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Research Article

The Effect of Time Delay on Dynamical Behavior in an Ecoepidemiological Model

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A delayed predator-prey model with disease in the prey is investigated. The conditions for the local stability and the existence of Hopf bifurcation at the positive equilibrium of the system are derived. The effect of the two different time delays on the dynamical behavior has been given. Numerical simulations are performed to illustrate the theoretical analysis. Finally, the main conclusions are drawn.

1. Introduction

During the past decades, epidemiological models have received considerate attention since the seminal SIR model of Kermack and McKendrich [1]. Great attention has been paid to the dynamics properties of the predator-prey models which have significant biological background. Numerous excellent and interesting results have been reported. For example, Bhattacharyya and Mukhopadhyay [2] studied the spatial dynamics of nonlinear prey-predator models with prey migration and predator switching, Bhattacharyya and Mukhopadhyay [3] analyzed the local and global dynamical behavior of an ecoepidemiological model, Kar and Ghorai [4] made a detailed discussion on the local stability, global stability, influence of harvesting and bifurcation of a delayed predator-prey model with harvesting, Chakraborty et al. [5] focused on the bifurcation and control of a bio-economic model of a delayed prey-predator model. For more related research, one can see [6–19].

In 2005, Song et al. [20] investigated the stability and Hopf bifurcation of a delayed ecoepidemiological model as follows:

$$\begin{aligned}
\dot{S}(t) &= rS\left(1 - \frac{S+I}{K}\right) - \beta SI, \\
\dot{I}(t) &= \beta SI - cI - pIY(t - \overline{\tau}_1), \\
\dot{Y}(t) &= -dY + kpYI(t - \overline{\tau}_2),
\end{aligned} \tag{1.1}$$

where $S(t)$, $I(t)$, $Y(t)$ represent the susceptible prey, infected prey and predator population, respectively. K $(K > 0)$ can be interpreted as the prey carrying capacity with an intrinsic birth rate constant r $(r > 0)$. β $(\beta > 0)$ is called the transmission coefficient. The predator has a death rate constant d $(d > 0)$ and the predation coefficient p $(p > 0)$. The death rate of infected prey is positive constant c. The coefficient in conversing prey into predator is k $(0 < k \leq 1)$. $\overline{\tau}_1$ and $\overline{\tau}_2$ are the time required for mature of predator and the time required for the gestation of predator, respectively. The more detail biological meaning of the coefficients of system (1.1), one can see [20].

For the sake of simplicity, Song et al. [20] rescales time $t \to \beta kt$, then system (1.1) can be transformed into the following form:

$$\begin{aligned}
\dot{s}(t) &= as[1 - (s + i)] - si, \\
\dot{i}(t) &= -b_2 i + si - liy(t - \tau_1), \\
\dot{y}(t) &= -b_1 y + klyi(t - \tau_2),
\end{aligned} \tag{1.2}$$

where $s = S/K$, $i = I/K$, $y = Y/K$, $a = r/K\beta$, $b_2 = c/K\beta$, $b_1 = d/K\beta$, $l = p/\beta$, $\tau_1 = \beta K\overline{\tau}_1$, $\tau_2 = \beta K\overline{\tau}_2$.

We would like to point out that although Song et al. [20] investigated the local stability and Hopf bifurcation of system (1.2) under the assumption $\tau_1 + \tau_2 = \tau$ and obtained some good results, but they did not discuss what the different time delay τ_1 and τ_2 have effect on the stability and Hopf bifurcation behavior of system (1.2). Thus it is important for us to deal with the effect of time delay on the dynamics of system (1.2). There are some work which deal with this topic [21–24]. In this paper, we will further investigate the stability and bifurcation of model (1.2) as a complementarity. We will show that the two different time delay τ_1 and τ_2 have different effect on the stability and Hopf bifurcation behavior of system (1.2).

The remainder of the paper is organized as follows. In Section 2, we investigate the stability of the positive equilibrium and the occurrence of local Hopf bifurcations. In Section 3, numerical simulations are carried out to illustrate the validity of the main results. Some main conclusions are drawn in Section 4.

2. Stability and Local Hopf Bifurcations

In this section, we will study the stability of the positive equilibrium and the existence of local Hopf bifurcations.

