

*Research Article*

# Approximate Solutions of Fractional Nonlinear Equations Using Homotopy Perturbation Transformation Method

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A homotopy perturbation transformation method (HPTM) which is based on homotopy perturbation method and Laplace transform is first applied to solve the approximate solution of the fractional nonlinear equations. The nonlinear terms can be easily handled by the use of He's polynomials. Illustrative examples are included to demonstrate the high accuracy and fast convergence of this new algorithm.

## 1. Introduction

In recent years, system of fractional nonlinear partial differential equations [1–3] have attracted much attention in a variety of applied sciences. The importance of obtaining the exact and approximate solutions of fractional nonlinear equations in physics and mathematics is still a significant problem that needs new methods to discover exact and approximate solutions. But these nonlinear fractional differential equations are difficult to get their exact solutions [4–7]. So, numerical methods have been used to handle these equations [8–11], and some semianalytical techniques have also largely been used to solve these equations. Such as, Adomian decomposition method [12, 13], variational iteration method [14, 15], differential transform method [16, 17], Laplace decomposition method [18, 19], and homotopy perturbation method [20–25]. Most of these methods have their inbuilt deficiencies like the calculation of Adomian's polynomials, the Lagrange multiplier, divergent results, and huge computational work.

In this work, we will use homotopy perturbation transformation method introduced by Khan [26, 27] to solve fractional nonlinear partial differential equations. This new method basically illustrates how two powerful algorithms, homotopy perturbation method and Laplace transform method, can be combined and used to approximate the solutions of nonlinear equation. The proposed algorithm provides the solution in a rapid convergent series which may lead to the solution in a closed form. This paper considers the effectiveness of the homotopy perturbation transformation method in solving fractional nonlinear equations.

## 2. Description of the HPTM

To illustrate the basic idea of this method [26, 27], we consider a general fractional nonlinear nonhomogeneous partial differential equation with initial conditions of the form

$$D_t^\alpha u(x, t) + Ru(x, t) + Nu(x, t) = g(x, t), \quad (2.1)$$

$$u(x, 0) = h(x), \quad u_t(x, 0) = f(x), \quad (2.2)$$

where  $g(x, t)$  is the source term,  $N$  represents the general nonlinear differential operator and  $R$  is the linear differential operator, and  $D_t^\alpha u(x, t)$  is the Caputo fractional derivative of function  $u(x, t)$  which is defined as

$${}_0D_t^\alpha u(x, t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t \frac{u^{(n)}(x, \tau) d\tau}{(t - \tau)^{\alpha+1-n}}, \quad (n - 1 < \text{Re}(\alpha) < n, n \in \mathbb{N}), \quad (2.3)$$

where  $\Gamma(\cdot)$  denotes the Gamma function. The properties of fractional derivative can be found in [1, 2]. Laplace transform (denoted throughout this paper by  $L$ ) of the Caputo operator is an important property will be used in this paper

$$L[{}_0D_t^\alpha u(x, t)] = s^\alpha u(x, s) - \sum_{k=0}^{n-1} u^{(k)}(x, 0^+) s^{\alpha-1-k}, \quad (n - 1 < \alpha \leq n). \quad (2.4)$$

Taking the Laplace transform on both sides of (2.1),

$$L[{}_0D_t^\alpha u(x, t)] + L[Ru(x, t)] + L[Nu(x, t)] = L[g(x, t)], \quad (2.5)$$

Using the property of the laplace transform, we have

$$L[u(x, t)] = \frac{h(x)}{s} + \frac{f(x)}{s^2} - \frac{1}{s^\alpha} L[Ru(x, t)] - \frac{1}{s^\alpha} L[Nu(x, t)] + \frac{1}{s^\alpha} L[g(x, t)]. \quad (2.6)$$

Operating with the Laplace inverse on both sides of (2.6) gives

$$u(x, t) = G(x, t) - L^{-1} \left[ \frac{1}{s^\alpha} L[Ru(x, t)] + \frac{1}{s^\alpha} L[Nu(x, t)] \right], \quad (2.7)$$

where  $G(x, t)$  represent the term arising from the source term and the prescribed initial conditions. Then, we apply the homotopy perturbation method; the basic assumption is that the solutions can be written as a power series in  $p$

$$u(x, t) = \sum_{n=0}^{\infty} p^n u_n(x, t) = u_0 + pu_1 + p^2u_2 + p^3u_3 + \dots, \tag{2.8}$$

and the nonlinear term can be decomposed as

$$Nu(x, t) = \sum_{n=0}^{\infty} p^n \mathcal{H}_n(u), \tag{2.9}$$

where  $p \in [0, 1]$  is an embedding parameter.  $\mathcal{H}_n(u)$  is He's polynomials [28, 29] and can be generated by

$$\mathcal{H}_n(u_0, \dots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[ N \left( \sum_{i=0}^{\infty} p^i u_i \right) \right]_{p=0}, \quad n = 0, 1, 2, \dots \tag{2.10}$$

