, x_{n+1}\right) \in E(G) \quad \forall n \in \mathbb{N} \quad \text{imply } Tx_n \longrightarrow Tx. \end{gathered} \tag{83}$$

Definition 33. Let (X, d) be a metric space endowed with a graph G and let $T : X \to X$ be a self-mapping. We say T is a graphic convex contraction if

$$x, y \in X \text{ with } (x, y) \in E(G) \quad \text{implies } (Tx, Ty) \in E(G), \tag{84}$$

$$d\left(T^2x, T^2y\right) \le a d(Tx, Ty) + b d(x, y) \tag{85}$$

holds for all $x, y \in X$ with $(x, y) \in E(G)$, where $a, b \ge 0$, $a + b < 1$.

Definition 34. Let (X, d) be a metric space endowed with a graph G and let $T : X \to X$ be a self-mapping. One says T is a graphic convex contraction of order 2 if

$$\begin{aligned} &x, y \in X \text{ with } (x, y) \in E(G) \quad \text{implies } (Tx, Ty) \in E(G), \\ &d\left(T^2x, T^2y\right) \le a_1 d(x, Tx) + a_2 d\left(Tx, T^2x\right) \\ &\qquad\qquad + b_1 d(y, Ty) + b_2 d\left(Ty, T^2y\right) \end{aligned} \tag{86}$$

holds for all $x, y \in X$ with $(x, y) \in E(G)$, where $a_1, a_2, b_1, b_2 \ge 0$, $a_1 + a_2 + b_1 + b_2 < 1$.

Definition 35. Let (X, d) be a metric space endowed with a graph G and let T be a self-mapping on X. Assume there exists $\gamma : X \times X \to [0, 1]$ with $\theta(a, b) := \sup\{\gamma(x, y) : a \le d(x, y) \le b\} < 1$ for all $0 < a \le b$, such that

$$\begin{aligned} d(Tx, Ty) \le \gamma(x, y) \max\Bigg\{ &d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \\ &\frac{d(x, Ty) + d(y, Tx)}{2} \Bigg\} \end{aligned} \tag{87}$$

holds for all $x, y \in X$ with $(x, y) \in E(G)$ and

$$x, y \in X \text{ with } (x, y) \in E(G) \quad \text{implies } (Tx, Ty) \in E(G), \tag{88}$$

then T is a graphic weakly Zamfirescu mapping.

Definition 36. Let (X, d) be a metric space endowed with a graph G. A mapping $T : X \to X$ is called graphic Ćirić strong almost contraction, if there exist a constant $r \in [0, 1)$ such that

$$d(Tx, Ty) \le rM(x, y) + Ld(y, Tx) \tag{89}$$

holds for all $x, y \in X$ with $(x, y) \in E(G)$, where $L \geq 0$:

$$M(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\}, \tag{90}$$

$$x, y \in X \text{ with } (x, y) \in E(G) \quad \text{implies } (Tx, Ty) \in E(G). \tag{91}$$

Theorem 37. *Let (X, d) be a metric space endowed with a graph G and let T be a graphic convex contraction on X. If $(x, Tx) \in E(G)$ for all $x \in X$, then T has an approximate fixed point.*

Proof. Define $\alpha : X^2 \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in E(G) \\ \dfrac{1}{2}, & \text{otherwise.} \end{cases} \tag{92}$$

At first we prove that T is an α-admissible mapping. Let $\alpha(x, y) \geq 1$; then $(x, y) \in E(G)$. Now since T is a graphic convex contraction, we have $(Tx, Ty) \in E(G)$. That is, $\alpha(Tx, Ty) \geq 1$. Also, clearly, T is a modified generalized convex contraction.

Let $(x, Tx) \in E(G)$ for all $x \in X$. Then, $\alpha(x, Tx) \geq 1$ for all $x \in X$. Hence, all conditions of Theorem 15 hold and T has an approximate fixed point. □

Similarly, we can deduce the following results.