If the following condition:

$$\text{(H1)}\ akl > b_1(k+l), \qquad akl(1-b_2) > b_1(1+a), \tag{2.1}$$

holds, then system (1.2) has a unique equilibrium point $E_0(s^*, i^*, y^*)$, where

$$s^* = \frac{akl - b_1(k+l)}{akl}, \qquad i^* = \frac{b_1}{kl}, \qquad y^* = \frac{akl(1-b_2) - b_1(1+a)}{akl^2}. \tag{2.2}$$

Let $\bar{s}(t) = s(t) - s^*$, $\bar{i}(t) = i(t) - i^*$, $\bar{y}(t) = y(t) - y^*$ and still denote $\bar{s}(t)$, $\bar{i}(t)$, $\bar{y}(t)$ by $s(t)$, $i(t)$, $y(t)$, respectively, then (1.2) reads as

$$\begin{aligned} \dot{s}(t) &= -as^*s - s^*(a+1)i, \\ \dot{i}(t) &= i^*s - li^*y(t-\tau_1), \\ \dot{y}(t) &= -(b_1 - kli^*)y + kly^*i(t-\tau_2). \end{aligned} \tag{2.3}$$

The characteristic equation of (2.3) is given by

$$\det\begin{pmatrix} a - 2as^* - (a+1)i^* - \lambda & -(a+1)s^* & 0 \\ i^* & s^* - b_2 - ly^* - \lambda & -li^*e^{-\lambda\tau_1} \\ 0 & kly^*e^{-\lambda\tau_2} & -b_1 + kli^* - \lambda \end{pmatrix} = 0. \tag{2.4}$$

That is

$$\lambda^3 + \theta_2\lambda^2 + \theta_1\lambda + \theta_0 + (\gamma_1\lambda + \gamma_0)e^{-\lambda(\tau_1+\tau_2)} = 0, \tag{2.5}$$

where

$$\begin{aligned} \theta_0 &= m_1m_2m_3 + m_5, \qquad \theta_1 = -(m_1m_2 + m_1m_3 + m_2m_3 + m_6), \\ \theta_2 &= -(m_1 + m_2 + m_3), \qquad \gamma_0 = m_1m_4, \qquad \gamma_1 = m_4, \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} m_1 &= a - 2as^* - (a+1)i^*, \qquad m_2 = b_2 + ly^* - s^*, \qquad m_3 = b_1 - kli^*, \\ m_4 &= kl^2i^*y^*, \qquad m_5 = -(a+1)s^*i^*(b_1 - kli^*), \qquad m_6 = -i^*(a+1)s^*. \end{aligned} \tag{2.7}$$

The following lemma is important for us to analyze the distribution of roots of the transcendental equation (2.5).

Lemma 2.1 (see [13]). *For the transcendental equation*

$$P\left(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m}\right) = \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} + \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right]e^{-\lambda\tau_1} + \cdots + \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right]e^{-\lambda\tau_m} = 0, \tag{2.8}$$

as $(\tau_1, \tau_2, \tau_3, \ldots, \tau_m)$ *vary, the sum of orders of the zeros of* $P(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m})$ *in the open right half plane can change, and only a zero appears on or crosses the imaginary axis.*

In the sequel, we consider four cases.

Case 1. $\tau_1 = \tau_2 = 0$, (2.5) becomes

$$\lambda^3 + \theta_2\lambda^2 + (\theta_1 + \gamma_1)\lambda + \theta_0 + \gamma_0 = 0. \tag{2.9}$$

All roots of (2.9) have a negative real part if the following condition holds:

$$\text{(H2)}\ \theta_0 + \gamma_0 > 0, \qquad \theta_2(\theta_1 + \gamma_1) > \theta_0 + \gamma_0. \tag{2.10}$$

Then the equilibrium point $E_0(s^*, i^*, y^*)$ is locally asymptotically stable when the conditions (H1) and (H2) are satisfied.

Case 2. $\tau_1 = 0$, $\tau_2 > 0$, (2.5) becomes

$$\lambda^3 + \theta_2\lambda^2 + \theta_1\lambda + \theta_0 + (\gamma_1\lambda + \gamma_0)e^{-\lambda\tau_2} = 0. \tag{2.11}$$

For $\omega > 0$, $i\omega$ be a root of (2.11), then it follows that

$$\begin{aligned} \gamma_1\omega\sin\omega\tau_2 + \gamma_0\cos\omega\tau_2 &= \theta_2\omega^2 - \theta_0, \\ \gamma_1\omega\cos\omega\tau_2 - \gamma_0\sin\omega\tau_2 &= \omega^3 - \theta_1\omega \end{aligned} \tag{2.12}$$

which is equivalent to

$$\omega^6 + \left(\theta_2^2 - 2\theta_0\theta_2 - 2\theta_1\right)\omega^4 + \left(\theta_1^2 - \gamma_1^2\right)\omega^2 - \gamma_0^2 = 0. \tag{2.13}$$