Substituting (2.8) and (2.9) in (2.7) we get

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = G(x, t) - p \left( L^{-1} \left[ \frac{1}{s^\alpha} L \left[ R \sum_{n=0}^{\infty} p^n u_n(x, t) \right] + \frac{1}{s^\alpha} L \left[ \sum_{n=0}^{\infty} p^n \mathcal{H}_n(u) \right] \right] \right). \tag{2.11}$$

Equating the terms with identical powers in  $p$ , we obtain the following approximations:

$$\begin{aligned} p^0 : u_0(x, t) &= G(x, t), \\ p^1 : u_1(x, t) &= -L^{-1} \left[ \frac{1}{s^\alpha} L [Ru_0(x, t)] + \frac{1}{s^\alpha} L [\mathcal{H}_0(u)] \right], \\ p^2 : u_2(x, t) &= -L^{-1} \left[ \frac{1}{s^\alpha} L [Ru_1(x, t)] + \frac{1}{s^\alpha} L [\mathcal{H}_1(u)] \right], \\ &\vdots \end{aligned} \tag{2.12}$$

The best approximations for the solution are

$$u(x, t) = u_0 + u_1 + u_2 + u_3 + \dots \tag{2.13}$$

This method does not resort to linearization or assumptions of weak nonlinearity, the solution generated in the form of general solution, and it is more realistic compared to the method of simplifying the physical problems.

### 3. Approximate Solutions of Fractional Equations

In order to assess the advantages and the accuracy of the homotopy perturbation transform method for fractional nonlinear equations, we have applied it to the following several problems.

*Case 1.* Consider the following time fractional advection nonhomogeneous equation [27]:

$$D_t^\alpha u(x, t) + uu_x = 2t + x + t^3 + xt^2, \quad (3.1)$$

$$u(x, 0) = 0, \quad (3.2)$$

where  $0 < \alpha \leq 1$ , taking the Laplace transform on both sides of (3.1)-(3.2)

$$L[D_t^\alpha u(x, t)] + L[uu_x] = L[2t + x + t^3 + xt^2]. \quad (3.3)$$

Using the property of the Laplace transform, we have

$$L[u(x, t)] = \frac{1}{s^\alpha} \left( \frac{2}{s^2} + \frac{x}{s} + \frac{6}{s^4} + \frac{2x}{s^3} \right) - \frac{1}{s^\alpha} L[uu_x]. \quad (3.4)$$

Operating with the Laplace inverse on both sides of (3.4) gives

$$u(x, t) = \frac{2t^{\alpha+1}}{\Gamma(2+\alpha)} + \frac{xt^\alpha}{\Gamma(1+\alpha)} + \frac{6t^{\alpha+3}}{\Gamma(4+\alpha)} + \frac{2xt^{\alpha+2}}{\Gamma(3+\alpha)} - L^{-1} \left[ \frac{1}{s^\alpha} L[uu_x] \right]. \quad (3.5)$$

Then, we apply the homotopy perturbation method, and substituting (2.8) and (2.9) in (3.5) we get

$$\sum_{n=0}^{\infty} p^n u_n = \frac{2t^{\alpha+1}}{\Gamma(2+\alpha)} + \frac{xt^\alpha}{\Gamma(1+\alpha)} + \frac{6t^{\alpha+3}}{\Gamma(4+\alpha)} + \frac{2xt^{\alpha+2}}{\Gamma(3+\alpha)} - p \left( L^{-1} \left[ \frac{1}{s^\alpha} L \left[ \sum_{n=0}^{\infty} p^n \mathcal{H}_n(u) \right] \right] \right), \quad (3.6)$$

where  $\mathcal{H}_n(u)$  is He's polynomials that represents nonlinear term  $uu_x$ ; we have a few terms of the He's polynomials for  $uu_x$  which are given by

$$\begin{aligned} \mathcal{H}_0(u) &= u_0 u_{0x}, \\ \mathcal{H}_1(u) &= u_0 u_{1x} + u_1 u_{0x}, \\ \mathcal{H}_2(u) &= u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x}, \\ &\vdots \end{aligned} \quad (3.7)$$

