

Research Article

Reentrant Phase Transitions and Triple Points of Topological AdS Black Holes in Born-Infeld-Massive Gravity

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Motivated by recent developments of black hole thermodynamics in de Rham, Gabadadze, and Tolley (dRGT) massive gravity, we study the critical behaviors of topological Anti-de Sitter (AdS) black holes in the presence of Born-Infeld nonlinear electrodynamics. Here the cosmological constant appears as a dynamical pressure of the system and its corresponding conjugate quantity is interpreted as thermodynamic volume. This shows that, besides the Van der Waals-like SBH/LBH phase transitions, the so-called reentrant phase transition (RPT) appears in four-dimensional space-time when the coupling coefficients $c_1 m^2$ of massive potential and Born-Infeld parameter b satisfy some certain conditions. In addition, we also find the triple critical points and the small/intermediate/large black hole phase transitions for $d = 5$.

1. Introduction

Einstein's General Relativity (GR), which describes that the graviton is a massless spin-2 particle, helped us to understand the dynamics of the Universe [1–3]. However, there are some fundamental issues, such as the hierarchy problem in particle physics, the old cosmological constant problem, and the origin of late-time acceleration of the Universe, that still exist in GR [4]. One of the alternating theories of gravity is known as a massive gravity, where mass terms are added to the GR action. A graviton mass has the advantage to potentially provide a theory of dark energy which could explain the present day acceleration of our Universe [5]. On the other hand, since the quantum theory of massless gravitons is nonrenormalizable, a natural question is whether one can build a self-consistent gravity theory if the graviton is massive. The first attempt toward constructing the theory of massive gravity was done by Fierz and Pauli (FP) [6]. With the quadratic order, the FP mass term is the only ghost-free term describing a gravity theory with five degrees of freedom [7]. However, due to the existence of the van Dam-Veltman-Zakharov (vDVZ)

discontinuity, this theory cannot recover linearized Einstein gravity in the limit of vanishing graviton mass [8, 9].

In particular, Vainshtein [10] proposed that the linear massive gravity can be recovered to GR through the “Vainshtein Mechanism” at small scales by including nonlinear terms in the massive gravity action. Nevertheless, it usually brings various instabilities for the gravitational theories on the nonlinear level by adding generic mass terms, since this model suffers from a pathology called a “Boulware-Deser” (BD) ghost. Later, a new nonlinear massive gravity theory was proposed by de Rham, Gabadadze, and Tolley (dRGT) [11–13], where the BD ghost [14] was eliminated by introducing higher order interaction terms in the action. Then, Vegh et al. [15, 16] constructed a nontrivial black hole solution with a Ricci flat horizon in four-dimensional dRGT massive gravity. The spherically symmetric solutions were also addressed in [17–19]; the corresponding charged black hole solution was found in [20, 21].

Recent development on the thermodynamics of black holes in extended phase space shows that the cosmological constant can be interpreted as the thermodynamic pressure

and treated as a thermodynamic variable in its own right [22, 23]:

$$P = -\frac{\Lambda}{8\pi} \quad (1)$$

in the geometric units $G_N = \hbar = c = 1$. Such operation assumes that gravitational theories including different values of the cosmological constants fall in the same class, with unified thermodynamic relations. For black hole thermodynamics, the variation of the cosmological constant ensures the consistency between the first law of black hole thermodynamics and the Smarr formula. Moreover, the classical theory of gravity may be an effective theory which follows from a yet unknown fundamental theory, in which all the presently “physical constants” are actually moduli parameters that can run from place to place in the moduli space of the fundamental theory. Since the fundamental theory is yet unknown, it is more reasonable to consider the extended thermodynamics of gravitational theories involving only a single action, and then all variables will appear in the thermodynamical relations. In the extended phase space, the charged AdS black hole admits a more direct and precise coincidence between the first-order small/large black holes (SBH/LBH) phase transition and the Van der Waals liquid-gas phase transition, and both systems share the same critical exponents near the critical point [24]. More discussions in various gravity theories can be found in [25–46]. Recently, some investigations for thermodynamics of AdS black holes have been also generalized to the extended phase space in the dRGT massive gravity [47–50], which show the Van der Waals-like SBH/LBH phase transition in the charged topological AdS black holes. In addition, the deep relation between the dynamical perturbation and the Van der Waals-like SBH/LBH phase transition in the four-dimensional dRGT massive gravity has been also recovered in [51]. In particular, for neutral AdS black holes in all $d \geq 6$ dimensional space-time, there exist peculiar behaviors of intermediate/small/large black hole phase transitions reminiscent of reentrant phase transitions (RPTs) when the coupling coefficients $c_i m^2$ of massive potential satisfy some certain conditions [52]. A system undergoes an RPT if a monotonic variation of any thermodynamic quantity results in two (or more) phase transitions such that the final state is macroscopically similar to the initial state. The RPT is usually observed in multicomponent fluid systems, ferroelectrics, gels, liquid crystals, and binary gases [53].

In Maxwell’s electromagnetic field theory, a point-like charge which allowed a singularity at the charge position usually brings about infinite self-energy. In order to overcome this problem, Born and Infeld [54] and Hoffmann [55] introduced Born-Infeld electromagnetic field to solve infinite self-energy problem by imposing a maximum strength of the electromagnetic field. In addition, BI type effective action arises in an open superstring theory and D-branes are free of physical singularities. In recent two decades, exact solutions of gravitating black objects in the presence of BI theory have been vastly investigated. In the extended phase space, [56, 57] recovered the RPT in the four-dimensional Einstein-Born-Infeld AdS black hole with spherical horizon. However,

for the higher-dimensional Einstein-Born-Infeld AdS black holes, there is no RPT. What about AdS black holes in the Born-Infeld-massive gravity? In this paper, we will generalize the discussion to topological AdS black holes for $d = 4$ and 5 in the Born-Infeld-massive gravity.