Let $z = \omega^2$, then (2.13) takes the form

$$z^3 + r_1z^2 + r_2z + r_3 = 0, \tag{2.14}$$

where $r_1 = \theta_2^2 - 2\theta_0\theta_2 - 2\theta_1$, $r_2 = \theta_1^2 - \gamma_1^2$, $r_3 = -\gamma_0^2$. Denote

$$h(z) = z^3 + r_1z^2 + r_2z + r_3. \tag{2.15}$$

Let $M = (q/2)^2 + (r/3)^3$, where $r = r_2 - (1/3)r_1^2$, $q = (2/27)r_1^3 - (1/3)r_1r_2 + r_3$. There are three cases for the solutions of (2.15).

(i) If $M > 0$, (2.15) has a real root and a pair of conjugate complex roots. The real root is positive and is given by

$$\mu_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{M}} + \sqrt[3]{-\frac{q}{2} - \sqrt{M}} - \frac{1}{3}r_1. \tag{2.16}$$

(ii) If $M = 0$, (2.15) has three real roots, of which two are equal. In particular, if $r_1 > 0$, there exists only one positive root, $\mu_1 = 2\sqrt[3]{-q/2} - r_1/3$; If $r_1 < 0$, there exists only one positive root, $\mu_1 = 2\sqrt[3]{-q/2} - r_1/3$ for $\sqrt[3]{-q/2} > -r_1/3$, and there exist three positive roots for $r_1/6 < \sqrt[3]{-q/2} < -r_1/3$, $\mu_1 = 2\sqrt[3]{-q/2} - r_1/3$, $\mu_2 = \mu_3 = -\sqrt[3]{-q/2} - r_1/3$.

(iii) If $M < 0$, there are three distinct real roots, $\mu_1 = 2\sqrt{(|r|/3)\cos(\varphi/3)} - r_1/3$, $\mu_2 = 2\sqrt{(|r|/3)\cos(\varphi/3 + 2\pi/3)} - r_1/3$, $\mu_3 = 2\sqrt{(|r|/3)\cos(\varphi/3 + 4\pi/3)} - r_1/3$, where $\cos\varphi = -q/2\sqrt{(|r|/3)^3}$. Furthermore, if $r_1 > 0$, there exists only one positive root. Otherwise, if $r_1 < 0$, there may exist either one or three positive real roots. If there is only one positive real root, it is equal to $\max(\mu_1, \mu_2, \mu_3)$.

Obviously, the number of positive real roots of (2.15) depends on the sign of r_1. If $r_1 \geq 0$, (2.15) has only one positive real root. Otherwise, there may exist three positive roots.

Without loss of generality, we assume that (2.14) has three positive roots, defined by z_1, z_2, z_3, respectively. Then (2.13) has three positive roots,

$$\omega_1 = \sqrt{z_1}, \qquad \omega_2 = \sqrt{z_2}, \qquad \omega_3 = \sqrt{z_3}. \tag{2.17}$$

By (2.12), we have

$$\cos\omega_k\tau_2 = \frac{(\theta_2\omega_k^2 - \theta_0)\gamma_0 + (\omega_k^3 - \theta_1\omega_k)\gamma_1\omega_k}{\gamma_0^2 + \gamma_1^2\omega_k^2}. \tag{2.18}$$

Thus, if we denote

$$\tau_{2_k}^{(j)} = \frac{1}{\omega_k}\left\{\arccos\left[\frac{(\theta_2\omega_k^2 - \theta_0)\gamma_0 + (\omega_k^3 - \theta_1\omega_k)\gamma_1\omega_k}{\gamma_0^2 + \gamma_1^2\omega_k^2}\right] + 2j\pi\right\}, \tag{2.19}$$

where $k = 1, 2, 3$; $j = 0, 1, \ldots$, then $\pm i\omega_k$ are a pair of purely imaginary roots of (2.11) with $\tau_{2_k}^{(j)}$. Define

$$\tau_{2_0} = \tau_{k_0}^{(0)} = \min_{k\in\{1,2,3\}}\left\{\tau_{2_k}^{(0)}\right\}, \qquad \omega_0 = \omega_{k_0}. \tag{2.20}$$

Based on above analysis, we have the following result.