Comparing the coefficient of like powers of  $p$ , we have

$$\begin{aligned}
 u_0(x, t) &= \frac{2t^{\alpha+1}}{\Gamma(2+\alpha)} + \frac{xt^\alpha}{\Gamma(1+\alpha)} + \frac{6t^{\alpha+3}}{\Gamma(4+\alpha)} + \frac{2xt^{\alpha+2}}{\Gamma(3+\alpha)}, \\
 u_1(x, t) &= -L^{-1} \left[ \frac{1}{s^\alpha} L[u_0 u_{0x}] \right] \\
 &= -\frac{x\Gamma(1+2\alpha)t^{3\alpha}}{\Gamma^2(1+\alpha)\Gamma(1+3\alpha)} - \frac{2\Gamma(2+2\alpha)t^{1+3\alpha}}{\Gamma(1+\alpha)\Gamma(2+\alpha)\Gamma(2+3\alpha)} - \frac{4x\Gamma(3+2\alpha)t^{2+3\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)\Gamma(3+3\alpha)} \\
 &\quad - \frac{4\Gamma(3+3\alpha)t^{4+2\alpha}}{\Gamma(2+\alpha)\Gamma(3+\alpha)\Gamma(4+3\alpha)} - \frac{6\Gamma(4+2\alpha)t^{3+3\alpha}}{\Gamma(1+\alpha)\Gamma(4+\alpha)\Gamma(4+3\alpha)} \\
 &\quad - \frac{4x\Gamma(5+2\alpha)t^{4+3\alpha}}{\Gamma^2(3+\alpha)\Gamma(5+3\alpha)} - \frac{12\Gamma(6+2\alpha)t^{5+3\alpha}}{\Gamma(3+\alpha)\Gamma(4+\alpha)\Gamma(6+3\alpha)}, \\
 u_2(x, t) &= -L^{-1} \left[ \frac{1}{s^\alpha} L[u_0 u_{1x} + u_1 u_{0x}] \right] \\
 &= \frac{2x\Gamma(1+2\alpha)\Gamma(1+4\alpha)t^{5\alpha}}{\Gamma^2(1+\alpha)\Gamma(1+3\alpha)\Gamma(1+5\alpha)} + \frac{2\Gamma(1+2\alpha)\Gamma(2+4\alpha)t^{1+5\alpha}}{\Gamma^2(1+\alpha)\Gamma(2+\alpha)\Gamma(1+3\alpha)\Gamma(2+5\alpha)} \\
 &\quad + \frac{2\Gamma(2+2\alpha)\Gamma(2+4\alpha)t^{1+5\alpha}}{\Gamma^2(1+\alpha)\Gamma(2+\alpha)\Gamma(2+3\alpha)\Gamma(2+5\alpha)} + \frac{4x\Gamma(1+2\alpha)\Gamma(3+4\alpha)t^{2+5\alpha}}{\Gamma^2(1+\alpha)\Gamma(3+\alpha)\Gamma(1+3\alpha)\Gamma(3+5\alpha)} + \dots \\
 &\quad \vdots
 \end{aligned} \tag{3.8}$$

and so on; in this manner the rest of component of the solution can be obtained. The solution of (3.1) in series form is given by

$$u(x, t) = \frac{2t^{\alpha+1}}{\Gamma(2+\alpha)} + \frac{xt^\alpha}{\Gamma(1+\alpha)} + \frac{6t^{\alpha+3}}{\Gamma(4+\alpha)} + \frac{2xt^{\alpha+2}}{\Gamma(3+\alpha)} - \frac{x\Gamma(1+2\alpha)t^{3\alpha}}{\Gamma^2(1+\alpha)\Gamma(1+3\alpha)} + \dots \tag{3.9}$$

If we take  $\alpha = 1$ , the first few components the solution of (3.1) are as follows:

$$\begin{aligned}
 u_0(x, t) &= t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3}, \\
 u_1(x, t) &= -\frac{t^4}{4} - \frac{xt^3}{3} - \frac{2xt^5}{15} - \frac{7t^6}{72} - \frac{xt^7}{63} - \frac{t^8}{96}, \\
 u_2(x, t) &= \frac{5t^12}{8064} + \frac{2xt^{11}}{2079} + \frac{2783t^{10}}{302400} + \frac{38xt^9}{2835} + \frac{143t^8}{2880} + \frac{22xt^7}{315} + \frac{7t^6}{12} + \frac{2xt^5}{15}, \\
 &\quad \vdots
 \end{aligned} \tag{3.10}$$

The noise terms  $-t^4/4 - xt^3/3$  between the components  $u_0$  and  $u_1$  can be canceled and the remaining term of  $u_0$  still satisfies the equation. For this special case, the exact solution is therefore

$$u(x, t) = t^2 + xt. \quad (3.11)$$

which was given in [27].

