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Research Article

On the L^1 Stability to a Generalized Degasperis-Procesi Equation

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A nonlinear generalized Degasperis-Procesi equation is investigated. Assuming that the strong solution of the equation is bounded in the sense of $L^\infty(R)$-norm and the initial data belong to the space $L^1(R) \cap L^2(R)$, we prove that the solutions are stable in the space $L^1(R)$.

1. Introduction

Coclite and Karlsen [1] investigated the following generalized Degasperis-Procesi equation:

$$u_t - u_{txx} + 4f'(u)u_x = f'''(u)u_x^3 + 3f''(u)u_xu_{xx} + f'(u)u_{xxx}. \tag{1}$$

When $f(u) \in C^3$ and satisfies

$$\left|f'(u)\right| \le c|u|, \qquad \left|f(u)\right| \le c|u|^2, \tag{2}$$

or

$$\left|f'(u)\right| \le c, \qquad \left|f(u)\right| \le c|u|, \tag{3}$$

where c is a positive constant, the existence and L^1 stability of entropy weak solutions belonging to the class $L^1(R) \bigcap BV(R)$ are established for (1) in paper [1].

The objective of this paper is to study the generalized Degasperis-Procesi equation

$$u_t - u_{txx} + mg'(u)u_x = g'''(u)u_x^3 + 3g''(u)u_xu_{xx} + g'(u)u_{xxx}, \tag{4}$$

where m is a positive constant, $g(u)$ is a polynomial of order n ($n \ge 2$), and $g(0) = 0$. When $m = 4$ and $g(u) = u^2/2$, (4) reduces to the classical Degasperis-Procesi model [2–10]. Assuming that there exists a strong solution to (4), which is bounded in its existence time interval $[0, T)$, and the initial value of (4) lies in $L^1(R) \cap L^2(R)$, we will prove that the strong solutions of the equation are stable in the space $L^1(R)$ (see Theorem 8 in Section 3). From the authors' knowledge, this is a new result for (4).

This paper is organized as follows. Section 2 gives several lemmas. The main result and its proof are presented in Section 3.

2. Several Lemmas

We consider the Cauchy problem of (4) in the following form:

$$\begin{aligned} u_t - u_{txx} + mg'(u)u_x &= g'''(u)u_x^3 + 3g''(u)u_xu_{xx} + g'(u)u_{xxx}, \\ u(0,x) &= u_0(x). \end{aligned} \tag{5}$$

Applying the operator $(1 - \partial_x^2)^{-1}$ to the first equation of problem (5), we obtain

$$\begin{aligned} u_t + g'(u)u_x + \partial_x P_u &= 0, \\ u(0,x) &= u_0(x), \end{aligned} \tag{6}$$

where $P_u = ((m-1)/2)\int_R e^{-|x-y|} g(u(t,y))dy$. Letting $\Psi_u(u) = \partial_x P_u$, we get

$$u_t + g'(u)u_x + \Psi_u(u) = 0,$$
$$u(0,x) = u_0(x). \tag{7}$$

Lemma 1. *The solution of problem* (5) *with* $m > 0$ *satisfies*

$$\int_R y_1 y\,dx = \int_R \frac{1+\xi^2}{m+\xi^2}|\hat{u}(\xi)|^2 d\xi = \int_R \frac{1+\xi^2}{m+\xi^2}|\widehat{u_0}(\xi)|^2 d\xi, \tag{8}$$

where $y_1 = u - \partial_{xx}^2 u$ *and* $y = (m-\partial_{xx}^2)^{-1}u$. *Moreover, there exist two constants* $c_1 > 0$ *and* $c_2 > 0$ *depending only on m such that*

$$c_1\|u_0\|_{L^2(R)} \le c_1\|u\|_{L^2(R)} \le c_2\|u_0\|_{L^2(R)}. \tag{9}$$

Proof. Letting $y_1 = u - \partial_{xx}^2 u$ and $y = (m-\partial_{xx}^2)^{-1}u$ and using (4), we obtain $u = my - y_{xx}$ and