Lemma 2.2. *If* $(H1)$ *and* $(H2)$ *hold, then all roots of* (1.2) *have a negative real part when* $\tau_2 \in [0, \tau_{2_0})$ *and* (1.2) *admits a pair of purely imaginary roots* $\pm\omega_k i$ *when* $\tau_2 = \tau_{2_k}^{(j)}$ $(k = 1, 2, 3;\ j = 0, 1, 2, \ldots)$.

Let $\lambda(\tau_2) = \alpha(\tau_2) + i\omega(\tau_2)$ be a root of (2.11) near $\tau_2 = \tau_{2_k}^{(j)}$, and $\alpha(\tau_{2_k}^{(j)}) = 0$, and $\omega(\tau_{2_k}^{(j)}) = \omega_0$. Due to functional differential equation theory, for every $\tau_{2_k}^{(j)}$, $k = 1, 2, 3$; $j = 0, 1, 2, \ldots$, there exists $\varepsilon > 0$ such that $\lambda(\tau_2)$ is continuously differentiable in τ_2 for $|\tau_2 - \tau_{2_k}^{(j)}| < \varepsilon$. Substituting $\lambda(\tau_2)$ into the left hand side of (2.11) and taking derivative with respect to τ_2, we have

$$\left(\frac{d\lambda}{d\tau_2}\right)^{-1} = -\frac{(3\lambda^2 + 2\theta_2\lambda + \theta_1)e^{\lambda\tau_2}}{\lambda(\gamma_1\lambda + \gamma_0)} + \frac{\gamma_1}{\lambda(\gamma_1\lambda + \gamma_0)} - \frac{\tau_2}{\lambda}. \tag{2.21}$$

We can easily obtain

$$\begin{aligned}
\left[\frac{d(\operatorname{Re}\lambda(\tau_2))}{d\tau_2}\right]^{-1}_{\tau_2=\tau_{2_k}^{(j)}} &= \operatorname{Re}\left\{-\frac{(3\lambda^2 + 2\theta_2\lambda + \theta_1)e^{\lambda\tau_2}}{\lambda(\gamma_1\lambda + \gamma_0)}\right\} + \operatorname{Re}\left\{\frac{\gamma_1}{\lambda(\gamma_1\lambda + \gamma_0)}\right\} \\
&= \frac{1}{\Lambda}\Big\{\gamma_1\omega_k^2\Big[\Big(\theta_1 - 3\omega_k^2\Big)\cos\omega_k\tau_{2_k}^{(j)} - 2\theta_2\omega_k\sin\omega_k\tau_{2_k}^{(j)}\Big] \\
&\qquad + (-\gamma_0\omega_k)\Big[2\theta_2\omega_k\cos\omega_k\tau_{2_k}^{(j)} + \Big(\theta_1 - 3\omega_k^2\Big)\sin\omega_k\tau_{2_k}^{(j)}\Big] + \gamma_1^2\omega_k^2\Big\} \\
&= \frac{1}{\Lambda}\Big\{\Big(\theta_1 - 3\omega_k^2\Big)\omega_k\Big[\gamma_0\sin\omega_k\tau_{2_k}^{(j)} + \gamma_1\omega_k\cos\omega_k\tau_{2_k}^{(j)}\Big] \\
&\qquad + 2\theta_2\omega_k^2\Big[-\gamma_0\cos\omega_k\tau_{2_k}^{(j)} - \gamma_1\omega_k\sin\omega_k\tau_{2_k}^{(j)}\Big] + \gamma_1^2\omega_k^2\Big\} \\
&= \frac{1}{\Lambda}\Big[\Big(3\omega_k^6 + \Big(2\theta_2^2 - 4\theta_1\Big)\omega_k^4 + \Big(\theta_1^2 - 2\theta_0\theta_2 + \gamma_1^2\Big)\omega_k^2\Big)\Big] \\
&= \frac{1}{\Lambda}\Big(3\omega_k^6 + 2r_1\omega_k^4 + r_2\omega_k^2\Big) = \frac{1}{\Lambda}\Big[z_k\Big(3z_k^2 + 2r_1z_k + r_2\Big)\Big] = \frac{z_k}{\Lambda}h'(z_k),
\end{aligned} \tag{2.22}$$

where $\Lambda = (\gamma_1\omega_k^2)^2 + (\gamma_0\omega_k)^2 > 0$. Thus, we have

$$\operatorname{sign}\left\{\frac{d(\operatorname{Re}\lambda(\tau_2))}{d\tau_2}\right\}_{\tau_2=\tau_{2_k}^{(j)}} = \operatorname{sign}\left\{\frac{d(\operatorname{Re}\lambda(\tau_2))}{d\tau_2}\right\}^{-1}_{\tau_2=\tau_{2_k}^{(j)}} = \operatorname{sign}\left\{\frac{z_k}{\Lambda}h'(z_k)\right\} \neq 0. \tag{2.23}$$

Since Λ, $z_k > 0$, we can conclude that the sign of $[d(\operatorname{Re}\lambda(\tau_2))/d\tau_2]_{\tau_2=\tau_{2_k}^{(j)}}$ is determined by that of $h'(z_k)$.