*Case 2.* Consider the following time-space fractional nonlinear Fokker-Planck equation [30]:

$$D_t^\alpha u(x, t) = -D_x^\beta \left( \frac{4u^2}{x} - \frac{xu}{3} \right) + D_x^{2\beta} (u^2), \quad (3.12)$$

$$u(x, 0) = x^2, \quad (3.13)$$

where  $0 < \alpha, \beta \leq 1$ , and  $\alpha$  and  $\beta$  are parameters describing the order of the time- and space-fractional derivatives, respectively.  $D_x^\beta$  is also the Caputo fractional derivative with respect to  $x$  and is defined as

$$D_x^\beta u(x, t) = \frac{1}{\Gamma(m - \beta)} \int_0^x \frac{u^{(m)}(\xi, t) d\xi}{(x - \xi)^{\beta+1-m}}, \quad (m - 1 < \text{Re}(\beta) \leq m, m \in \mathbb{N}) \quad (3.14)$$

taking the Laplace transform on both sides of (3.12)-(3.13)

$$s^\alpha L[u(x, t)] - x^2 s^{\alpha-1} = L \left[ -D_x^\beta \left( \frac{4u^2}{x} - \frac{xu}{3} \right) + D_x^{2\beta} (u^2) \right]. \quad (3.15)$$

We have

$$L[u(x, t)] = \frac{x^2}{s} + \frac{1}{s^\alpha} L \left[ -D_x^\beta \left( \frac{4u^2}{x} - \frac{xu}{3} \right) + D_x^{2\beta} (u^2) \right]. \quad (3.16)$$

Operating with the Laplace inverse on both sides of (3.16) gives

$$u(x, t) = x^2 + L^{-1} \left[ \frac{1}{s^\alpha} L \left[ -D_x^\beta \left( \frac{4u^2}{x} - \frac{xu}{3} \right) + D_x^{2\beta} (u^2) \right] \right]. \quad (3.17)$$

Then, we apply the homotopy perturbation method, and substituting (2.8) and (2.9) in (3.17) we get

$$\sum_{n=0}^{\infty} p^n u_n = x^2 + p \left( L^{-1} \left[ \frac{1}{s^\alpha} L \left[ D_x^{2\beta} \left( \sum_{n=0}^{\infty} p^n \mathcal{H}_n(u) \right) - D_x^\beta \left( \frac{4 \sum_{n=0}^{\infty} p^n \mathcal{H}_n(u)}{x} - \frac{x \sum_{n=0}^{\infty} p^n u_n}{3} \right) \right] \right] \right), \quad (3.18)$$

where  $\mathcal{H}_n(u)$  is He's polynomials that represents nonlinear term  $u^2$ ; we have a few terms of the He's polynomials for  $u^2$  which are given by

$$\mathcal{H}_0(u) = u_0^2, \tag{3.19}$$

$$\mathcal{H}_1(u) = 2u_0u_1, \tag{3.20}$$

$$\mathcal{H}_2(u) = u_1^2 + 2u_0u_2, \tag{3.21}$$

⋮

Comparing the coefficient of like powers of  $p$ , we have (3.22)

$$\begin{aligned} u_0(x, t) &= x^2, \\ u_1(x, t) &= L^{-1} \left[ \frac{1}{s^\alpha} L \left[ D_x^{2\beta} (u_0^2) - D_x^\beta \left( \frac{4u_0^2}{x} - \frac{xu_0}{3} \right) \right] \right], \\ &= \frac{24t^\alpha x^{4-2\beta}}{\Gamma(1+\alpha)\Gamma(5-2\beta)} - \frac{22t^\alpha x^{3-\beta}}{\Gamma(1+\alpha)\Gamma(4-\beta)}, \\ u_2(x, t) &= L^{-1} \left[ \frac{1}{s^\alpha} L \left[ D_x^{2\beta} (2u_0u_1) - D_x^\beta \left( \frac{8u_0u_1}{x} - \frac{xu_1}{3} \right) \right] \right] \tag{3.22} \\ &= -\frac{184t^{2\alpha} x^{5-3\beta}\Gamma(6-2\beta)}{\Gamma(1+2\alpha)\Gamma(6-3\beta)\Gamma(5-2\beta)} + \frac{48t^{2\alpha} x^{6-4\beta}\Gamma(7-2\beta)}{\Gamma(1+2\alpha)\Gamma(7-4\beta)\Gamma(5-2\beta)} \\ &\quad + \frac{506t^{2\alpha} x^{4-2\beta}\Gamma(5-\beta)}{3\Gamma(1+2\alpha)\Gamma(5-2\beta)\Gamma(4-\beta)} - \frac{44t^{2\alpha} x^{5-3\beta}\Gamma(6-\beta)}{\Gamma(1+2\alpha)\Gamma(6-3\beta)\Gamma(4-\beta)}, \\ &\quad \vdots \end{aligned}$$