$$\begin{aligned}\frac{d}{dt}\int_R y_1 y\,dx &= \int_R \frac{\partial y_1}{\partial t} y\,dx + \int_R y_1\frac{\partial y}{\partial t}dx = 2\int_R \frac{\partial y_1}{\partial t} y\,dx\\ &= 2\int_R \left[-mg'(u)u_x + g'''(u)u_x^3 \right.\\ &\qquad \left. + 3g''(u)u_x u_{xx} + g'(u)u_{xxx}\right] y\,dx\\ &= 2\int_R \left[-m\partial_x[g(u)] + [g(u)]_{xxx}\right] y\,dx\\ &= \int_R [mg(u)]\,y_x - g(u)\,y_{xxx}dx\\ &= \int_R [mg(u)]\,y_x - g(u)(my_x - u_x)\,dx\\ &= \int_R g(u)u_x dx,\\ &= 0.\end{aligned} \tag{10}$$

Using the Parseval identity and (10), we obtain (8) and (9). ◻

Remark 2. When $m \le 0$, from (8), we cannot obtain inequality (9).

Lemma 3. *If* $u_0 \in L^2(R)$ *and* $\|u\|_{L^\infty(R)} < M$, *it holds that*

$$\|P_u\|_{L^\infty(R_+\times R)}, \qquad \|\Psi_u(u)\|_{L^\infty(R_+\times R)} < c_0 M^{n-2}, \tag{11}$$

where c_0 *is a constant independent of t and* $n \ge 2$.

Proof. Using the assumption $u_0 \in L^2(R)$ and Lemma 1, we have $u \in L^2(R)$. Using (7), we get

$$P_u(t,x) = \frac{m-1}{2}\int_R e^{-|x-y|} g(u)\,dy,$$
$$\Psi_u(u(t,x)) = \frac{m-1}{2}\int_R e^{-|x-y|}\,\text{sign}(y-x)\,g(u)\,dy. \tag{12}$$

Since the function $g(u)$ is a polynomial of order n and $\|u\|_{L^\infty(R)} < M$, combining Lemma 1 derives that (11) holds. ◻

Lemma 4. *Assume that* $\|u\|_{L^\infty(R)} < M$ *and* $\|v\|_{L^\infty(R)} < M$ *are two solutions of* (4) *with initial data* $u_0, v_0 \in L^2(R)$, *respectively. Then, for any* $\phi(t,x) \in C_0^\infty([0,\infty)\times R)$, *it holds that*

$$\int_{-\infty}^{\infty}\left|\Psi_u(u) - \Psi_v(v)\right|\left|\phi(t,x)\right|dx \le c_0\int_{-\infty}^{\infty}|u-v|\,dx, \tag{13}$$

where $c_0 > 0$ *depends on m,n, M,* ϕ, $\|u_0\|_{L^2(R)}$, *and* $\|v_0\|_{L^2(R)}$.

Proof. We have

$$\begin{aligned}&\int_{-\infty}^{\infty}\left|\Psi_u(u) - \Psi_v(v)\right|\left|\phi(t,x)\right|dx\\ &\le (m-1)\int_{-\infty}^{\infty}\left|\partial_x\Lambda^{-2}(g(u)-g(v))\right|\left|\phi(t,x)\right|dx\\ &= \frac{|m-1|}{2}\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-|x-y|}\left|\text{sign}(x-y)\right|\right| g(u)\\ &\quad - g(v)\,|dy\,|\phi(t,x)|\,dx|\\ &\le c_0\int_{-\infty}^{\infty}|u-v|\,M^{n-2}dy\left|\int_{-\infty}^{\infty}|\phi(t,x)|\,dx\right|\\ &\le c_0\int_{-\infty}^{\infty}|u-v|\,dy,\end{aligned} \tag{14}$$

which completes the proof. ◻

We define $\delta(\sigma)$ as a function which is infinitely differentiable on $(-\infty,+\infty)$ such that $\delta(\sigma) \ge 0$, $\delta(\sigma) = 0$ for $|\sigma| \ge 1$, and $\int_{-\infty}^{\infty}\delta(\sigma)d\sigma = 1$. For any number $\varepsilon > 0$, we let $\delta_\varepsilon(\sigma) = \delta(\varepsilon^{-1}\sigma)/\varepsilon$. Then we know that $\delta_\varepsilon(\sigma)$ is a function in $C^\infty(-\infty,\infty)$ and

$$\delta_\varepsilon(\sigma) \ge 0, \quad \delta_\varepsilon(\sigma) = 0 \quad \text{if } |\sigma| \ge \varepsilon,$$
$$|\delta_\varepsilon(\sigma)| \le \frac{c}{\varepsilon}, \qquad \int_{-\infty}^{\infty}\delta_\varepsilon(\sigma) = 1. \tag{15}$$