The analysis above leads to the following result.

Theorem 2.3. *Suppose that* $z_k = \omega_k^2$ *and* $h'(z_k) \neq 0$, *where* $h(z)$ *is defined by* (2.15). *Then*

$$\left[\frac{d(\operatorname{Re}\lambda(\tau_2))}{d\tau}\right]_{\tau_2=\tau_{2_k}^{(j)}} \neq 0 \tag{2.24}$$

and the sign *of* $[d(\operatorname{Re}\lambda(\tau_2))/d\tau_2]_{\tau_2=\tau_{2_k}^{(j)}}$ *is consistent with that of* $h'(z_k)$.

In the sequel, we assume that

$$\text{(H3)}\ \ h'(z_k) \neq 0. \tag{2.25}$$

According to above analysis and the results of Kuang [25] and Hale [26], we have the following.

Theorem 2.4. *For* $\tau_1 = 0$, *if* $(H1)$ *and* $(H2)$ *hold, then the positive equilibrium* $E_0(s^*, i^*, y^*)$ *of system* (1.2) *is asymptotically stable for* $\tau_2 \in [0, \tau_{2_0})$. *In addition to the conditions* $(H1)$ *and* $(H2)$, *we further assume that* $(H3)$ *holds, then system* (1.2) *undergoes a Hopf bifurcation at the positive equilibrium* $E_0(s^*, i^*, y^*)$ *when* $\tau_2 = \tau_{2_k}^{(j)}$, $k = 1, 2, 3$; $j = 0, 1, 2, \ldots$.

Case 3. $\tau_1 > 0$, $\tau_2 = 0$, (2.5) takes the form

$$\lambda^3 + \theta_2\lambda^2 + \theta_1\lambda + \theta_0 + (\gamma_1\lambda + \gamma_0)e^{-\lambda\tau_1} = 0. \tag{2.26}$$

For $\omega_* > 0$, $i\omega_*$ be a root of (2.26), then it follows that

$$\begin{aligned} \gamma_1\omega_* \sin\omega_*\tau_1 + \gamma_0 \cos\omega_*\tau_1 &= \theta_2\omega_*^2 - \theta_0, \\ \gamma_1\omega_* \cos\omega_*\tau_1 - \gamma_0 \sin\omega_*\tau_1 &= \omega_*^3 - \theta_1\omega_* \end{aligned} \tag{2.27}$$

which is equivalent to

$$\omega_*^6 + \left(\theta_2^2 - 2\theta_0\theta_2 - 2\theta_1\right)\omega_*^4 + \left(\theta_1^2 - \gamma_1^2\right)\omega_*^2 - \gamma_0^2 = 0. \tag{2.28}$$

Let $z_* = \omega_*^2$, then (2.13) takes the form

$$z_*^3 + p_2 z_*^2 + p_1 z_* + p_0 = 0, \tag{2.29}$$

where $p_0 = -\gamma_0^2$, $p_1 = \theta_1^2 - \gamma_1^2$, $p_2 = \theta_2^2 - 2\theta_0\theta_2 - 2\theta_1$. Denote

$$h_*(z_*) = z_*^3 + r_1 z_*^2 + r_2 z_* + r_3. \tag{2.30}$$

Let $M = (q/2)^2 + (r/3)^3$, where $r = r_2 - (1/3)r_1^2$, $q = (2/27)r_1^3 - (1/3)r_1 r_2 + r_3$. For (2.15), Similar analysis on the solutions of system (2.30) as that in Case 2. Here we omit it.