and so on; in this manner the rest of component of the solution can be obtained. The solution of (3.12) in series form is given by

$$u(x, t) = x^2 + \frac{24t^\alpha x^{4-2\beta}}{\Gamma(1+\alpha)\Gamma(5-2\beta)} - \frac{22t^\alpha x^{3-\beta}}{\Gamma(1+\alpha)\Gamma(4-\beta)} - \frac{184t^{2\alpha} x^{5-3\beta}\Gamma(6-2\beta)}{\Gamma(1+2\alpha)\Gamma(6-3\beta)\Gamma(5-2\beta)} + \dots \tag{3.23}$$

If we take  $\alpha = \beta = 1$ , the first few components the solution of (3.12) are as follows:

$$\begin{aligned} u_0(x, t) &= x^2, \\ u_1(x, t) &= x^2 t, \\ u_2(x, t) &= x^2 \frac{t^2}{2!}, \\ &\vdots \end{aligned} \tag{3.24}$$

and so on. Hence, for this special case, we have

$$u(x, t) = x^2 \left( 1 + t + \frac{t^2}{2!} + \cdots \right) = x^2 e^t \tag{3.25}$$

which was given in [30].

Figure 1 shows the approximate solution for (3.12)-(3.13) by using the homotopy perturbation transformation method when choosing  $x = 0.8$ ,  $\alpha = 1$ . From the figure, it is clear to see the time evolution of fractional Fokker-Planck equation and we also know the approximate solution of the model is continuous with the fractional parameter  $\beta$ . Figure 2 shows the approximate solution for (3.12)-(3.13) when  $t = 0.8$ ,  $\alpha = 1$ , and the approximate solution of the model is continuous with the fractional parameter  $\beta$ . Figures 3 and 4 show the approximate solution for (3.12)-(3.13) when the parameter  $\beta = 1$ , and from the figures, we also know that the solution of the fractional nonlinear equation changes with the parameters  $\alpha$  and  $x, t$ .

Table 1 shows the approximate solutions for (3.12) by using the homotopy perturbation transformation method, Adomian decomposition method, variational iteration method and the exact solution  $u(x, t) = x^2 e^t$  when  $\alpha = \beta = 1$ . It is noted that only the third-order of the homotopy perturbation transformation solution is used in evaluating the approximate solutions in Table 1, and it is evident that the method used in this paper and the Adomian decomposition method have high accuracy compare with the variational iteration method, and we take 15 terms of the VIM solution. And for nonlinear equations, Adomian's polynomials are very difficult to calculate. In brief, the homotopy perturbation transformation method is an effectiveness tool to solve fractional nonlinear equation only using Mathematica symbol computing software.

*Case 3.* Consider the following time fractional nonhomogeneous nonlinear system [31]:

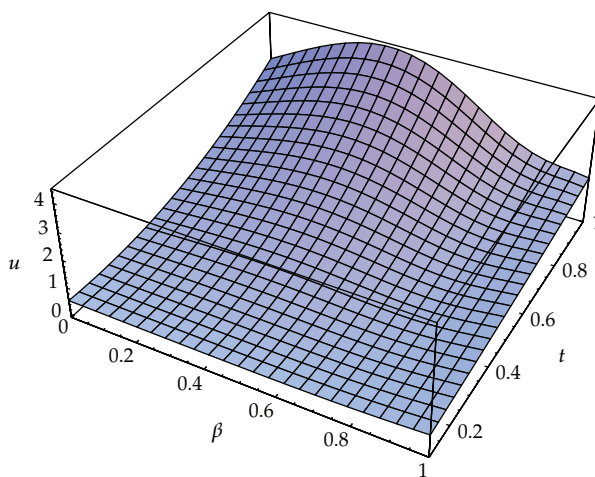
$$D_t^\alpha u - u_x v - u = 1, \tag{3.26}$$

