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Research Article

Stability Analysis of Nonlinear Systems with Slope Restricted Nonlinearities

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The problem of absolute stability of Lur'e systems with sector and slope restricted nonlinearities is revisited. Novel time-domain and frequency-domain criteria are established by using the Lyapunov method and the well-known Kalman-Yakubovich-Popov (KYP) lemma. The criteria strengthen some existing results. Simulations are given to illustrate the efficiency of the results.

1. Introduction

Absolute stability of nonlinear systems has been investigated comprehensively for the past several decades [1–12]. It is well known that the Popov criterion and the circle criterion are two classical results with the forms of frequency-domain inequalities (FDIs), which are turned out to be equivalent to some linear matrix inequalities (LMIs). This not only gives the opportunity to use the powerful LMI toolbox [13] to study absolute stability, but also gives the opportunity to consider the controller design problems. In [14], absolute stability of single-input and single-output Lur'e systems with a sector and slope restricted nonlinearity is brought forward. It is pointed out that the slope restriction on the nonlinearity strengthens the Popov criterion by adding an additional term to the original FDI of the criterion. Much work [15–22] on the slope restricted and multivariable problem has been done by using a Lur'e-Postnikov function or an extended Lur'e-Postnikov function.

In this paper, both time-domain criterion and frequency-domain criterion for absolute stability of Lur'e systems with sector and slope restricted nonlinearities are presented based on the Lyapunov method and the KYP lemma. Some mathematical tools are used through the derivation of the absolute stability criterion. Compared with some existing results, the proposed results are less conservative. This should be owed to the effect of the slope restricted conditions on the nonlinearities. The rest of the paper is organized as follows. In Section 2, the system description and some preliminaries are presented. Time-domain and frequency-domain criteria for absolute stability of the system are given in Section 3. Numerical examples are given in Section 4 and some concluding remarks are given in Section 5.

Throughout this paper, the superscript $*$ means transpose of real matrices and conjugate transpose of complex matrices. For a Hermitian matrix W, $W > 0$ ($W \geq 0$) denotes that W is a positive definite (semidefinite) matrix and $W < 0$ denotes that W is a negative definite matrix. $\mathrm{Re}\{Y\}$ means $(1/2)(Y + Y^*)$ for any real or complex square matrix Y.

2. Problem Statement

Consider the following multi-input and multioutput Lur'e system

$$\dot{x}(t) = Ax(t) + B\varphi(\sigma(t)), \qquad \sigma(t) = C^* x(t), \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{n\times m}$ are real matrices, $\varphi(0) = 0$, $\sigma(t) = \begin{bmatrix} \sigma_1(t) \\ \vdots \\ \sigma_m(t) \end{bmatrix}$ is the output, $\varphi(\sigma(t)) = \begin{bmatrix} \varphi_1(\sigma_1(t)) \\ \vdots \\ \varphi_m(\sigma_m(t)) \end{bmatrix}$

is piecewise continuously differentiable on $\mathbb{R}^m$, and $\varphi_i(\sigma_i(t))$ $(i = 1, 2, \ldots, m)$ are assumed to satisfy

$$\gamma_{1i}\sigma_i^2(t) \leq \varphi_i(\sigma_i(t))\sigma_i(t) \leq \delta_{1i}\sigma_i^2(t), \quad (2)$$

$$\gamma_{2i} \leq \frac{d\varphi_i(\sigma_i(t))}{d\sigma_i(t)} \leq \delta_{2i}, \quad (3)$$

where $\gamma_{2i} \leq \gamma_{1i}$, $\delta_{2i} \geq \delta_{1i}$, $\gamma_{2i} \leq 0$, and $\delta_{2i} \geq 0$. The inequalities (2) and (3) denote sector restriction and slope restriction on $\varphi(\sigma(t))$, respectively. Let $\Gamma_1 = \operatorname{diag}(\gamma_{11}, \ldots, \gamma_{1m})$, $\Delta_1 = \operatorname{diag}(\delta_{11}, \ldots, \delta_{1m})$, $\Gamma_2 = \operatorname{diag}(\gamma_{21}, \ldots, \gamma_{2m})$, $\Delta_2 = \operatorname{diag}(\delta_{21}, \ldots, \delta_{2m})$. Then $\Gamma_2 - \Gamma_1 \leq 0$, $\Delta_2 - \Delta_1 \geq 0$, $\Gamma_2 \leq 0$, and $\Delta_2 \geq 0$. Setting $\psi_i(\sigma_i(t)) = d\varphi_i(\sigma_i(t))/dt$, (3) is formulated as follows:

$$\gamma_{2i} \leq \frac{\psi_i(\sigma_i(t))}{\dot{\sigma}_i(t)} \leq \delta_{2i}. \quad (4)$$

The transfer function from $\varphi(\sigma(t))$ to $-\sigma(t)$ is denoted as $\chi(s) = C^*(A - sI)^{-1}B$.