This paper is organized as follows. In Section 2, we review the thermodynamics of Born-Infeld-massive black holes in the extended phase space. In Section 3, we study the critical behaviors of four- and five-dimensional topological AdS black holes in context of $P - V$ criticality and phase diagrams. We end the paper with conclusions and discussions in Section 4.

2. Thermodynamics of d -Dimensional Born-Infeld AdS Black Holes

We start with the action of d -dimensional massive gravity in presence of Born-Infeld field [58]:

$$\mathcal{F} = \frac{1}{16\pi} \int d^d x \sqrt{-g} \left[R - 2\Lambda + \mathcal{L}(\mathcal{F}) + m^2 \sum_{i=1}^4 c_i \mathcal{U}_i(g, f) \right], \quad (2)$$

where the last four terms are the massive potential associated with graviton mass, c_i are the negative constants [21], and f is a fixed rank-2 symmetric tensor. Moreover, \mathcal{U}_i are symmetric polynomials of the eigenvalues of the $d \times d$ matrix $\mathcal{K}_\nu^\mu \equiv \sqrt{g^{\mu\alpha} f_{\alpha\nu}}$:

$$\begin{aligned} \mathcal{U}_1 &= [\mathcal{K}], \\ \mathcal{U}_2 &= [\mathcal{K}]^2 - [\mathcal{K}^2], \\ \mathcal{U}_3 &= [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3], \\ \mathcal{U}_4 &= [\mathcal{K}]^4 - 6[\mathcal{K}^2][\mathcal{K}]^2 + 8[\mathcal{K}^3][\mathcal{K}] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4]. \end{aligned} \quad (3)$$

The square root in \mathcal{K} is understood as the matrix square root, that is, $(\sqrt{A})^\mu_\nu (\sqrt{A})_\lambda^\nu = A^\mu_\lambda$, and the rectangular brackets denote traces $[\mathcal{K}] = \mathcal{K}^\mu_\mu$. In addition, b is the Born-Infeld parameter and $\mathcal{L}(\mathcal{F})$ with

$$\mathcal{L}(\mathcal{F}) = 4b^2 \left(1 - \sqrt{1 + \frac{F^{\mu\nu} F_{\mu\nu}}{2b^2}} \right). \quad (4)$$

In the limit $b \rightarrow \infty$, it reduces to the standard Maxwell field $\mathcal{L}(\mathcal{F}) = -F^{\mu\nu} F_{\mu\nu} + \mathcal{O}(F^4)$. If taking $b = 0$, $\mathcal{L}(\mathcal{F})$ disappears.

Consider the metric of d -dimensional space-time in the following form:

$$ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 h_{ij} dx^i dx^j, \quad (5)$$

where $h_{ij} dx^i dx^j$ is the line element for an Einstein space with constant curvature $(d-2)(d-3)k$. The constant k

characterizes the geometric property of hypersurface, which takes values $k = 0$ for flat, $k = -1$ for negative curvature, and $k = 1$ for positive curvature, respectively.

By using the reference metric [21]

$$f_{\mu\nu} = \text{diag}(0, 0, c_0^2 h_{ij}) \quad (6)$$

with a positive constant c_0 , we can obtain

$$\begin{aligned} \mathcal{U}_1 &= \frac{(d-2)c_0}{r}, \\ \mathcal{U}_2 &= \frac{(d-2)(d-3)c_0^2}{r^2}, \\ \mathcal{U}_3 &= \frac{(d-2)(d-3)(d-4)c_0^3}{r^3}, \\ \mathcal{U}_4 &= \frac{(d-2)(d-3)(d-4)(d-5)c_0^4}{r^4}. \end{aligned} \quad (7)$$

Obviously, the terms related to c_3 and c_4 only appear in the black hole solutions for $d \geq 5$ and $d \geq 6$, respectively [21].

In addition, the electromagnetic field tensor in d -dimensions is given by $F_{tr} = \sqrt{d_2 d_3 / (1 + \Gamma)} (q/r^{d_2})$, and the metric function $f(r)$ is obtained as follows [58]:

$$\begin{aligned} f(r) &= k - \frac{m_0}{r^{d_3}} + \frac{(4b^2 - 2\Lambda)}{d_1 d_2} r^2 - \frac{4b^2 r^2}{d_1 d_2} \sqrt{1 + \Gamma} \\ &+ \frac{4d_2 q^2}{d_1 r^{2d_3}} \mathcal{H} \\ &+ m^2 c_0 \left(\frac{c_1 r}{d_2} + c_0 c_2 + \frac{d_3 c_0^2 c_3}{r} + \frac{d_3 d_4 c_0^3 c_4}{r^2} \right), \end{aligned} \quad (8)$$

where $d_i = d - i$ and

$$\begin{aligned} \Gamma &= \frac{d_2 d_3 q^2}{b^2 r^{2d_2}}, \\ \mathcal{H} &= {}_2F_1 \left[\frac{1}{2}, \frac{d_3}{2d_2}, \frac{3d_7/3}{2d_2}, -\Gamma \right]. \end{aligned} \quad (9)$$

Moreover, m_0 and q are related to the mass M and charge Q of black holes as

$$\begin{aligned} Q &= \frac{\sqrt{d_2 d_3} \Sigma_k}{4\pi} q, \\ M &= \frac{d_2 \Sigma_k}{16\pi} m_0, \end{aligned} \quad (10)$$

where Σ_k represents the volume of constant curvature hypersurface described by $h_{ij} dx^i dx^j$. The electromagnetic potential difference (Φ) between the horizon and infinity reads as $\Phi = \sqrt{d_2/d_3} (q/r_+^{d_3}) \mathcal{H}_+$.