Without loss of generality, we assume that (2.30) has three positive roots, defined by z_{*1}, z_{*2}, z_{*3}, respectively. Then (2.29) has three positive roots

$$\omega_{*1} = \sqrt{z_{*1}}, \qquad \omega_{*2} = \sqrt{z_{*2}}, \qquad \omega_{*3} = \sqrt{z_{*3}}. \tag{2.31}$$

By (2.27), we have

$$\cos \omega_{*k}\tau_1 = \frac{(\theta_2\omega_{*k}^2 - \theta_0)\gamma_0 + (\omega_{*k}^3 - \theta_1\omega_{*k})\gamma_1\omega_{*k}}{\gamma_0^2 + \gamma_1^2\omega_{*k}^2}. \tag{2.32}$$

Thus, if we denote

$$\tau_{1_k}^{(j)} = \frac{1}{\omega_{*k}}\left\{\arccos\left[\frac{(\theta_2\omega_{*k}^2 - \theta_0)\gamma_0 + (\omega_{*k}^3 - \theta_1\omega_{*k})\gamma_1\omega_{*k}}{\gamma_0^2 + \gamma_1^2\omega_{*k}^2}\right] + 2j\pi\right\}, \tag{2.33}$$

where $k = 1,2,3$; $j = 0,1,\ldots$, then $\pm i\omega_{*k}$ are a pair of purely imaginary roots of (2.26) with $\tau_{1_k}^{(j)}$. Define

$$\tau_{1_0} = \tau_{k_0}^{(0)} = \min_{k\in\{1,2,3\}}\left\{\tau_{1_k}^{(0)}\right\}, \qquad \omega_{*0} = \omega_{*k_0}. \tag{2.34}$$

The above analysis leads to the following result.

Lemma 2.5. *If* $(H1)$ *and* $(H2)$ *hold, then all roots of* (1.2) *have a negative real part when* $\tau_1 \in [0, \tau_{1_0})$ *and* (1.2) *admits a pair of purely imaginary roots* $\pm\omega_k i$ *when* $\tau_1 = \tau_{1_k}^{(j)}$ $(k = 1,2,3;\ j = 0,1,2,\ldots)$.

Let $\lambda(\tau_1) = \alpha(\tau_1) + i\omega(\tau_1)$ be a root of (2.26) near $\tau_1 = \tau_{1_k}^{(j)}$, and $\alpha(\tau_{1_k}^{(j)}) = 0$, and $\omega(\tau_{1_k}^{(j)}) = \omega_{*0}$. Due to functional differential equation theory, for every $\tau_{1_k}^{(j)}$, $k = 1,2,3$; $j = 0,1,2,\ldots$, there exists $\varepsilon_* > 0$ such that $\lambda(\tau_1)$ is continuously differentiable in τ_1 for $|\tau_1 - \tau_{1_k}^{(j)}| < \varepsilon$. Substituting $\lambda(\tau_1)$ into the left hand side of (2.26) and taking derivative with respect to τ_1, we have

$$\left(\frac{d\lambda}{d\tau_1}\right)^{-1} = -\frac{(3\lambda^2 + 2\theta_2\lambda + \theta_1)e^{\lambda\tau_2}}{\lambda(\gamma_1\lambda + \gamma_0)} + \frac{\gamma_1}{\lambda(\gamma_1\lambda + \gamma_0)} - \frac{\tau}{\lambda}. \tag{2.35}$$