Assume that the function $v(x)$ is locally integrable on $(-\infty,\infty)$. We define an approximation function of v as

$$v^\varepsilon(x) = \frac{1}{\varepsilon}\int_{-\infty}^{\infty}\delta\left(\frac{x-y}{\varepsilon}\right)v(y)\,dy, \quad \varepsilon > 0. \tag{16}$$

We get $v^\varepsilon(x) \to v(x)$ as $\varepsilon \to 0$ almost everywhere.

We state the concept of a characteristic cone. For any $R_0 > 0$, we define $N > \max_{t\in[0,T]}\|u\|_{L^\infty} < \infty$. Let $\mho$ represent the cone $\{(t,x) : |x| < R_0 - Nt, 0 \le t \le T_0 = \min(T, R_0N^{-1})\}$. We let S_τ represent the cross section of the cone $\mho$ by the plane $t = \tau, \tau \in [0,T_0]$. Set $K_{r+2\rho} = \{x : |x| \le r + 2\rho\}$, where $r > 0, \rho > 0$, and $\pi_T = [0,T]\times R$ for an arbitrary $T > 0$. The space of all infinitely differentiable functions $\phi(t,x)$ with compact support in $[0,T]\times R$ is denoted by $C_0^\infty(\pi_T)$.

Lemma 5 (see [11]). *Let the function $v(t,x)$ be bounded and measurable in cylinder $\Omega_T = [0,T]\times K_r$. If $\rho \in (0, \min[r,T])$ and $\varepsilon \in (0,\rho)$, then the function*

$$V\varepsilon = \frac{1}{\varepsilon^2} \times \iiiint_{\substack{|(t-\tau)/2|\le\varepsilon,\\ \rho\le(t+\tau)/2\le T-\rho,\\ |(x-y)/2|\le\varepsilon,\\ |(x+y)/2|\le r-\rho}} \left|v(t,x)-v(\tau,y)\right| dx\,dt\,dy\,d\tau \tag{17}$$

satisfies $\lim_{\varepsilon\to 0} V_\varepsilon = 0$.

Lemma 6 (see [11]). *Let $|\partial F(u)/\partial u|$ be bounded. Then the function*

$$H(u,v) = \operatorname{sign}(u-v)(F(u)-F(v)) \tag{18}$$

satisfies the Lipschitz condition in u and v, respectively.

Using the methods presented in [11], we have the following result.

Lemma 7. *If u is a strong solution of problem* (6), $\phi(t,x) \in C_0^\infty(\pi_T)$, *and* $\phi(0,x)=0$, *it holds that*

$$\iint_{\pi_T} \{|u-k|\phi_t + \operatorname{sign}(u-k)[g(u)-g(k)]\phi_x - \operatorname{sign}(u-k)\Psi_u(t,x)\phi\}\,dx\,dt = 0, \tag{19}$$

where k is an arbitrary constant.

Proof. Let $\Phi(u)$ be a twice differential function on the line $-\infty < u < \infty$. We multiply the first equation of problem (6) by the function $\Phi'(u)\phi(t,x)$, where $\phi(t,x) \in C_0^\infty(\pi_T)$. Integrating over π_T and transferring the derivatives with respect to t and x to the test function ϕ, for any constant k, we obtain

$$\iint_{\pi_T} \left\{\Phi(u)\phi_t + \left[\int_k^u \Phi'(z)g'(z)\,dz\right]\phi_x - \Phi'(u)\Psi_u(t,x)\phi\right\} dx\,dt = 0, \tag{20}$$

in which we have used $\int_{-\infty}^{\infty}[\int_k^u \Phi'(z)g'(z)dz]\phi_x dx = -\int_{-\infty}^{\infty}[\Phi'(u)g'(u)u_x]\phi(t,x)dx$.