Then the mass M of the Born-Infeld AdS black hole for massive gravity is given by

$$\begin{aligned} M &= \frac{d_2 \Sigma_k r_+^{d_3}}{16\pi} \left[k + \frac{16\pi P}{d_1 d_2} r_+^2 + \frac{4b^2 r_+^2}{d_1 d_2} \left(1 - \sqrt{1 + \Gamma_+} \right) \right. \\ &+ \frac{4d_2 q^2}{d_1 r_+^{2d_3}} \mathcal{H}_+ \\ &\left. + m^2 \left(\frac{c_0 c_1 r_+}{d_2} + c_0^2 c_2 + \frac{d_3 c_0^3 c_3}{r_+} + \frac{d_3 d_4 c_0^4 c_4}{r_+^2} \right) \right], \end{aligned} \quad (11)$$

in terms of the horizon radius r_+ . Due to existence of the pressure in obtained relation for total mass of the black holes, here the black hole mass M can be considered as the enthalpy H rather than the internal energy of the gravitational system [59].

In addition, the Hawking temperature which is related to the definition of surface gravity on the outer horizon r_+ can be obtained as

$$\begin{aligned} T &= \frac{d_3 k}{4\pi r_+} + \frac{4r_+ P}{d_2} + \frac{b^2 r_+}{d_2 \pi} \left(1 - \sqrt{1 + \Gamma_+} \right) \\ &+ \frac{m^2 c_0}{4\pi} \left(c_1 + \frac{d_3 c_0 c_2}{r_+} + \frac{d_3 d_4 c_0^2 c_3}{r_+^2} + \frac{d_3 d_4 d_5 c_0^3 c_4}{r_+^3} \right), \end{aligned} \quad (12)$$

and the entropy S of the Born-Infeld AdS black hole reads as

$$S = \frac{\Sigma_k}{4} r_+^{d_2}. \quad (13)$$

It is easy to check that those thermodynamic quantities obey the (extended phase space) first law of black hole thermodynamics:

$$\begin{aligned} dH &= TdS + VdP + \mathfrak{B}db + \frac{c_0 m^2 \Sigma_k r_+^{d_2}}{16\pi} dc_1 \\ &+ \frac{d_2 c_0^2 m^2 \Sigma_k r_+^{d_3}}{16\pi} dc_2 + \frac{d_2 d_3 c_0^3 m^2 \Sigma_k r_+^{d_4}}{16\pi} dc_3 \\ &+ \frac{d_2 d_3 d_4 c_0^4 m^2 \Sigma_k r_+^{d_5}}{16\pi} dc_4, \end{aligned} \quad (14)$$

where \mathfrak{B} , which is a quantity conjugate to b , is called the ‘‘Born-Infeld vacuum polarization’’:

$$\begin{aligned} \mathfrak{B} &= \left(\frac{\partial H}{\partial b} \right) \Big|_{(S, P, c_1, c_2, c_3, c_4)} \\ &= \frac{\Sigma_k b r_+^{d_1}}{2d_1 \pi} \left(1 - \sqrt{1 + \Gamma_+} \right) + \frac{d_2 d_3 \Sigma_k}{4\pi d_1 b} \frac{\mathcal{H}_+ q^2}{r_+^{d_3}}, \end{aligned} \quad (15)$$

and the thermodynamic volume V [60], which is the corresponding conjugate quantity of P , can be written as

$$V = \frac{\Sigma_k r_+^{d_1}}{d_1}. \quad (16)$$

The behavior of free energy G is important to determine the thermodynamic phase transition in the canonical ensemble. We can calculate the free energy from the thermodynamic relation:

$$\begin{aligned} G &= H - TS \\ &= \frac{r_+^{d_1}}{d_1 d_2} \left(P + \frac{b^2}{4\pi} \sqrt{1 + \Gamma_+} \right) + \frac{d_2^2 q^2 \mathcal{H}_+}{2\pi d_1 r_+^{d_3}} + \frac{r_+^{d_3}}{16\pi} \\ &\quad + \frac{m^2 c_0^2 r_+^{d_5}}{16\pi} (c_2 r_+^2 + 2d_3 c_0 c_3 r_+ + 3d_3 d_4 c_0^2 c_4). \end{aligned} \quad (17)$$

3. Phase Transitions of Topological AdS Black Holes in Born-Infeld-Massive Gravity

For further convenience, we denote

$$\begin{aligned} \hat{T} &= T - \frac{c_0 c_1 m^2}{4\pi}, \\ W_2 &= -\frac{k + c_0^2 c_2 m^2}{8\pi}, \\ W_3 &= -\frac{c_0^3 c_3 m^2}{8\pi}, \\ W_4 &= -\frac{c_0^4 c_4 m^2}{8\pi}, \end{aligned} \quad (18)$$

where \hat{T} denotes the shifted temperature and can be negative according to the value of $c_0 c_1 m^2$. Then, the equation of state of the black hole can be obtained from (12):

$$\begin{aligned} P &= \frac{d_2}{4r_+} \left[\hat{T} + \frac{2d_3 W_2}{r_+} + \frac{2d_3 d_4 W_3}{r_+^2} + \frac{2d_3 d_4 d_5 W_4}{r_+^3} \right. \\ &\quad \left. - \frac{b^2 r_+}{d_2 \pi} \left(1 - \sqrt{1 + \Gamma_+} \right) \right]. \end{aligned} \quad (19)$$

To compare with the VdW fluid equation, we can translate the ‘‘geometric’’ equation of state to physical one by identifying the specific volume ν of the fluid with the horizon radius of the black hole as $\nu = 4r_+/d_2$. Evidently, the specific volume ν is proportional to the horizon radius r_+ ; therefore we will just use the horizon radius in the equation of state for the black hole hereafter in this paper.