System (1) is called to be absolutely stable if the equilibrium point $x(t) = 0$ is globally asymptotically stable for all nonlinear vector valued functions $\varphi(\sigma(t))$ satisfying (2) and (3). In the following sections, less conservative absolute stability criteria including time-domain criterion and frequency-domain criterion for system (1) are given. Before studying these problems, first we introduce the KYP lemma and Schur complement. These lemmas will be used repeatedly in this paper to get our main results.

Lemma 1 (KYP lemma [23]). *Given that $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and symmetric matrix $\Sigma \in \mathbb{R}^{(n+m)\times(n+m)}$, with $\det(j\omega I - A) \neq 0$ for $\omega \in \mathbb{R}$, and the pair (A, B) is controllable, the following two statements are equivalent.*

(i) $\begin{bmatrix} (j\omega I-A)^{-1}B \\ I \end{bmatrix}^* \Sigma \begin{bmatrix} (j\omega I-A)^{-1}B \\ I \end{bmatrix} \leq 0$, *for all $\omega \in \mathbb{R}$.*

(ii) *There exists a matrix $P = P^*$ such that $\begin{bmatrix} A^*P+PA & PB \\ B^*P & 0 \end{bmatrix} + \Sigma \leq 0$. The equivalence for strict inequalities holds even if (A, B) is not controllable.*

Lemma 2 (Schur complement [24]). *The LMI $\begin{bmatrix} S_{11} & S_{12} \\ S_{12}^* & -S_{22} \end{bmatrix} < 0$ is equivalent to one of the following statements:*

(i) $S_{22} > 0$ *and* $S_{11} + S_{12}S_{22}^{-1}S_{12}^* < 0$;

(ii) $S_{11} < 0$ *and* $S_{22} + S_{12}^*S_{11}^{-1}S_{12} > 0$.

3. Main Results

We choose the following Lur'e-Postnikov function:

$$V(x(t)) = x^*(t)Px(t) + \sum_{i=1}^{m}\lambda_i \int_0^{\sigma_i(t)} \varphi_i(s)\,ds \quad (5)$$

as the Lyapunov function, where $P = P^*$ and $\lambda_i \in \mathbb{R}$ $(i = 1, 2, \ldots, m)$ are necessary to be determined. It should be pointed out that P is not necessary to be positive definite and λ_i $(i = 1, 2, \ldots, m)$ are not necessary to be nonnegative.

Theorem 3. *System (1) is absolutely stable for all $\varphi(\sigma(t))$ satisfying (2) and (3) if $A + B\Gamma_1C^*$ is Hurwitzian and there exist diagonal matrices $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$, $T_1 \geq 0$, $T_2 > 0$, and symmetric matrices P such that the LMI is feasible:*

$$\begin{bmatrix} A^*P + PA + \Sigma_{11} & PB + \Sigma_{12} & \Sigma_{13} \\ B^*P + \Sigma_{12}^* & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{13}^* & \Sigma_{23}^* & -T_2 \end{bmatrix} < 0, \quad (6)$$

where

$$\begin{aligned}
\Sigma_{11} &= -C\Gamma_1T_1\Delta_1C^* - A^*C\Gamma_2T_2\Delta_2C^*A, \\
\Sigma_{12} &= \frac{1}{2}A^*C\Lambda + \frac{1}{2}CT_1(\Gamma_1 + \Delta_1) - A^*C\Gamma_2T_2\Delta_2C^*B, \\
\Sigma_{13} &= \frac{1}{2}A^*CT_2(\Gamma_2 + \Delta_2), \qquad \Sigma_{23} = \frac{1}{2}B^*CT_2(\Gamma_2 + \Delta_2), \\
\Sigma_{22} &= \frac{1}{2}\Lambda C^*B + \frac{1}{2}B^*C\Lambda - T_1 - B^*C\Gamma_2T_2\Delta_2C^*B.
\end{aligned} \quad (7)$$

Proof. We will demonstrate that the given conditions imply the negative definiteness of $\dot{V}(x(t))$ and the positive definiteness of $V(x(t))$.