Integration by parts yields that

$$\int_{-\infty}^{\infty}\left[\int_k^u \Phi'(z)g'(z)\,dz\right]\phi_x dx = \int_{-\infty}^{\infty}\left[\Phi'(u)[g(u)-g(k)] - \int_k^u [g(z)-g(k)]\Phi''(z)\,dz\right]\phi_x\,dx. \tag{21}$$

Let $\Phi^\varepsilon(u)$ be an approximation of the function $|u-k|$ and set $\Phi(u) = \Phi^\varepsilon(u)$. Using the properties of the sign$(u-k)$, from (20) and (21), and sending $\varepsilon \to 0$, we have

$$\iint_{\pi_T} \{|u-k|\phi_t + \operatorname{sign}(u-k)[g(u)-g(k)]\phi_x - \operatorname{sign}(u-k)\Psi_u(t,x)\phi\}\,dx\,dt = 0, \tag{22}$$

which completes the proof. □

3. Main Result

Generally speaking, we cannot get the boundedness of strong solutions for problem (6). This is why we assume that the strong solutions of problem (6) possess boundedness in order to establish the L^1 stability for the problem. Now we state our main result as follows.

Theorem 8. *Assume that there exist strong solutions u and v for problem* (5) *or* (6). *Let T be the maximum existence time for the solutions. If* $\|u\|_{L^\infty(R)} < M$, $\|v\|_{L^\infty(R)} < M$, *and the initial data* $u_0, v_0 \in L^1(R)\cap L^2(R)$, *it holds that*

$$\|u(t,\cdot)-v(t,\cdot)\|_{L^1(R)} \le ce^{ct}\int_{-\infty}^{\infty}|u_0(x)-v_0(x)|\,dx, \quad t\in[0,T], \tag{23}$$

where c depends on $\|u_0\|_{L^2(R)}$, $\|v_0\|_{L^2(R)}$, *M, T, and the coefficients of polynomial g(u).*

Proof. For $\phi(t,x) \in C_0^\infty(\pi_T)$, we assume that $\phi(t,x) = 0$ outside the cylinder

$$\uplus = \{(t,x)\} = [\rho, T-2\rho]\times K_{r-2\rho}, \quad 0 < 2\rho \le \min(T,r). \tag{24}$$

Let

$$\psi = \phi\left(\frac{t+\tau}{2}, \frac{x+y}{2}\right)\delta_\varepsilon\left(\frac{t-\tau}{2}\right)\delta_\varepsilon\left(\frac{x-y}{2}\right) = \phi(\cdots)\lambda_\varepsilon(*), \tag{25}$$

where $(\cdots) = ((t+\tau)/2, (x+y)/2)$ and $(*) = ((t-\tau)/2, (x-y)/2)$. The function $\delta_\varepsilon(\sigma)$ is defined in (15). Note that

$$\psi_t + \psi_\tau = \phi_t(\cdots)\lambda_\varepsilon(*), \qquad \psi_x + \psi_y = \phi_x(\cdots)\lambda_\varepsilon(*). \tag{26}$$

Following Kruzkov's device of doubling the variables presented in [11], from Lemma 7, and choosing $k = v(\tau,y)$, we have

$$\iiiint_{\pi_T\times\pi_T} \{|u(t,x)-v(\tau,y)|\psi_t + \operatorname{sign}(u(t,x)-v(\tau,y)) \times (g(u(t,x)) - g(v(\tau,y)))\psi_x + \operatorname{sign}(u(t,x)-v(\tau,y)) \times\Psi_u(t,x)\psi\}\,dx\,dt\,dy\,d\tau = 0. \tag{27}$$

Similarly, it has

$$\iiiint_{\pi_T\times\pi_T} \{|v(\tau,y)-u(t,x)|\psi_\tau + \operatorname{sign}(v(\tau,y)-u(t,x)) \times (g(u(t,x))-g(v(\tau,y)))\psi_y + \operatorname{sign}(v(\tau,y)-u(t,x)) \times \Psi_v(\tau,y)\psi\}\,dx\,dt\,dy\,d\tau = 0. \quad (28)$$