We know that the critical point occurs when P has an inflection point:

$$\left. \frac{\partial P}{\partial r_+} \right|_{\hat{T}=\hat{T}_c, r_+=r_c} = \left. \frac{\partial^2 P}{\partial r_+^2} \right|_{\hat{T}=\hat{T}_c, r_+=r_c} = 0, \quad (20)$$

where the subscript stands for the quantities at the critical point. The critical shifted temperature is obtained as

$$\begin{aligned} \hat{T}_c &= -\frac{2d_3}{r_c} \left(2w_2 + \frac{3d_4 W_3}{r_c} + \frac{4d_4 d_5 W_4}{r_c^2} \right) \\ &\quad - \frac{d_2 d_3 q^2}{\pi r_c^{2d_5/2}} (1 + \Gamma_+)^{-1/2}, \end{aligned} \quad (21)$$

and the equation for critical horizon radius r_c is given by

$$\begin{aligned} F(r_c) &= 6d_4 d_5 W_4 + 3d_4 W_3 r_c + W_2 r_c^2 \\ &\quad + \frac{d_5/2 d_2 q^2}{2\pi r_c^{2d_4}} (1 + \Gamma_+)^{-1/2} \\ &\quad - \frac{d_3 d_2^3 q^4}{4\pi b^2 r_c^{4d_3}} (1 + \Gamma_+)^{-3/2} = 0. \end{aligned} \quad (22)$$

For later discussions, it is convenient to rescale some quantities in the following way:

$$\begin{aligned} W_2 &= q^{2/(d-2)} \cdot b^{2(d-3)/(d-2)} w_2, \\ W_3 &= q^{3/(d-2)} \cdot b^{(2d-7)/(d-2)} w_3, \\ W_4 &= q^{4/(d-2)} \cdot b^{2(d-4)/(d-2)} w_4, \\ r_+ &= \left(\frac{q}{b} \right)^{1/(d-2)} \cdot x, \\ P &= b^2 \cdot p, \\ \hat{T} &= q^{1/(d-2)} \cdot b^{(2d-5)/(d-2)} \cdot t, \\ G &= q^{(d-1)/(d-2)} \cdot b^{-2/(d-2)} \Sigma_k \cdot g. \end{aligned} \quad (23)$$

In terms of quantities above, (19), (21), and (22) can be written as

$$\begin{aligned} p &= \frac{d_2}{4x} \left[t + \frac{2d_3 w_2}{x} + \frac{2d_3 d_4 w_3}{x^2} + \frac{2d_3 d_4 d_5 w_4}{x^3} \right. \\ &\quad \left. - \frac{x}{d_2 \pi} \left(1 - \sqrt{1 + \frac{d_2 d_3}{x^{2d_2}}} \right) \right], \end{aligned} \quad (24)$$

$$\begin{aligned} t_c &= -\frac{2d_3}{x_c} \left(2w_2 + \frac{3d_4 w_3}{x_c} + \frac{4d_4 d_5 w_4}{x_c^2} \right) - \frac{d_2 d_3}{\pi x_c^{2d_5/2}} \left(1 \right. \\ &\quad \left. + \frac{d_2 d_3}{x_c^{2d_2}} \right)^{-1/2}, \end{aligned} \quad (25)$$

$$\begin{aligned} F(x_c) &= 6d_4 d_5 w_4 + 3d_4 w_3 x_c + w_2 x_c^2 + \frac{d_2 d_5/2}{2\pi x_c^{2d_4}} \left(1 \right. \\ &\quad \left. + \frac{d_2 d_3}{x_c^{2d_2}} \right)^{-1/2} - \frac{d_3 d_2^3}{4\pi x_c^{4d_3}} \left(1 + \frac{d_2 d_3}{x_c^{2d_2}} \right)^{-3/2} = 0, \end{aligned} \quad (26)$$

where x_c denotes the critical value of x . For arbitrary parameter d , it is hard to obtain the exact solution of (26).

In what follows we shall specialize to $d = 4$ and 5 and then perform a detailed study of the thermodynamics of these black holes.

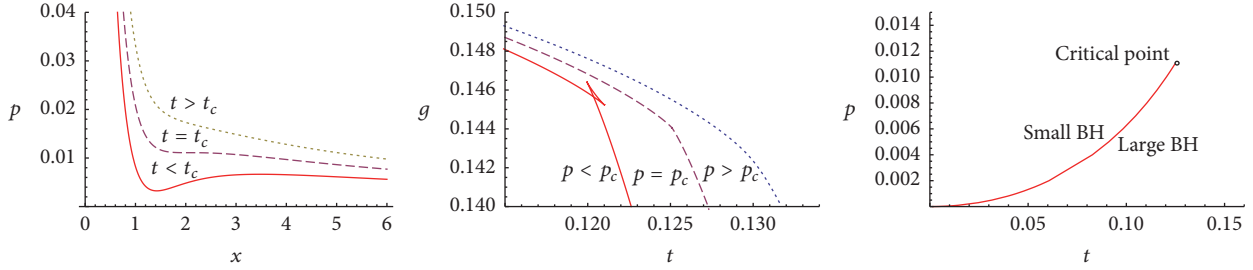


FIGURE 1: Born-Infeld AdS black holes for $d = 4$ and $w_2 = -0.1$. There is one critical point, which corresponds to VdW-like SBH/LBH phase transition when $t < t_c$. Here the critical pressure and temperature read $p = p_c \approx 0.011986$ and $t = t_c \approx 0.125087$, respectively.

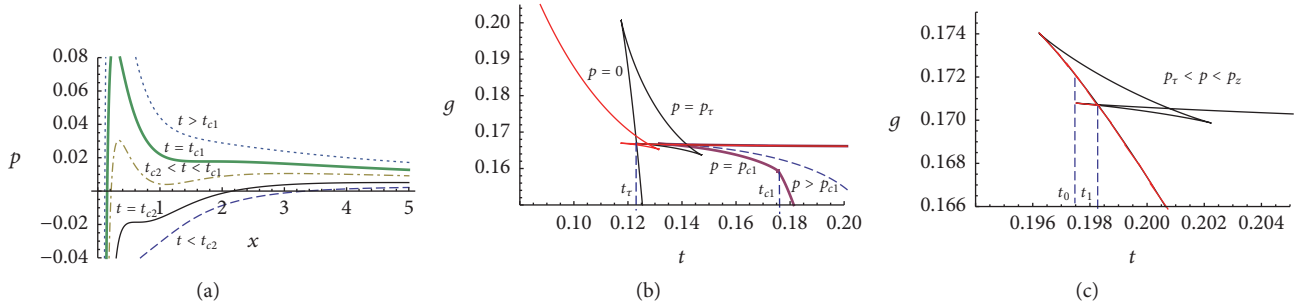


FIGURE 2: Born-Infeld AdS black holes for $d = 4$ and $w_2 = -0.124$. (a) The $p - x$ diagram shows the existence of two critical points, one at positive pressure $p_{c1} \approx 0.0177545$, the other at negative pressure $p_{c2} \approx -0.0084877$. (b) The Gibbs free energy shows one physical (with positive pressure) critical point and the corresponding first-order SBH/LBH phase transition, occurring for $t \in (t_r, t_{c1})$ and $p \in (p_r, p_{c1})$. (c) There is a *reentrant phase transition* (RPT) corresponding to the *zeroth-order phase transition* at $t = t_0$ followed by a first-order VdW-like SBH/LBH phase transition at the intersection $t = t_1$ with the swallowtail structure.