Taking the derivative of $V(x(t))$ along the trajectory of (1), we have

$$\dot{V}(x(t)) = x^*(t)P\dot{x}(t) + \dot{x}^*(t)Px(t) + \varphi^*(\sigma(t))\Lambda C^*\dot{x}(t). \quad (8)$$

Conditions (2) and (4) for $\varphi_i(\sigma_i(t))$ are equivalent to

$$\begin{aligned}
u_{1i}(x_i) &= (\varphi_i(\sigma_i(t)) - \gamma_{1i}\sigma_i(t))(\varphi_i(\sigma_i(t)) - \delta_{1i}\sigma_i(t)) \leq 0, \\
u_{2i}(x_i) &= (\psi_i(\sigma_i(t)) - \gamma_{2i}\dot{\sigma}_i(t))(\psi_i(\sigma_i(t)) - \delta_{2i}\dot{\sigma}_i(t)) \leq 0.
\end{aligned} \quad (9)$$

For any $t_{1i} \geq 0$ and $t_{2i} > 0$, $i = 1, 2, \ldots, m$, it follows

$$\begin{aligned}
\sum_{i=1}^{m} t_{1i}u_{1i}(x_i) &= \varphi^*(\sigma(t))T_1\varphi(\sigma(t)) \\
&\quad - \frac{1}{2}\varphi^*(\sigma(t))(\Gamma_1 + \Delta_1)T_1C^*x \\
&\quad - \frac{1}{2}x^*CT_1(\Gamma_1 + \Delta_1)\varphi(\sigma(t)) \\
&\quad + x^*C\Gamma_1T_1\Delta_1C^*x \leq 0, \\
\sum_{i=1}^{m} t_{2i}u_{2i}(x_i) &= \psi^*(\sigma(t))T_2\psi(\sigma(t)) \\
&\quad - \frac{1}{2}\psi^*(\sigma(t))(\Gamma_2 + \Delta_2)T_2C^*\dot{x} \\
&\quad - \frac{1}{2}\dot{x}^*CT_2(\Gamma_2 + \Delta_2)\psi(\sigma(t)) \\
&\quad + \dot{x}^*C\Gamma_2T_2\Delta_2C^*\dot{x} \leq 0,
\end{aligned} \quad (10)$$

where $T_1 = \operatorname{diag}(t_{11}, \ldots, t_{1m}) \geq 0$ and $T_2 = \operatorname{diag}(t_{21}, \ldots, t_{2m}) > 0$. Then

$$\dot{V}(x(t)) \leq x^*(t) P\dot{x}(t) + \dot{x}^*(t) Px(t) + \varphi^*(\sigma(t)) \Lambda C^* \dot{x}(t) - \sum_{i=1}^{m} t_{1i} u_{1i}(x_i) - \sum_{i=1}^{m} t_{2i} u_{2i}(x_i). \quad (11)$$

The given condition (6) guarantees the negative definiteness of the right hand of (11). Consequently, $\dot{V}(x(t))$ is negative definite.

Now we are only left to demonstrate that $V(x(t))$ is positive definite. In (5), P is only a symmetric matrix but not a positive definite matrix and λ_i may be a positive or negative number. Therefore, the proof of the positive definiteness of $V(x(t))$ is a little difficult and complex. Without loss of generality, letting $\lambda_i < 0$ $(i = 1, 2, \ldots, k)$ and $\lambda_i \geq 0$ $(i = k+1, \ldots, m)$ $(0 \leq k \leq m)$, then $V(x(t))$ has the following form:

$$\begin{aligned} V(x(t)) &= x^*(t) Px(t) + \sum_{i=1}^{m} \lambda_i \int_0^{\sigma_i(t)} (\gamma_{1i}s + \varphi_i(s) - \gamma_{1i}s)\, ds \\ &\geq x^*(t) \left[P + \frac{1}{2} C\Lambda\Gamma_1 C^* + \frac{1}{2} C\Lambda(\Delta_{1k} - \Gamma_{1k}) C^*\right] x(t) \\ &\quad + \sum_{i=k+1}^{m} \lambda_i \int_0^{\sigma_i(t)} (\varphi_i(s) - \gamma_{1i}s)\, ds, \end{aligned} \quad (12)$$

where $\Delta_{1k} = \operatorname{diag}(\delta_{11}, \ldots, \delta_{1k}, 0, \ldots, 0)$ and $\Gamma_{1k} = \operatorname{diag}(\gamma_{11}, \ldots, \gamma_{1k}, 0, \ldots, 0)$. Since (2) implies $\sigma_i(t)(\varphi_i(\sigma_i(t)) - \gamma_{1i}\sigma_i(t)) \geq 0$, $\sum_{i=k+1}^{m} \lambda_i \int_0^{\sigma_i(t)} (\varphi_i(s) - \gamma_{1i}s)\, ds \geq 0$ is satisfied. Then $V(x(t))$ is positive definite if $P + (1/2)C\Lambda\Gamma_1 C^* + (1/2)C\Lambda(\Delta_{1k} - \Gamma_{1k})C^*$ is positive definite, which is proved in what follows.