$$D_t^\beta v + uv_x + v = 1, \tag{3.27}$$

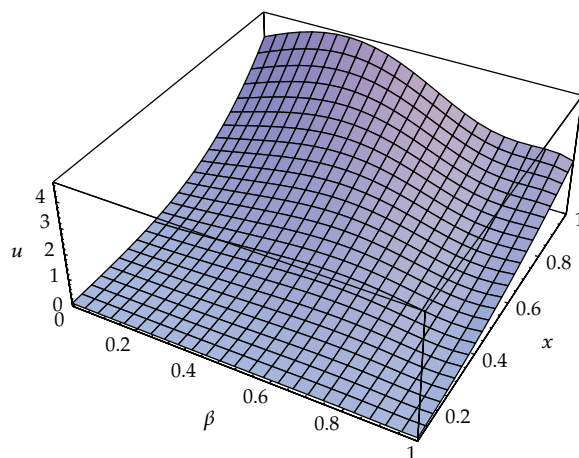
with the initial conditions

$$u(x, 0) = e^{-x}, \quad v(x, 0) = e^x, \tag{3.28}$$





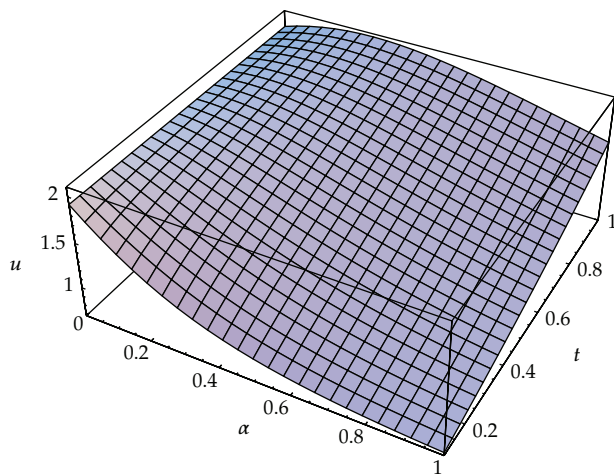
**Figure 1:** The surface of second-order approximate solution of (3.12) when  $x = 0.8, \alpha = 1$ .



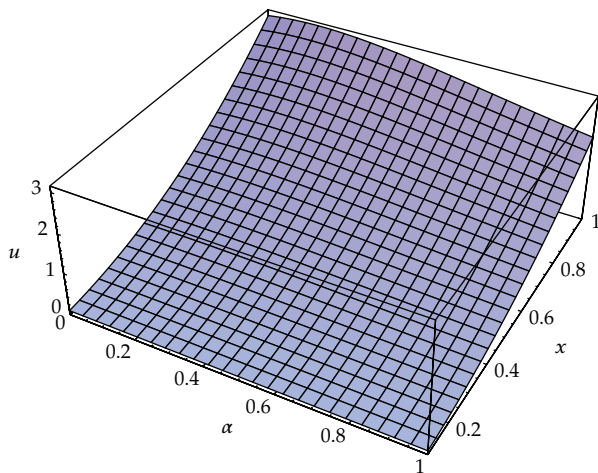
**Figure 2:** The surface of second-order approximate solution of (3.12) when  $t = 0.8, \alpha = 1$ .

**Table 1:** Approximate values and exact solutions when  $\alpha = 1, \beta = 1$  for (3.12).

| $t$  | $x$  | solution <sub>HPTM</sub> | solution <sub>ADM</sub> | solution <sub>VIM</sub> | exact solution |
|------|------|--------------------------|-------------------------|-------------------------|----------------|
| 0.06 | 0.25 | 0.066367                 | 0.066367                | 0.066363                | 0.066365       |
|      | 0.50 | 0.265468                 | 0.265468                | 0.265450                | 0.265459       |
|      | 0.75 | 0.597303                 | 0.597303                | 0.597262                | 0.597283       |
|      | 1.0  | 1.061970                 | 1.061970                | 1.061800                | 1.061840       |
| 0.2  | 0.25 | 0.076417                 | 0.076416                | 0.076250                | 0.076338       |
|      | 0.50 | 0.305667                 | 0.305667                | 0.305000                | 0.305351       |
|      | 0.75 | 0.687750                 | 0.687750                | 0.686250                | 0.687039       |
|      | 1.0  | 1.222670                 | 1.222670                | 1.220000                | 1.221400       |
| 0.4  | 0.25 | 0.093833                 | 0.093833                | 0.092500                | 0.093239       |
|      | 0.50 | 0.375330                 | 0.375330                | 0.370000                | 0.372956       |
|      | 0.75 | 0.844500                 | 0.844500                | 0.832500                | 0.839151       |
|      | 1.0  | 1.501330                 | 1.501330                | 1.480000                | 1.491820       |