3.1. $P - V$ Criticality for $d = 4$. For $d = 4$, (26) will reduce to the cubic equation:

$$F(y) = y^3 - \frac{3y}{4} - \frac{\pi w_2}{2} = 0 \quad (27)$$

with $y = (x_c^4 + 2)^{-1/2}$.

Depending on different values of w_2 , (27) admits one or more positive real roots for x , which can be also reflected by

$$\frac{\partial F(y)}{\partial y} = 3y^2 - \frac{3}{4}. \quad (28)$$

When $|w_2| \leq 1/2\pi$, three real roots occur, which are given by

$$y_i = \cos\left(\frac{1}{3} \arccos(2\pi w_2) - \frac{2\pi i}{3}\right), \quad i = 0, 1, 2. \quad (29)$$

Moreover, in order that $x_c = (1/y^2 - 2)^{1/4}$ be positive, we require an additional constraint $|y| \leq 1/\sqrt{2}$. Then, we have $y_0 > 0$ in case of $-1/2\pi \leq w_2 \leq -1/\sqrt{8\pi}$, and $y_1 > 0$ in the region of $-1/2\pi \leq w_2 \leq 0$, while the solution y_2 is always negative.

Now by inserting solutions of y_0 and y_1 into (24) and (25), we analyze the critical behaviors. Notice that analytic methods cannot be applied in our analysis because of the complexity of the Gibbs free energy and equation of state, we resort to graphical and numerical methods.

- (1) $w_2 \in (-1/\sqrt{8\pi}, 0)$. As shown in Figure 1, the $p - x$ diagram displays that the dashed curve represents critical isotherm at $t = t_c$ and the dotted and solid curves correspond to $t > t_c$ and $t < t_c$, respectively. In the $g - t$ diagram, the solid curve represents $p < p_c$, the dotted curve correspond to $p > p_c$, and the dashed curve is for $p = p_c$. We observe standard swallowtail behavior. Moreover, the $p - t$ diagram shows the coexistence line of the first-order phase transition terminating at a critical point. These plots are analogous to typical behavior of the liquid-gas phase transition of the Van der Waals fluid.
- (2) $w_2 \in (-0.132795, -1/\sqrt{8\pi})$, there only exist one physical (with positive pressure) critical point and the corresponding VdW-like SBH/LBH phase transition, which occurs for the pressures $p \in (p_r, p_{c1})$ and temperatures $t \in (t_r, t_{c1})$; see Figure 2. For the $p - t$ diagram in Figure 3, three separate phases of black holes emerge in the region of $p_r < p \leq p_z < p_{c1}$: intermediate black holes (IBH) (on the left), small (on the middle), and large (on the right), where small and large black holes are separated by the SBH/LBH phase transition, but the intermediate and small are separated by a finite jump in g , which is so-called *zeroth-order phase transition* [61].

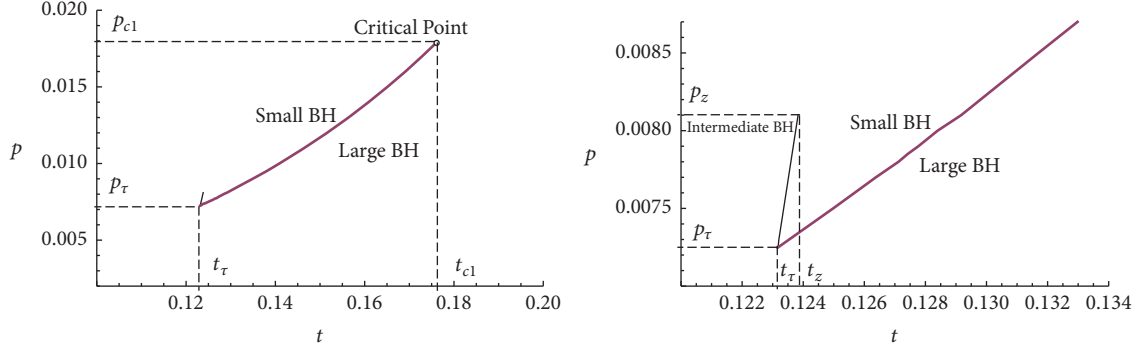


FIGURE 3: The coexistence line of the VdW-like SBH/LBH phase transition is depicted by a thick solid line. It initiates from the critical point (p_{c1}, t_{c1}) and terminates at (p_r, t_r) . The solid line describes the “coexistence line” of small and intermediate black holes, separated by a finite gap in g , indicating the zeroth-order phase transition. It commences from (t_z, p_z) and terminates at (p_r, t_r) .