Denote $A_1 = A + B\Gamma_1 C^*$, $P_1 = P + (1/2)C\Lambda\Gamma_1 C^*$, $A_k = A_1 + B(\Delta_{1k} - \Gamma_{1k})C^*$, and $P_k = P_1 + (1/2)C\Lambda(\Delta_{1k} - \Gamma_{1k})C^*$. Firstly, the given conditions imply that $A + B\Gamma_1 C^* + B\widetilde{\Delta}C^*$ is Hurwitzian for any diagonal matrix $\widetilde{\Delta}$ satisfying $0 \leq \widetilde{\Delta} \leq \Delta_1 - \Gamma_1$. Actually, the matrix $A + B\Gamma_1 C^* + B\widetilde{\Delta}C^*$ is Hurwitzian for $\widetilde{\Delta} = 0$ in virtue of the given conditions. So we will demonstrate that $A + B\Gamma_1 C^* + B\widetilde{\Delta}C^*$ is Hurwitzian for any diagonal matrix $\widetilde{\Delta}$ satisfying $0 < \widetilde{\Delta} \leq \Delta_1 - \Gamma_1$. We assume there exists a diagonal matrix $\widetilde{\Delta}$ satisfying $0 < \widetilde{\Delta} \leq \Delta_1 - \Gamma_1$ such that the matrix $A + B\Gamma_1 C^* + B\widetilde{\Delta}C^* = A_1 + B\widetilde{\Delta}C^*$ is not Hurwitzian. On the one hand, a number α satisfying $0 < \alpha \leq 1$ can be found such that

$$\begin{aligned} &\det\left(j\omega_0 I - A_1 - \alpha B\widetilde{\Delta}C^*\right) \\ &\quad = \det(j\omega_0 I - A_1) \det\left(I - \alpha C^* (j\omega_0 I - A_1)^{-1} B\widetilde{\Delta}\right) = 0 \end{aligned} \quad (13)$$

holds for certain $\omega_0 \in \mathbb{R}$. Since A_1 is Hurwitzian, $\det(j\omega_0 I - A_1) \neq 0$ and $\det(I - \alpha C^*(j\omega_0 I - A_1)^{-1} B\widetilde{\Delta}) = 0$ are followed. The latter formula indicates that there exists a vector $\nu \neq 0$ such that

$$\nu^* \left(I - \alpha\widetilde{\Delta}G^*(j\omega_0)\right) = 0, \quad (14)$$

where $\nu^*\widetilde{\Delta} \neq 0$ and $G(j\omega_0) = C^*(j\omega_0 I - A_1)^{-1} B$. Then we derive

$$\begin{aligned} \nu^*\widetilde{\Delta} \Big\{ &-T_1 + \frac{1}{2}(\Delta_1 - \Gamma_1) T_1 G(j\omega_0) + \frac{1}{2} G^*(j\omega_0) T_1 (\Delta_1 - \Gamma_1) \\ &+ \frac{1}{2} j\omega_0 \Lambda G(j\omega_0) + \frac{1}{2}[j\omega_0 \Lambda G(j\omega_0)]^* \\ &- \omega_0^2 G^*(j\omega_0) \Gamma_2 T_2 \Delta_2 G(j\omega_0) \Big\} \widetilde{\Delta}\nu \geq 0. \end{aligned} \quad (15)$$

On the another hand, pre- and postmultiplying both sides of (6) by $W_1 = \begin{bmatrix} I & C\Gamma_1 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$ and W_1^*, we have

$$\begin{bmatrix} A_1^* P_1 + P_1 A_1 - A_1^* C\Gamma_2 T_2 \Delta_2 C^* A_1 & \overline{\Sigma}_{12} & \overline{\Sigma}_{13} \\ \overline{\Sigma}_{12}^* & \Sigma_{22} & \Sigma_{23} \\ \overline{\Sigma}_{13}^* & \Sigma_{23}^* & -T_2 \end{bmatrix} < 0, \quad (16)$$

where

$$\begin{aligned} \overline{\Sigma}_{12} &= P_1 B + \frac{1}{2} A_1^* C\Lambda + \frac{1}{2} CT_1(\Delta_1 - \Gamma_1) - A_1^* C\Gamma_2 T_2 \Delta_2 C^* B, \\ \overline{\Sigma}_{13} &= \frac{1}{2} A_1^* CT_2 (\Gamma_2 + \Delta_2). \end{aligned} \quad (17)$$