**Figure 3:** The surface of second-order approximate solution of (3.12) when  $x = 0.8$ ,  $\beta = 1$ .



**Figure 4:** The surface of second-order approximate solution of (3.12) when  $t = 0.8$ ,  $\beta = 1$ .

where  $0 < \alpha, \beta \leq 1$ ; in a similar way as above we have

$$\begin{aligned} \sum_{n=0}^{\infty} p^n u_n &= e^{-x} + \frac{t^\alpha}{\Gamma(\alpha+1)} + p \left( L^{-1} \left[ \frac{1}{s^\alpha} L \left[ \sum_{n=0}^{\infty} p^n \mathcal{H}_{1n}(u, v) + \sum_{n=0}^{\infty} p^n u_n \right] \right] \right), \\ \sum_{n=0}^{\infty} p^n v_n &= e^x + \frac{t^\beta}{\Gamma(\beta+1)} - p \left( L^{-1} \left[ \frac{1}{s^\alpha} L \left[ \sum_{n=0}^{\infty} p^n \mathcal{H}_{2n}(u, v) + \sum_{n=0}^{\infty} p^n v_n \right] \right] \right), \end{aligned} \quad (3.29)$$

where  $\mathcal{H}_{1n}(u, v)$  and  $\mathcal{H}_{2n}(u, v)$  are He's polynomials that represent nonlinear term  $vu_x$  and  $uv_x$  respectively, and we have a few terms of the He's polynomials for  $vu_x$  and  $uv_x$  which are given by

$$\begin{aligned}
 \mathcal{H}_{10}(u, v) &= v_0 u_{0x}, \\
 \mathcal{H}_{11}(u, v) &= v_1 u_{0x} + v_0 u_{1x}, \\
 \mathcal{H}_{12}(u, v) &= v_2 u_{0x} + v_1 u_{1x} + v_0 u_{2x}, \\
 &\vdots \\
 \mathcal{H}_{20}(u, v) &= u_0 v_{0x}, \\
 \mathcal{H}_{21}(u, v) &= u_1 v_{0x} + u_0 v_{1x}, \\
 \mathcal{H}_{22}(u, v) &= u_2 v_{0x} + u_1 v_{1x} + u_0 v_{2x}, \\
 &\vdots
 \end{aligned} \tag{3.30}$$

Comparing the coefficient of like powers of  $p$ , we have

$$\begin{aligned}
 u_0(x, t) &= e^{-x} + \frac{t^\alpha}{\Gamma(1 + \alpha)}, \\
 v_0(x, t) &= e^x + \frac{t^\beta}{\Gamma(1 + \beta)}, \\
 u_1(x, t) &= L^{-1} \left[ \frac{1}{s^\alpha} L[v_0 u_{0x} + u_0] \right], \\
 &= -\frac{t^\alpha}{\Gamma(1 + \alpha)} + \frac{e^{-x} t^\alpha}{\Gamma(1 + \alpha)} + \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)} - \frac{e^{-x} t^{\alpha+\beta}}{\Gamma(1 + \alpha + \beta)}, \\
 v_1(x, t) &= -L^{-1} \left[ \frac{1}{s^\alpha} L[u_0 v_{0x} + v_0] \right], \\
 &= -\frac{t^\beta}{\Gamma(1 + \beta)} - \frac{e^x t^\beta}{\Gamma(1 + \beta)} - \frac{t^{2\beta}}{\Gamma(1 + 2\beta)} - \frac{e^x t^{\alpha+\beta}}{\Gamma(1 + \alpha + \beta)}, \\
 u_2(x, t) &= L^{-1} \left[ \frac{1}{s^\alpha} L[v_1 u_{0x} + v_0 u_{1x} + u_1] \right] \\
 &= -\frac{2t^{2\alpha}}{\Gamma(1 + 2\alpha)} + \frac{e^{-x} t^{2\alpha}}{\Gamma(1 + 2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1 + 3\alpha)} + \frac{t^{\alpha+\beta}}{\Gamma(1 + \alpha + \beta)} \\
 &\quad + \frac{e^{-x} t^{\alpha+\beta}}{\Gamma(1 + \alpha + \beta)} + \frac{2t^{2\alpha+\beta}}{\Gamma(1 + 2\alpha + \beta)} - \frac{e^{-x} t^{2\alpha+\beta}}{\Gamma(1 + 2\alpha + \beta)} + \frac{e^{-x} t^{\alpha+2\beta}}{\Gamma(1 + \alpha + 2\beta)} \\
 &\quad - \frac{e^{-x} t^{2\alpha+\beta} \Gamma(1 + \alpha + \beta)}{\Gamma(1 + \alpha) \Gamma(1 + \beta) \Gamma(1 + 2\alpha + \beta)} + \frac{e^{-x} t^{2\alpha+2\beta} \Gamma(1 + \alpha + 2\beta)}{\Gamma(1 + \beta) \Gamma(1 + \alpha + \beta) \Gamma(1 + 2\alpha + 2\beta)},
 \end{aligned}$$