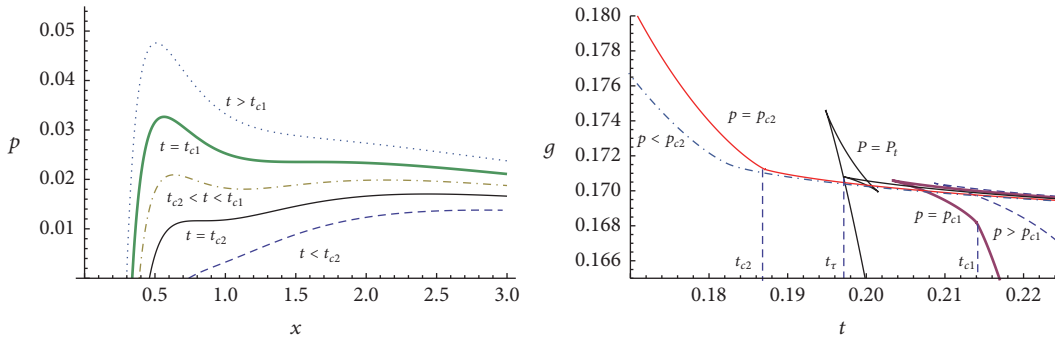


FIGURE 4: Born-Infeld AdS black holes for $d = 4$ and $w_2 = -0.14$. There are two critical points at positive pressure.

For $p < p_r$ only one phase of large black holes exists. When taking $w_2 = -0.124$, we obtain

$$\begin{aligned} (t_r, t_z, t_{c1}) &\approx (0.12316, 0.123825, 0.175593), \\ (p_r, p_z, p_{c1}) &\approx (0.0072472, 0.008104, 0.0177545). \end{aligned} \quad (30)$$

(3) $w_2 \in (-1/2\pi, -0.132795)$, there exist two critical points with positive pressure, and the similar RPT also occurs. As shown in Figure 4, we obtain

$$\begin{aligned} (t_{c2}, t_r, t_z, t_{c1}) &\approx (0.187113, 0.197121, 0.198064, 0.2139999), \\ (p_{c2}, p_r, p_z, p_{c1}) &\approx (0.0116313, 0.018695, 0.0194174, 0.0235228), \end{aligned} \quad (31)$$

when taking $w_2 = -0.14$.

With regard to $|w_2| > 1/2\pi$, the solution of (27) is given by

$$y_3 = -\cosh\left(\frac{1}{3} \arccos(-2\pi w_2)\right), \quad (32)$$

which violates the constraint condition $|y| \leq 1/\sqrt{2}$.

All in all, when the parameter w_2 satisfies $-1/2\pi < w_2 < 0$, the Van der Waals-like SBH/LBH phase transition appears. In addition, the interesting RPT happens in case of $-1/2\pi < w_2 < -1/\sqrt{8\pi}$.

3.2. $P - V$ Criticality for $d = 5$. Then (26) can be rewritten as

$$\begin{aligned} F(x_c) &= \frac{1}{x_c} (x_c^6 + 6)^{-3/2} - \frac{5}{18x_c} (x_c^6 + 6)^{-1/2} \\ &\quad - \frac{2\pi}{27} \left(w_2 + \frac{3w_3}{x_c} \right) = 0. \end{aligned} \quad (33)$$

Evidently, it is not possible to obtain analytic solution of above equation. To see more closely the phase transition of the Born-Infeld AdS black hole, here we analyze the asymptotic property of the function $F(x_c)$. In addition, the function $dF(x_c)/dx_c$ reads

$$\begin{aligned} \frac{dF(x_c)}{dx_c} &= 72 \left(1 - \frac{5x_c^6}{24} \right)^2 + \frac{135x_c^{12}}{8} \\ &\quad + 4\pi w_3 (x_c^6 + 6)^{5/2}. \end{aligned} \quad (34)$$

Evidently, (33) has more than one real roots. For different values of w_2 and w_3 , we will investigate the phase structure and criticality in the extended phase space.

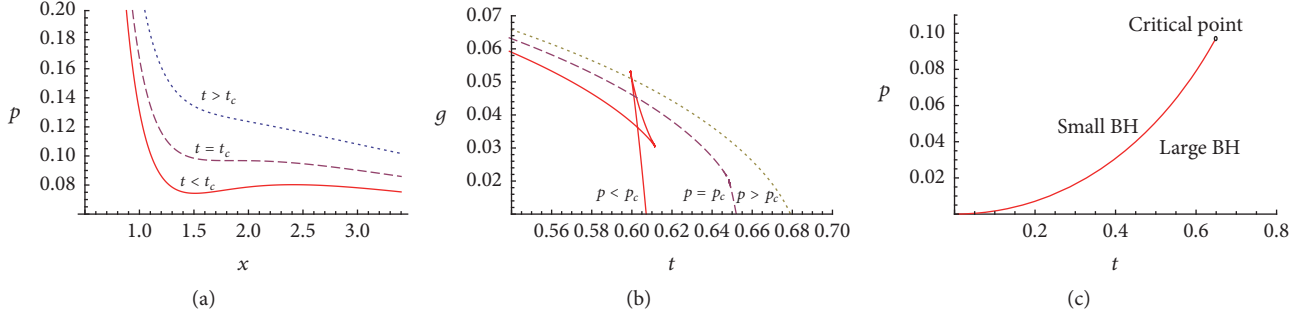


FIGURE 5: Born-Infeld AdS black holes for $d = 5$, $w_2 = -0.25$, and $w_3 = 0.1$. (a) The $p - x$ diagram. The dashed curve represents critical isotherm at $t = t_c$. The dotted and solid curves correspond to $t > t_c$ and $t < t_c$, respectively. (b) The $g - t$ diagram. The solid curve represents $p < p_c$, the dotted curve correspond to $p > p_c$, and the dashed curve is for $p = p_c$. We observe standard swallowtail behavior. (c) The $p - t$ diagram, showing the coexistence line of SBH/LBH phase transition terminating at a critical point. These plots are analogous to typical behavior of the liquid-gas phase transition of the VdW fluid.