By the Schur complement, (16) implies

$$\begin{bmatrix} A_1^* P_1 + P_1 A_1 - A_1^* C\Gamma_2 T_2 \Delta_2 C^* A_1 & \overline{\Sigma}_{12} \\ \overline{\Sigma}_{12}^* & \Sigma_{22} \end{bmatrix} < 0. \quad (18)$$

From the KYP lemma, we derive that (18) holds if and only if

$$\begin{bmatrix} (j\omega I - A_1)^{-1} B \\ I \end{bmatrix}^* \begin{bmatrix} \widehat{\Sigma}_{11} & \widehat{\Sigma}_{12} \\ \widehat{\Sigma}_{12}^* & \Sigma_{22} \end{bmatrix} \begin{bmatrix} (j\omega I - A_1)^{-1} B \\ I \end{bmatrix} < 0, \quad \forall \omega \in \mathbb{R}, \quad (19)$$

where $\widehat{\Sigma}_{11} = -A_1^* C\Gamma_2 T_2 \Delta_2 C^* A_1$ and $\widehat{\Sigma}_{12} = (1/2)A_1^* C\Lambda + (1/2)CT_1(\Delta_1 - \Gamma_1) - A_1^* C\Gamma_2 T_2 \Delta_2 C^* B$. Inequality (19) is equivalent to

$$\begin{aligned} &-T_1 + \frac{1}{2}(\Delta_1 - \Gamma_1) T_1 G(j\omega) + \frac{1}{2} G^*(j\omega) T_1 (\Delta_1 - \Gamma_1) \\ &\quad + \frac{1}{2} j\omega\Lambda G(j\omega) + \frac{1}{2}[j\omega\Lambda G(j\omega)]^* \\ &\quad - \omega^2 G^*(j\omega) \Gamma_2 T_2 \Delta_2 G(j\omega) < 0, \quad \forall \omega \in \mathbb{R} \end{aligned} \quad (20)$$

in terms of the equalities $G(j\omega) = C^*(j\omega I - A_1)^{-1} B$ and $j\omega G(j\omega) = C^* A_1 (j\omega I - A_1)^{-1} B + C^* B$. Letting $\omega = \omega_0$ in

(20) and pre- and postmultiplying both sides of the resulting inequality by $\nu^* \widetilde{\Delta}$ and $\widetilde{\Delta}\nu$, it follows that

$$\begin{aligned}
&\nu^* \widetilde{\Delta} \Big\{ -T_1 + \frac{1}{2}(\Delta_1 - \Gamma_1) T_1 G(j\omega_0) \\
&\quad + \frac{1}{2} G^*(j\omega_0) T_1 (\Delta_1 - \Gamma_1) + \frac{1}{2} j\omega_0 \Lambda \\
&\quad \times G(j\omega_0) + \frac{1}{2}[j\omega_0 \Lambda G(j\omega_0)]^* \\
&\quad - \omega_0^2 G^*(j\omega_0) \Gamma_2 T_2 \Delta_2 G(j\omega_0) \Big\} \widetilde{\Delta} \nu \\
&= \nu^* \widetilde{\Delta} T_1 \left[\frac{1}{\alpha}(\Delta_1 - \Gamma_1) - \widetilde{\Delta} \right] \nu + \frac{\omega_0^2}{\alpha^2} \nu^* (-\Gamma_2 T_2 \Delta_2) \nu < 0.
\end{aligned} \tag{21}$$

We can observe that (15) and (21) are contradictive, which means that the assumption is not true and $A + B\Gamma_1 C^* + B\widetilde{\Delta} C^*$ is Hurwitzian for any diagonal matrix $\widetilde{\Delta}$ satisfying $0 \le \widetilde{\Delta} \le \Delta_1 - \Gamma_1$. Therefore, the matrix $A_k = A + B\Gamma_1 C^* + B(\Delta_{1k} - \Gamma_{1k})C^*$ is Hurwitzian. Secondly, the given conditions imply that $P + (1/2)C\Lambda\Gamma_1 C^* + (1/2)C\Lambda(\Delta_{1k} - \Gamma_{1k})C^*$ is positive definite. Actually, pre- and postmultiplying both sides of (16) by $W_2 = \left[\begin{smallmatrix} I & C(\Delta_{1k}-\Gamma_{1k}) & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{smallmatrix}\right]$ and W_2^* yield