Theorem 38. *Let (X, d) be a complete metric space endowed with a graph G and let T be a graphic convex contraction on X. Also suppose that T is G-continuous mapping. If there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$, then T has a fixed point. Moreover, T has a unique fixed point if, for all $x, y \in \mathrm{Fix}(T)$ with $(x, y) \notin E(G)$, there exists $z \in X$ such that $(x, z) \in E(G)$ and $(y, z) \in E(G)$.*

Theorem 39. *Let (X, d) be a metric space endowed with a graph G and let T be a graphic convex contraction of order 2 on X. If $(x, Tx) \in E(G)$ for all $x \in X$, then T has an approximate fixed point.*

Theorem 40. *Let (X, d) be a metric space endowed with a graph G and let T be a graphic convex contraction of order 2 on X. Also suppose that T is G-continuous mapping. If there exists a $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$, then T has a fixed point. Moreover, T has a unique fixed point if, for all $x, y \in \mathrm{Fix}(T)$ with $(x, y) \notin E(G)$, there exists $z \in X$ such that $(x, z) \in E(G)$ and $(y, z) \in E(G)$.*

Theorem 41. *Let (X, d) be a metric space endowed with a graph G and let T be a graphic weakly Zamfirescu mapping on X. If $(x, Tx) \in E(G)$ for all $x \in X$, then T has an approximate fixed point.*

Theorem 42. *Let (X, d) be a complete metric space endowed with a graph G and let T be a graphic weakly Zamfirescu mapping on X. Also suppose that T is G-continuous mapping. If there exists an $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$, then T has a fixed point.*

Theorem 43. *Let (X, d) be a metric space endowed with a graph G and let T be a self-mapping on X. Assume that the following assertions hold true:*

(i) *T is graphic Ćirić strong almost contraction on X;*

(ii) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E(G)$ with $x_n \to x$ as $n \to \infty$, then either*

$$(Tx_n, x) \in E(G) \quad \text{or} \quad (T^2 x_n, x) \in E(G) \tag{93}$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point.

Let $(X, d, \preceq)$ be a partially ordered metric space. Define the graph G by

$$E(G) := \{(x, y) \in X \times X : x \preceq y\}. \tag{94}$$

For this graph, the condition "$\forall x, y \in X, (x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G)$" in Definitions 32–35 translates into "$\forall x, y \in X, x \preceq y \Rightarrow T(x) \preceq T(y)$" which means T is nondecreasing with respect to this order [6]. Fixed point theorems for monotone operators in ordered metric spaces are widely investigated and have found various applications in differential and integral equations (see [2, 25–30] and references therein). From Theorems 37–43 we derive the following new results in partially ordered metric spaces.

Theorem 44. *Let $(X, d, \preceq)$ be a partially ordered metric space and let T be a nondecreasing ordered convex contraction on X. If $x \preceq Tx$ for all $x \in X$, then T has an approximate fixed point.*

Theorem 45. *Let $(X, d, \preceq)$ be a complete partially ordered metric space and let T be a nondecreasing and ordered convex contraction on X. Also suppose that T is continuous mapping. If there exists an $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point. Moreover, T has a unique fixed point if, for all $x, y \in \mathrm{Fix}(T)$ with $x \npreceq y$, there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$.*

Theorem 46. *Let $(X, d, \preceq)$ be a partially ordered metric space and let T be a nondecreasing, ordered convex contraction of order 2 on X. If $x \preceq Tx$ for all $x \in X$, then T has an approximate fixed point.*

Theorem 47. *Let $(X, d, \preceq)$ be a complete partially ordered metric space and let T be a nondecreasing and ordered convex contraction of order 2 on X. Also suppose that T is continuous mapping. If there exists an $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point. Moreover, T has a unique fixed point if, for all $x, y \in \mathrm{Fix}(T)$ with $x \npreceq y$, there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$.*

Theorem 48. *Let $(X, d, \preceq)$ be a partially ordered metric space and let T be a nondecreasing, ordered weakly Zamfirescu mapping on X. If $x \preceq Tx$ for all $x \in X$, then T has an approximate fixed point.*

Theorem 49. *Let $(X, d, \preceq)$ be a complete partially ordered metric space and let T be a nondecreasing and ordered weakly Zamfirescu on X. Also suppose that T is continuous mapping. If there exists an $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point.*

Theorem 50. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Assume that the following assertions hold true:*

(i) *T is nondecreasing and ordered Ćirić strong almost contraction on X;*

(ii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ with $x_n \to x$ as $n \to \infty$, then either*

$$Tx_n \preceq x \quad \text{or} \quad T^2 x_n \preceq x \tag{95}$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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