It follows from (27) and (28) that

$$0 \le \iiiint_{\pi_T\times\pi_T} \{|u(t,x)-v(\tau,y)|(\psi_t+\psi_\tau) + \operatorname{sign}(u(t,x)-v(\tau,y)) \times (g(u(t,x))-g(v(\tau,y))) \times (\psi_x+\psi_y)\}\,dx\,dt\,dy\,d\tau + \left|\iiiint_{\pi_T\times\pi_T} \operatorname{sign}(u(t,x)-v(t,x)) \times (\Psi_u(t,x)-\Psi_v(\tau,y)) \times\psi\,dx\,dt\,dy\,d\tau\right| = B_1+B_2+\left|\iiiint_{\pi_T\times\pi_T} B_3\,dx\,dt\,dy\,d\tau\right|. \quad (29)$$

We will prove the following inequality:

$$0 \le \iint_{\pi_T} \{|u(t,x)-v(t,x)|\phi_t + \operatorname{sign}(u(t,x)-v(t,x)) \times (g(u(t,x))-g(v(t,x)))\phi_x\}\,dx\,dt + \left|\iint_{\pi_T} \operatorname{sign}(u(t,x)-v(t,x)) \times [\Psi_u(t,x)-\Psi_v(t,x)]\phi\,dx\,dt\right|. \quad (30)$$

We observe that the first two terms of inequality (29) can be represented in the form

$$J_\varepsilon = J(t,x,\tau,y,u(t,x),v(\tau,y))\lambda_\varepsilon(*). \quad (31)$$

From Lemma 6, we know that J_ε satisfies the Lipschitz condition in u and v, respectively. By the choice of ϕ, we have $J_\varepsilon = 0$ outside the region

$$\{(t,x;\tau,y)\} = \left\{\rho \le \frac{t+\tau}{2} \le T-2\rho, \frac{|t-\tau|}{2} \le \varepsilon, \frac{|x+y|}{2} \le r-2\rho, \frac{|x-y|}{2} \le \varepsilon\right\}, \quad (32)$$

$$\iiiint_{\pi_T\times\pi_T} J_\varepsilon\,dx\,dt\,dy\,d\tau = \iiiint_{\pi_T\times\pi_T} [J(t,x,\tau,y,u(t,x),v(\tau,y)) - J(t,x,t,x,u(t,x),v(t,x))] \times\lambda_\varepsilon(*)\,dx\,dt\,dy\,d\tau + \iiiint_{\pi_T\times\pi_T} J(t,x,t,x,u(t,x),v(t,x)) \times\lambda_\varepsilon(*)\,dx\,dt\,dy\,d\tau = A_{11}(\varepsilon)+A_{12}. \quad (33)$$

Considering the estimate $|\lambda(*)| \le c/\varepsilon^2$ and the expression of function $A_{11}(\varepsilon)$, we have

$$|A_{11}(\varepsilon)| \le c\left[\varepsilon+\frac{1}{\varepsilon^2} \times \iiiint_{\substack{|(t-\tau)/2|\le\varepsilon,\\ \rho\le(t+\tau)/2\le T-\rho,\\ |(x-y)/2|\le\varepsilon,\\ |(x+y)/2|\le r-\rho}} |v(t,x)-v(\tau,y)|\,dx\,dt\,dy\,d\tau\right], \quad (34)$$

where the constant c does not depend on ε. Using Lemma 5, we obtain $A_{11}(\varepsilon) \to 0$ as $\varepsilon \to 0$. The integral A_{12} does not depend on ε. In fact, substituting $t=\alpha$, $(t-\tau)/2=\beta$, $x=\eta$, and $(x-y)/2=\xi$ and noting that

$$\int_{-\varepsilon}^{\varepsilon}\int_{-\infty}^{\infty} \lambda_\varepsilon(\beta,\xi)\,d\xi\,d\beta = 1, \quad (35)$$

we have

$$A_{12} = 2^2\iint_{\pi_T} J_\varepsilon(\alpha,\eta,\alpha,\eta,u(\alpha,\eta),v(\alpha,\eta)) \times\left\{\int_{-\varepsilon}^{\varepsilon}\int_{-\infty}^{\infty} \lambda_\varepsilon(\beta,\xi)\,d\xi\,d\beta\right\}d\eta\,d\alpha = 4\iint_{\pi_T} J(t,x,t,x,u(t,x),v(t,x))\,dx\,dt. \quad (36)$$

Hence

$$\lim_{\varepsilon\to 0}\iiiint_{\pi_T\times\pi_T} J_\varepsilon\,dx\,dt\,dy\,d\tau = 4\iint_{\pi_T} J(t,x,t,x,u(t,x),v(t,x))\,dx\,dt. \quad (37)$$