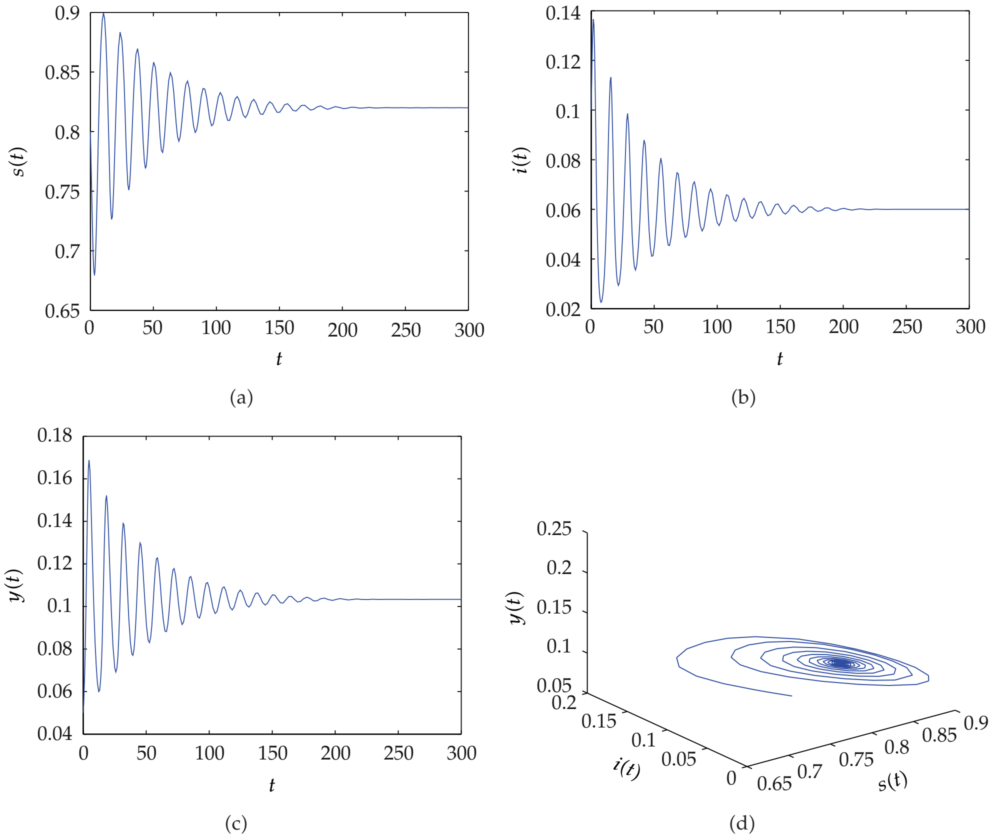

Figure 1: Trajectory portrait and phase portrait of system (3.1) with $\tau_1 = 0$, $\tau_2 = 0.2 < \tau_{2_0} \approx 0.32$. The positive equilibrium $E_0(0.82, 0.06, 0.1033)$ is asymptotically stable. The initial value is $(0.8, 0.1, 0.05)$.

We can easily obtain

$$\begin{aligned}
\left[\frac{d(\operatorname{Re}\lambda(\tau_1))}{d\tau_1}\right]^{-1}_{\tau_1=\tau_{1_k}^{(j)}} &= \operatorname{Re}\left\{-\frac{(3\lambda^2+2\theta_2\lambda+\theta_1)e^{\lambda\tau_1}}{\lambda(\gamma_1\lambda+\gamma_0)}\right\} + \operatorname{Re}\left\{\frac{\gamma_1}{\lambda(\gamma_1\lambda+\gamma_0)}\right\} \\
&= \frac{1}{\Lambda_*}\left\{\gamma_1\omega_{*k}^2\left[\left(\theta_1-3\omega_{*k}^2\right)\cos\omega_{*k}\tau_{1_k}^{(j)} - 2\theta_2\omega_{*k}\sin\omega_{*k}\tau_{1_k}^{(j)}\right]\right. \\
&\qquad \left. +(-\gamma_0\omega_{*k})\left[2\theta_2\omega_{*k}\cos\omega_{*k}\tau_{1_k}^{(j)} + \left(\theta_1-3\omega_{*k}^2\right)\sin\omega_{*k}\tau_{1_k}^{(j)}\right] + \gamma_1^2\omega_{*k}^2\right\} \\
&= \frac{1}{\Lambda_*}\left\{\left(\theta_1-3\omega_{*k}^2\right)\omega_{*k}\left[\gamma_0\sin\omega_{*k}\tau_{1_k}^{(j)} + \gamma_1\omega_{*k}\cos\omega_{*k}\tau_{1_k}^{(j)}\right]\right. \\
&\qquad \left. +2\theta_2\omega_{*k}^2\left[-\gamma_0\cos\omega_{*k}\tau_{1_k}^{(j)} - \gamma_1\omega_{*k}\sin\omega_{*k}\tau_{1_k}^{(j)}\right] + \gamma_1^2\omega_{*k}^2\right\} \\
&= \frac{1}{\Lambda_*}\left[\left(3\omega_{*k}^6 + \left(2\theta_2^2-4\theta_1\right)\omega_{*k}^4 + \left(\theta_1^2-2\theta_0\theta_2+\gamma_1^2\right)\omega_{*k}^2\right)\right] \\
&= \frac{1}{\Lambda_*}\left(3\omega_{*k}^6 + 2r_1\omega_{*k}^4 + r_2\omega_{*k}^2\right) = \frac{1}{\Lambda}\left[z_{*k}\left(3z_{*k}^2 + 2r_1 z_{*k} + r_2\right)\right] = \frac{z_{*k}}{\Lambda}h'(z_{*k}),
\end{aligned} \tag{2.36}$$