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} \\ \Xi_{12}^* & \Sigma_{22} & \Sigma_{23} \\ \Xi_{13}^* & \Sigma_{23}^* & -T_2 \end{bmatrix} < 0, \tag{22}$$

where

$$\begin{aligned}
\Xi_{11} &= A_k^* P_k + P_k A_k + C(\Delta_{1k} - \Gamma_{1k}) \\
&\quad \times T_1 [(\Delta_1 - \Gamma_1) - (\Delta_{1k} - \Gamma_{1k})] C^* \\
&\quad - A_k^* C \Gamma_2 T_2 \Delta_2 C^* A_k, \\
\Xi_{12} &= P_k B + \frac{1}{2} A_k^* C\Lambda - A_k^* C \Gamma_2 T_2 \Delta_2 C^* B \\
&\quad + \frac{1}{2} C T_1 (\Delta_1 - \Gamma_1) - C T_1 (\Delta_{1k} - \Gamma_{1k}), \\
\Xi_{13} &= \frac{1}{2} A_k^* C T_2 (\Gamma_2 + \Delta_2).
\end{aligned} \tag{23}$$

Inequality (22) implies $\Xi_{11} < 0$. According to $0 \le \Delta_k - \Gamma_k \le \Delta_1 - \Gamma_1$, $T_2 > 0$, $\Gamma_2 \le 0$, $\Delta_2 \ge 0$, $A_k^* P_k + P_k A_k < 0$ is followed. The matrix A_k is Hurwitzian, which results in the positive definiteness of P_k and $V(x(t))$. This completes the proof. □

It is found in the proof of Theorem 3, more exactly in inequality (16), that if (6) holds, then $A + B\Gamma_1 C^*$ is Hurwitzian if and only if $P + (1/2)C\Lambda\Gamma_1 C^* > 0$.

Theorem 4. *System* (1) *is absolutely stable for all* $\varphi(\sigma(t))$ *satisfying* (2) *and* (3) *if there exist diagonal matrices* Λ, $T_1 \ge 0$, $T_2 > 0$, *symmetric matrices* P, $Q > 0$ *such that* $P + (1/2)C\Lambda\Gamma_1 C^* > 0$ *and the LMI* (6) *holds.*

Remark 5. Theorem 3 is derived directly by using the time-domain method and can be used to study multi-input and multioutput Lur'e systems. Inequality (6) in Theorem 3 is in the form of LMI, which is easier to be solved by means of the LMI toolbox.

The LMI (6) can be transformed into an equivalent FDI. Thus, a frequency-domain criterion for (1) is given as follows.

Theorem 6. *System* (1) *is absolutely stable for all* $\varphi(\sigma(t))$ *satisfying* (2) *and* (3) *if the matrix* $A + B\Gamma_1 C^*$ *is Hurwitzian and there exist diagonal matrices* Λ, $T_1 \ge 0$, $T_2 > 0$ *such that the following frequency-domain inequality holds*

$$\begin{aligned}
&\operatorname{Re}\big\{[I + \Gamma_1 \chi(j\omega)]^* T_1 [I + \Delta_1 \chi(j\omega)] + j\omega\Lambda\chi(j\omega) \\
&\quad + \omega^2 [I + \Gamma_2 \chi(j\omega)]^* T_2 [I + \Delta_2 \chi(j\omega)]\big\} > 0,
\end{aligned} \tag{24}$$

$$\omega \in \mathbb{R}.$$

Proof. Let $\overline{P} = \left[\begin{smallmatrix} P & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $\overline{A} = \left[\begin{smallmatrix} A & B \\ 0 & 0 \end{smallmatrix}\right]$, and $L = \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$. Inequality (6) can be rewritten as

$$\begin{bmatrix} \overline{P}\,\overline{A} + \overline{A}^* \overline{P} + \Omega_{11} & \overline{P} L + \Omega_{12} \\ L^* \overline{P} + \Omega_{12}^* & -T_2 \end{bmatrix} < 0, \tag{25}$$

where

$$\Omega_{11} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^* & \Sigma_{22} \end{bmatrix}, \qquad \Omega_{12} = \begin{bmatrix} \Sigma_{13} \\ \Sigma_{23} \end{bmatrix}. \tag{26}$$