$$\begin{aligned}
v_2(x, t) &= -L^{-1} \left[ \frac{1}{s^\alpha} L[u_1 v_{0x} + u_0 v_{1x} + v_1] \right] \\
&= -\frac{t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} + \frac{e^x t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} - \frac{e^x t^{2\alpha+\beta}}{\Gamma(1+2\alpha+\beta)} + \frac{2t^{2\beta}}{\Gamma(1+2\beta)} \\
&\quad + \frac{e^x t^{2\beta}}{\Gamma(1+2\beta)} + \frac{2t^{\alpha+2\beta}}{\Gamma(1+\alpha+2\beta)} + \frac{e^x t^{\alpha+2\beta}}{\Gamma(1+\alpha+2\beta)} + \frac{t^{3\beta}}{\Gamma(3\beta)} \\
&\quad + \frac{e^x t^{\alpha+2\beta} \Gamma(1+\alpha+\beta)}{\Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(1+\alpha+2\beta)} + \frac{e^x t^{2\alpha+2\beta} \Gamma(1+2\alpha+\beta)}{\Gamma(1+\alpha)\Gamma(1+\alpha+\beta)\Gamma(1+2\alpha+2\beta)}, \\
&\quad \vdots
\end{aligned} \tag{3.31}$$

and so on; in this manner the rest of component of the solution can be obtained. The solution of (3.26) and (3.27) in series form is given by

$$\begin{aligned}
u(x, t) &= e^{-x} + \frac{e^{-x} t^\alpha}{\Gamma(1+\alpha)} - \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{e^{-x} t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} + \frac{e^{-x} t^{2\alpha}}{\Gamma(1+2\alpha)} + \dots, \\
v(x, t) &= e^x - \frac{e^x t^\beta}{\Gamma(1+\beta)} + \frac{t^{2\beta}}{\Gamma(1+2\beta)} - \frac{e^x t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} + \frac{e^x t^{2\beta}}{\Gamma(1+2\beta)} + \dots.
\end{aligned} \tag{3.32}$$

If we take  $\alpha = \beta = 1$ , the first few components the solution of (3.26) and (3.27) are as follows:

$$\begin{aligned}
u_0(x, t) &= e^{-x} + t, \\
v_0(x, t) &= e^x + t, \\
u_1(x, t) &= -t + e^{-x} t + \frac{t^2}{2} - \frac{e^{-x} t^2}{2}, \\
v_1(x, t) &= -t - e^x t - \frac{t^2}{2} - \frac{e^x t^2}{2}, \\
u_2(x, t) &= -\frac{t^2}{2} + e^{-x} t^2 + \frac{t^3}{2} - \frac{e^{-x} t^3}{3} + \frac{e^{-x} t^4}{8}, \\
v_2(x, t) &= \frac{t^2}{2} + e^x t^2 + \frac{t^3}{2} + \frac{e^x t^3}{3} + \frac{e^x t^4}{8}, \\
&\quad \vdots
\end{aligned} \tag{3.33}$$

and so on. Hence, for this special case, we have

$$\begin{aligned} u(x, t) &= e^{-x} \left( 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right) = e^{-x+t}, \\ v(x, t) &= e^x \left( 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots \right) = e^{x-t}, \end{aligned} \tag{3.34}$$

which was given in [31].

#### 4. Conclusion

In this work, a homotopy perturbation transformation method which is based on homotopy perturbation method and Laplace transform is used to solve fractional nonlinear partial equations. The nonlinear terms can be easily handled by the use of He's polynomials. It is worth mentioning that the method is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the result, and the size reduction amounts to an improvement of the performance of the approach. The HPTM can be applied for some other engineering system with less computational work.

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