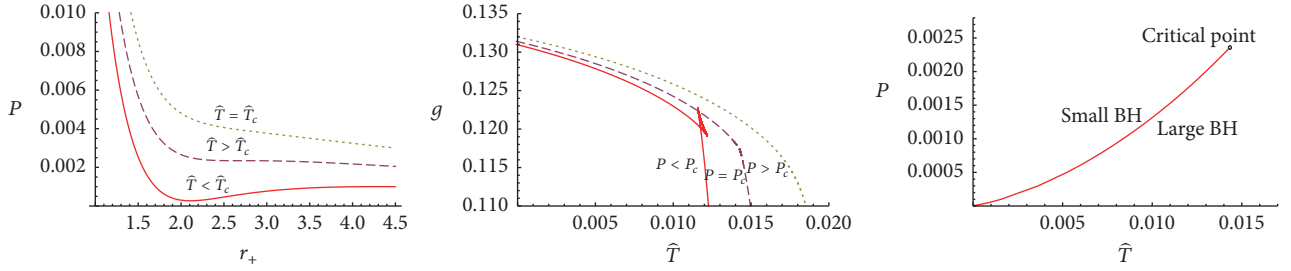


FIGURE 6: Born-Infeld AdS black holes for $d = 5$, $w_2 = 0.001$, and $w_3 = -0.02$.

3.2.1. $w_2 > 0$ and $w_3 > 0$. When $x_c \rightarrow +\infty$, $F(x_c)$ equals $-2\pi w_2/27$. Near the origin $x = 0$, we have

$$F(x_c) = -\frac{(\sqrt{6} + 12\pi w_3)}{54x_c}; \quad (35)$$

namely, $F(x_c)$ approaches $-\infty$ on account of $w_3 > -1/2\sqrt{6}\pi$. Moreover, function $dF(x_c)/dx_c$ is always positive, so there is no real solution for x_+ . Therefore, there is no critical point.

3.2.2. $w_2 < 0$ and $w_3 > 0$. Here we adopt similar discussions above. The function $F(x_c) = -2\pi w_2/27 > 0$ in case of $w_2 < 0$. However, $F(x_c)$ approaches $-\infty$ near the origin $x = 0$. Evidently, there is only one positive root of (33) on account of $dF(x_c)/dx_c > 0$. Then, a critical point occurs. In Figure 5, we display VdW-like small/large black hole phase transition in the system.

3.2.3. $w_2 > 0$ and $w_3 < 0$. In this case, it is hard work to discuss the asymptotic property of (33). Here we resort to graphical and numerical methods and also find the existence of VdW-like small/large black hole phase transition in the system; see Figure 6.

3.2.4. $w_2 < 0$ and $w_3 < 0$. In [58], Hendi et al. pointed out that the VdW-like SBH/LBH phase transition occurs when $w_2 < 0$ and $w_3 < 0$. Actually, there are some other interesting phase transitions.

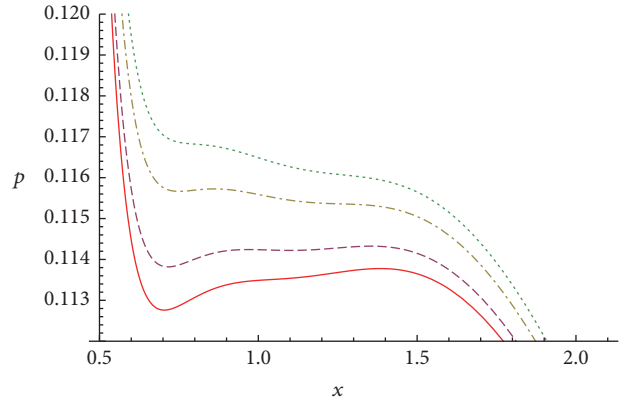


FIGURE 7: Behavior of p as a function of x for $t = 0.495, 0.497, 0.4988, \text{ and } 0.5$ from bottom to top. For small pressure, one can see that there are stable SBH and LBH branches, which implies the occurrence of VdW-like phase transition. With the increasing of the temperature, there appears a new stable IBH branch. Further increasing the pressure, this branch disappears.

For the case of $w_2 = -0.084825$ and $w_3 = -0.045$, the pressure p has three critical points, that is, $(p_{c1}, p_{c2}, p_{c3}) = (0.113983, 0.115076, 0.116354)$ and $(t_{c1}, t_{c2}, t_{c3}) = (0.496659, 0.498326, 0.499503)$. We plot the pressure p as a function of x for $t = 0.48, 0.497, 0.4988, \text{ and } 0.51$ (from bottom to top) in Figure 7. When $p < p_{c1}$, there exists a characteristic swallowtail behavior in the $g - t$ diagram, and