According to the KYP lemma, (25) is equivalent to

$$\begin{bmatrix} (j\omega I - \overline{A})^{-1} L \\ I \end{bmatrix}^* \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12}^* & -T_2 \end{bmatrix} \begin{bmatrix} (j\omega I - \overline{A})^{-1} L \\ I \end{bmatrix} < 0, \quad \forall \omega \in \mathbb{R}. \tag{27}$$

By simple computations, we have

$$\begin{aligned}
(j\omega I - \overline{A})^{-1} L &= \frac{1}{j\omega} \begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix}, \\
C^* A (A - j\omega I)^{-1} B &= C^* B + j\omega \chi(j\omega),
\end{aligned} \tag{28}$$

where $\chi(j\omega) = C^*(A - j\omega I)^{-1} B$. Substituting (28) into (27), the equivalence between (6) and (24) is derived. □

Remark 7. For the case $\Gamma_1 = 0$, the FDI (24) reduces to

$$\begin{aligned}
&T_1 + \operatorname{Re}\big\{(T_1 \Delta_1 + j\omega\Lambda)\chi(j\omega) + \omega^2 [I + \Gamma_2 \chi(j\omega)]^* \\
&\quad \times T_2 [I + \Delta_2 \chi(j\omega)]\big\} > 0, \quad \omega \in \mathbb{R},
\end{aligned} \tag{29}$$

which corresponds to the FDI as given in Theorem 1.15.1 in [4]. However, the results there only aim at single-input and single-output Lur'e systems.

If the slope restrictions on $\varphi(\sigma(t))$ are removed, another absolute stability criterion is derived by choosing (5) as the Lyapunov function.

Theorem 8. *System* (1) *is absolutely stable for all* $\varphi(\sigma(t))$ *satisfying* (2) *if the matrix* $A + B\Gamma_1 C^*$ *is Hurwitzian and there exist diagonal matrices* Λ, $T \geq 0$, *symmetric matrices* $P, Q > 0$, *such that the following LMI is feasible:*

$$\begin{bmatrix} A^*P + PA - C\Gamma_1 T\Delta_1 C^* & PB + \Omega_{12} \\ B^*P + \Omega_{12}^* & \Omega_{22} \end{bmatrix} < 0, \quad (30)$$

where $\Omega_{12} = (1/2)A^*C\Lambda + (1/2)CT(\Gamma_1 + \Delta_1)$, $\Omega_{22} = (1/2)\Lambda C^* B + (1/2)B^* C\Lambda - T$.

Proof. The proof is similar to that of Theorem 3. □

Remark 9. Theorem 8 gives absolute stability conditions for sector restricted Lur'e systems. In fact, the slope restricted condition (3) plays an important role in improving the condition of absolute stability. The forthcoming example shows that Theorem 3 is less conservative than Theorem 8.

Similar to Theorem 3, an equivalent frequency-domain criterion to Theorem 8 can be given as follows.

Theorem 10. *System* (1) *is absolutely stable for all* $\varphi(\sigma(t))$ *satisfying* (2) *if the matrix* $A + B\Gamma_1 C^*$ *is Hurwitzian and there exist diagonal matrices* Λ, $T \geq 0$ *such that the following FDI holds:*

$$\operatorname{Re}\left\{[I + \Gamma_1 \chi(j\omega)]^* T [I + \Delta_1 \chi(j\omega)] + j\omega\Lambda\chi(j\omega)\right\} > 0, \quad \omega \in \mathbb{R}. \quad (31)$$

Proof. From the KYP lemma, (30) is equivalent to

$$\begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix}^* \begin{bmatrix} -C\Gamma_1 T\Delta_1 C^* & \Omega_{12} \\ \Omega_{12}^* & \Omega_{22} \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix} < 0, \quad \forall \omega \in \mathbb{R}. \quad (32)$$

The equivalence between (30) and (31) is derived from $\chi(j\omega) = C^*(A - j\omega I)^{-1}B$ and $C^*A(A - j\omega I)^{-1}B = C^*B + j\omega\chi(j\omega)$. □

Remark 11. Theorem 10 includes two particular cases. For the case $\Lambda = 0$, (31) is reduced to

$$\operatorname{Re}\left\{(I + \Gamma_1 \chi(j\omega))^* T (I + \Delta_1 \chi(j\omega))\right\} > 0, \quad \omega \in \mathbb{R}. \quad (33)$$

Correspondingly, Theorem 10 is in the form of the circle criterion. For the case $\Gamma_1 = 0$, (31) reduces to

$$T + \operatorname{Re}\left\{(j\omega\Lambda + T\Delta_1)\chi(j\omega)\right\} > 0, \quad \omega \in \mathbb{R}. \quad (34)$$

Theorem 10 has the same form as the Popov criterion.