Since

$$\begin{aligned}
B_3 &= \operatorname{sign}(u(t,x)-v(\tau,y)) \\
&\quad\times(\Psi_u(t,x)-\Psi_v(\tau,y))\,\phi\lambda_\varepsilon(*) \\
&= \overline{B_3}(t,x,\tau,y)\,\lambda_\varepsilon(*), \\
&\iiiint_{\pi_T\times\pi_T} B_3\,dx\,dt\,dy\,d\tau \\
&\quad= \iiiint_{\pi_T\times\pi_T}\left[\overline{B_3}(t,x,\tau,y)-\overline{B_3}(t,x,t,x)\right] \\
&\qquad\times\lambda_\varepsilon(*)\,dx\,dt\,dy\,d\tau \\
&\quad+\iiiint_{\pi_T\times\pi_T}\overline{B_3}(t,x,t,x)\,\lambda_\varepsilon(*)\,dx\,dt\,dy\,d\tau \\
&\quad= A_{21}(\varepsilon)+A_{22},
\end{aligned}\tag{38}$$

we obtain

$$\begin{aligned}
&\left|A_{21}(\varepsilon)\right| \\
&\le c\Bigg(\varepsilon+\frac{1}{\varepsilon^2} \\
&\quad\times\iiiint_{\substack{|(t-\tau)/2|\le\varepsilon,\\ \rho\le(t+\tau)/2\le T-\rho,\\ |(x-y)/2|\le\varepsilon,\\ |(x+y)/2|\le r-\rho}}\left|\Psi_v(t,x)-\Psi_v(\tau,y)\right|dxdtdyd\tau\Bigg).
\end{aligned}\tag{39}$$

Using Lemma 5, we have $A_{21}(\varepsilon)\to 0$ as $\varepsilon\to 0$. Using (35), we have

$$\begin{aligned}
A_{22} &= 2^2\iint_{\pi_T}\overline{I_3}(\alpha,\eta,\alpha,\eta,u(\alpha,\eta),v(\alpha,\eta)) \\
&\qquad\times\left\{\int_{-h}^{h}\lambda_\varepsilon(\beta,\xi)\,d\xi\,d\beta\right\}d\eta\,d\alpha \\
&= 4\iint_{\pi_T}\overline{I_3}(t,x,t,x,u(t,x),v(t,x))\,dx\,dt \\
&= 4\iint_{\pi_T}\operatorname{sign}(u(t,x)-v(t,x)) \\
&\qquad\times(\Psi_u(t,x)-\Psi_v(t,x))\,\phi(t,x)\,dx\,dt.
\end{aligned}\tag{40}$$

From (33),(37), (39), and (40), we prove that inequality (30) holds.

Set

$$\omega(t)=\int_{-\infty}^{\infty}|u(t,x)-v(t,x)|\,dx. \tag{41}$$

We define

$$\theta_\varepsilon=\int_{-\infty}^{\sigma}\delta_\varepsilon(\sigma)\,d\sigma,\quad\left(\theta'_\varepsilon(\sigma)=\delta_\varepsilon(\sigma)\ge 0\right) \tag{42}$$

and choose two numbers ρ and $\tau\in(0,T_0)$, $\rho<\tau$. In (30), we choose

$$\begin{gathered}
\phi=\left[\theta_\varepsilon(t-\rho)-\theta_\varepsilon(t-\tau)\right]\chi(t,x), \\
h<\min(\rho,T_0-\tau),
\end{gathered}\tag{43}$$

where

$$\chi(t,x)=\chi_h(t,x)=1-\theta_h(|x|+Nt-R+h),\quad h>0. \tag{44}$$

When h is sufficiently small, we note that function $\chi(t,x)=0$ outside the cone $\mho$ and $\phi(t,x)=0$ outside the set $\uplus$. For $(t,x)\in\mho$, we have the relations

$$0=\chi_t+N\left|\chi_x\right|\ge\chi_t+N\chi_x. \tag{45}$$

Applying (41)–(45) and (30), we have the inequality

$$\begin{aligned}
0 &\le \iint_{\pi_{T_0}}\{[\delta_\varepsilon(t-\rho)-\delta_\varepsilon(t-\tau)]\,\chi_h \\
&\qquad\times|u(t,x)-v(t,x)|\}\,dx\,dt \\
&\quad+\int_0^{T_0}\int_{-\infty}^{\infty}\left[\theta_\varepsilon(t-\rho)-\theta_\varepsilon(t-\tau)\right] \\
&\qquad\times\left|\left[J_u(t,x)-J_v(t,x)\right]\chi_h(t,x)\right|dx\,dt.
\end{aligned}\tag{46}$$