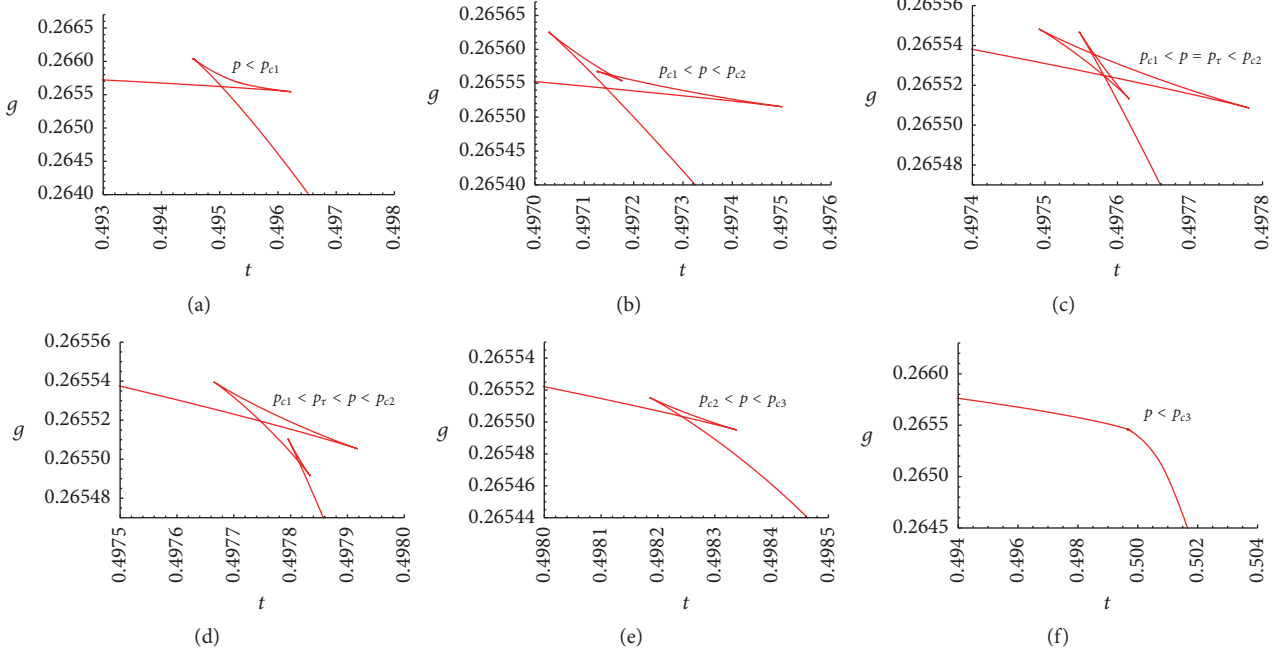


FIGURE 8: Gibbs free energy of five-dimensional AdS black holes for $\omega_2 = -0.084825$ and $\omega_3 = -0.045$.

a VdW-like SBH/LBH phase transition will occur. Further increasing p such that $p_{c1} < p < p_{c2}$, there appears a new stable IBH branch. For the corresponding Gibbs free energy in Figure 8, three black hole phases (i.e., small, large, and intermediate black holes) coexist together. Therefore, we observe a triple point characterized by $(p_r, t_r) = (0.01960, 0.11226)$. Slightly above this pressure, a standard SBH/IBH/LBH phase transition will appear in the system with the increase of t . And such phase transition disappears when p_{c2} is approached.

Further increasing p , the stable IBH branch vanishes in case of $p_{c2} < p < p_{c3}$. And only one stable branch survives when $p > p_{c3}$. In the ranges $p < p_{c1}$ and $p_{c2} < p < p_{c3}$, one characteristic swallowtail behavior is displayed in Figure 8. When $p > p_{c3}$, there is no such behavior.

4. Conclusions and Discussions

In the extended phase space, we have studied the phase transition and critical behavior of topological AdS black holes in the four- and five-dimensional Born-Infeld-massive gravity. For $d = 4$, we found that when the horizon topology is spherical ($k = 1$), Ricci flat ($k = 0$), or hyperbolic ($k = -1$), there always exist the Van der Waals-like SBH/LBH phase transitions when the coupling coefficients of massive potential are located in the region $-1/2\pi < \omega_2 < 0$. In addition, a monotonic lowering of the temperature yields a large-small-large black hole transition in the region $-1/2\pi < \omega_2 < -1/\sqrt{8}\pi$, where we refer to the former large state as an intermediate black hole (IBH), which is reminiscent of RPTs. Moreover, this process is also accompanied by a discontinuity in the global minimum of the Gibbs free energy, referred to as a zeroth-order phase transition.

In some range of the parameters, there are three critical points for five-dimensional Born-Infeld AdS black hole. In such range, the Gibbs free energy displays the behavior of two swallowtails. This phenomenon has been never recovered before.

Recent observations of gravitational waves have put an upper bound of 1.2×10^{-22} eV/ c^2 on the graviton's mass [62]. We can find in 4-dimensional case; the interesting RPTs can always appear as long as the parameters q and b take the suitable values with the constant k taking the values ± 1 . When the constant $k = 0$, the role of the graviton's mass is highlighted, the parameters q and b cannot take an acceptable range (means in the framework of the Born-Infeld theory) to make the parameter $\omega_2 \in (-1/2\pi, -1/\sqrt{8}\pi)$, which means only the VdW-like phase transitions might happen. In the 5-dimensional case, when the constant $k = 1$, this interesting phenomenon could appear as long as the parameters q and b take the suitable values. There are no three critical points when the constant k takes -1 or 0 , because the parameter ω_2 is always positive.

Reference [57] shows that the RPTs only exist in the 4-dimensional Born-Infeld AdS black hole with a spherical horizon and also gives the proof that there is no reentrant phase transition in the system of higher (≥ 5) dimensional Born-Infeld AdS black hole. Reference [48] demonstrated that there only exists the Van der Waals-like phase transition in the 4-dimensional AdS black hole in massive gravity with Maxwell's electromagnetic field theory. Our results reveal that the nonlinear electromagnetic field plays an important role in the phase transition of the 4-dimensional AdS black hole, and the massive gravity could bring richer phase structures and critical behavior (triple critical points) than that of the Born-Infeld term in the 5-dimensional AdS black hole.

Recently, the charged black hole [63], Born-Infeld black hole [64], and black hole in the Maxwell and Yang-Mills fields [65] have been constructed in Gauss-Bonnet-massive gravity. Only Van der Waals-like first-order SBH/LBH phase transition exists in these models. In addition, the RPT and triple points also occur in the higher-dimensional rotating AdS black holes [66, 67] and higher-dimensional Gauss-Bonnet AdS black hole [68–70]. It would be interesting to extend our discussion to these black holes in Gauss-Bonnet and 3rd-order Lovelock-massive gravity.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

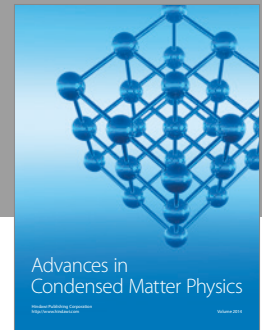
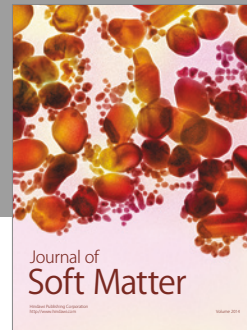
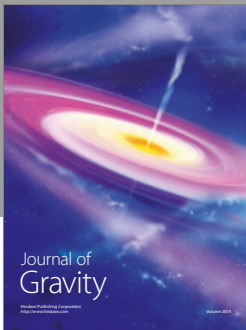
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