4. Numerical Example

In this section, a numerical example is presented to illustrate the effectiveness of the proposed results.

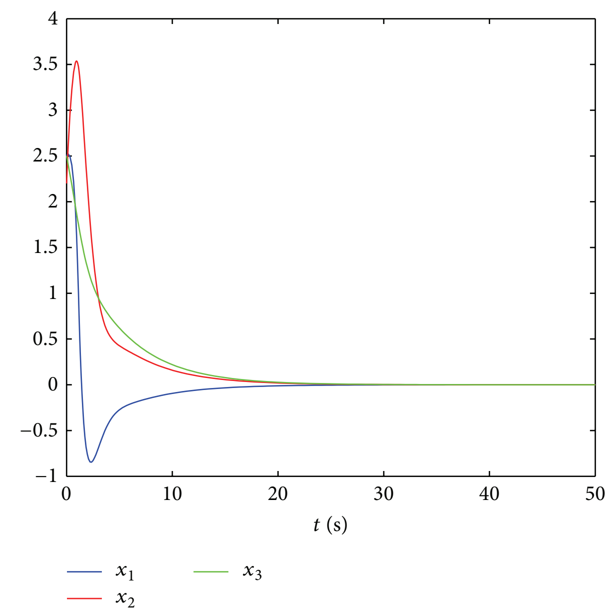

Figure 1: The states of system (35).

Consider Chua's oscillator [25] with the following dimensionless equations

$$\begin{aligned} \dot{x}_1(t) &= \alpha\left[x_2(t) - x_1(t) - f(x_1(t))\right], \\ \dot{x}_2(t) &= x_1(t) - x_2(t) + x_3(t), \\ \dot{x}_3(t) &= -\beta x_2(t) - \gamma x_3(t), \end{aligned} \quad (35)$$

where $f(x_1(t)) = m_1 x_1(t) + (1/2)(m_0 - m_1)(|x_1(t) + 1| - |x_1(t) - 1|)$, α, β, γ, m_0, and m_1 are numbers. System (35) can be reformulated in the form of (1) with $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$, $A = \begin{bmatrix} -\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{bmatrix}$, $B = \begin{bmatrix} -\alpha \\ 0 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^*$, $\sigma(t) = x_1(t)$, and $\varphi(\sigma(t)) = m_1\sigma(t) + (1/2)(m_0 - m_1)(|\sigma(t) + 1| - |\sigma(t) - 1|)$. The nonlinearity $\varphi(\sigma(t))$ satisfies

$$\begin{gathered} \min\{m_0, m_1\}\sigma(t)^2 \leq \varphi(\sigma(t))\sigma(t) \leq \max\{m_0, m_1\}\sigma(t)^2, \\ \min\{m_0, m_1\} \leq \frac{d\varphi(\sigma(t))}{d\sigma(t)} \leq \max\{m_0, m_1\}. \end{gathered} \quad (36)$$

Thus, $\Gamma_1 = \Gamma_2 = \min\{m_0, m_1\}$ and $\Delta_1 = \Delta_2 = \max\{m_0, m_1\}$.

When $\alpha = -0.8018$, $\beta = 0.136$, $\gamma = 0.1097$, and $m_0 = -2.96$ are taken, system (35) is absolutely stable for $m_1 \leq 2.009$ by applying Theorem 3. However, we derive that system (35) is absolutely stable for $m_1 \leq 1.81$ and $m_1 \leq 1.51$, respectively, by Theorem 8 and the Popov criterion. This shows that Theorem 3 is an improvement with respect to Theorem 8 and the Popov criterion, and the slope restrictions could improve the absolute stability condition. The states of system (35) with $m_1 = 2$ at the initial value $\begin{bmatrix} 2.5 & 2.2 & 2.5 \end{bmatrix}^*$ are given in Figure 1, from which it is illustrated that system (35) is absolutely stable.

5. Conclusion

We have proposed new absolute stability criteria for Lur'e systems with sector and slope restricted nonlinearities from time-domain and frequency-domain points of view. The slope restrictions on nonlinearities improve the absolute stability conditions. We have shown that the criteria are less conservative than some existing results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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