Using Lemma 4 and letting $h\to 0$ and $R_0\to\infty$, we obtain

$$\begin{aligned}
0 &\le \int_0^{T_0}\Bigg\{\left[\delta_\varepsilon(t-\rho)-\delta_\varepsilon(t-\tau)\right] \\
&\qquad\times\int_{-\infty}^{\infty}|u(t,x)-v(t,x)|\,dx\Bigg\}\,dt \\
&\quad+c\int_0^{T_0}\left[\theta_\varepsilon(t-\rho)-\theta_\varepsilon(t-\tau)\right] \\
&\qquad\times\int_{-\infty}^{\infty}|u(t,x)-v(t,x)|\,dx\,dt.
\end{aligned}\tag{47}$$

By the properties of the function $\delta_\varepsilon(\sigma)$ for $\varepsilon\le\min(\rho,T_0-\rho)$, we have

$$\begin{aligned}
&\left|\int_0^{T_0}\delta_\varepsilon(t-\rho)\,\omega(t)\,dt-\omega(\rho)\right| \\
&\quad=\left|\int_0^{T_0}\delta_\varepsilon(t-\rho)\left|\omega(t)-\omega(\rho)\right|dt\right| \\
&\quad\le c\frac{1}{\varepsilon}\int_{\rho-\varepsilon}^{\rho+\varepsilon}\left|\omega(t)-\omega(\rho)\right|dt\longrightarrow 0,\quad\text{as }\varepsilon\longrightarrow 0,
\end{aligned}\tag{48}$$

where c is independent of ε. Letting

$$F(\rho)=\int_0^{T_0}\theta_\varepsilon(t-\rho)\,\omega(t)\,dt=\int_0^{T_0}\int_{-\infty}^{t-\rho}\delta_\varepsilon(\sigma)\,d\sigma\,\omega(t)\,dt, \tag{49}$$

we get

$$F'(\rho) = -\int_0^{T_0} \delta_\varepsilon (t-\rho)\,\omega(t)\,dt \longrightarrow -\omega(\rho), \quad \text{as } \varepsilon \longrightarrow 0, \tag{50}$$

from which we obtain

$$F(\rho) \longrightarrow F(0) - \int_0^{\rho} \omega(\sigma)\,d\sigma, \quad \text{as } \varepsilon \longrightarrow 0. \tag{51}$$

Similarly, we have

$$F(\tau) \longrightarrow F(0) - \int_0^{\tau} \omega(\sigma)\,d\sigma, \quad \text{as } \varepsilon \longrightarrow 0. \tag{52}$$

It follows from (51) and (52) that

$$F(\rho) - F(\tau) \longrightarrow \int_{\rho}^{\tau} \omega(\sigma)\,d\sigma, \quad \text{as } \varepsilon \longrightarrow 0. \tag{53}$$

Sending $\rho \rightarrow 0$ and $\tau \rightarrow t$ and using

$$\begin{aligned} \left|u(\rho,x) - v(\rho,x)\right| &\le \left|u(\rho,x) - u_0(x)\right| \\ &\quad + \left|v(\rho,x) - v_0(x)\right| + \left|u_0(x) - v_0(x)\right|, \end{aligned} \tag{54}$$

from (47), (48), and (53)-(54), we have

$$\begin{aligned} \int_{-\infty}^{\infty} \left|u(t,x) - v(t,x)\right| dx &\le \int_{-\infty}^{\infty} \left|u_0 - v_0\right| dx \\ &\quad + c_0 \int_0^t \int_{-\infty}^{\infty} \left|u(t,x) - v(t,x)\right| dx\,dt. \end{aligned} \tag{55}$$

Applying the Gronwall inequality yields the desired result. □

Conflict of Interests

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