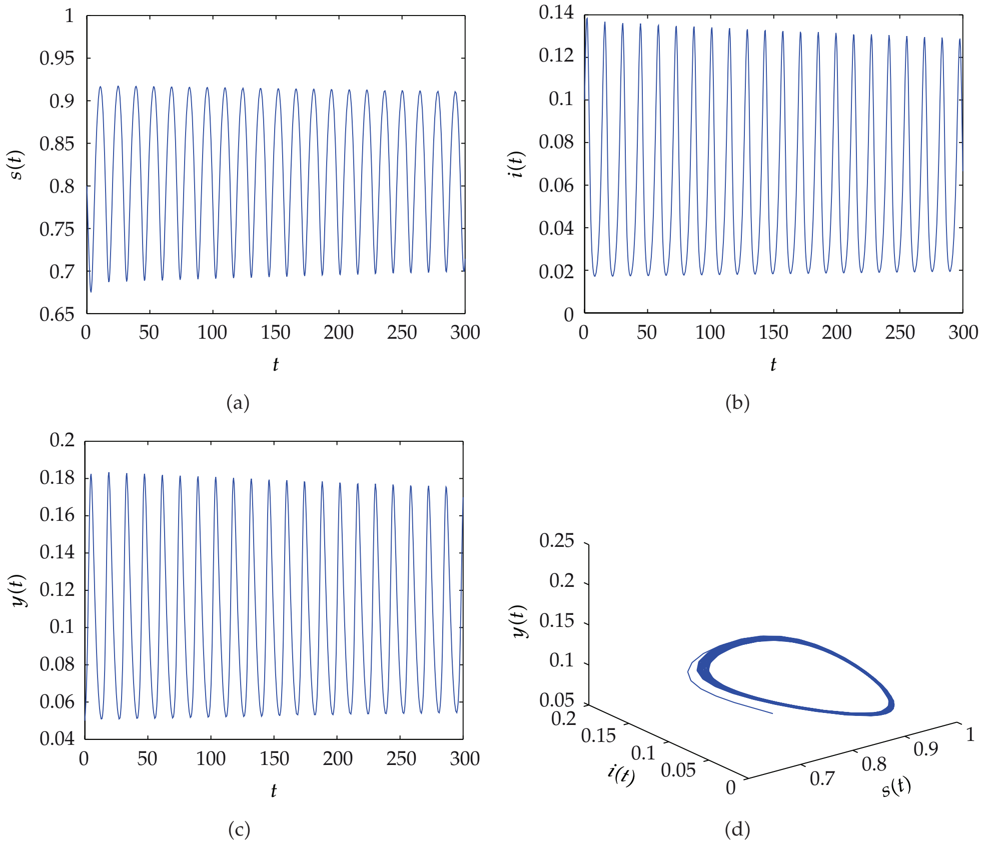

Figure 2: Trajectory portrait and phase portrait of system (3.1) with $\tau_1 = 0$, $\tau_2 = 0.415 > \tau_{2_0} \approx 0.32$. Hopf bifurcation occurs from the positive equilibrium $E_0(0.82, 0.06, 0.1033)$. The initial value is $(0.8, 0.1, 0.05)$.

where $\Lambda_* = (\gamma_1 \omega_{*k}^2)^2 + (\gamma_0 \omega_{*k})^2 > 0$. Thus, we have

$$\operatorname{sign}\left\{\frac{d(\operatorname{Re}\lambda(\tau_1))}{d\tau_1}\right\}_{\tau_1=\tau_{1_k}^{(j)}} = \operatorname{sign}\left\{\frac{d(\operatorname{Re}\lambda(\tau_1))}{d\tau_1}\right\}^{-1}_{\tau_1=\tau_{1_k}^{(j)}} = \operatorname{sign}\left\{\frac{z_{*k}}{\Lambda} h_*'(z_{*k})\right\} \neq 0. \tag{2.37}$$

Since $\Lambda_*, z_{*k} > 0$, we can conclude that the sign of $[d(\operatorname{Re}\lambda(\tau_1))/d\tau_1]_{\tau=\tau_{1_k}^{(j)}}$ is determined by that of $h_*'(z_{*k})$.

From the analysis above, we obtain the following result.

Theorem 2.6. *Suppose that $z_{*k} = \omega_{*k}^2$ and $h_*'(z_{*k}) \neq 0$, where $h_*(z_*)$ is defined by* (2.30). *Then*

$$\left[\frac{d(\operatorname{Re}\lambda(\tau_1))}{d\tau_1}\right]_{\tau_1=\tau_{1_k}^{(j)}} \neq 0 \tag{2.38}$$

and the sign *of* $[d(\operatorname{Re}\lambda(\tau_1))/d\tau_1]_{\tau=\tau_{1_k}^{(j)}}$ *is consistent with that of* $h_*'(z_{*k})$.

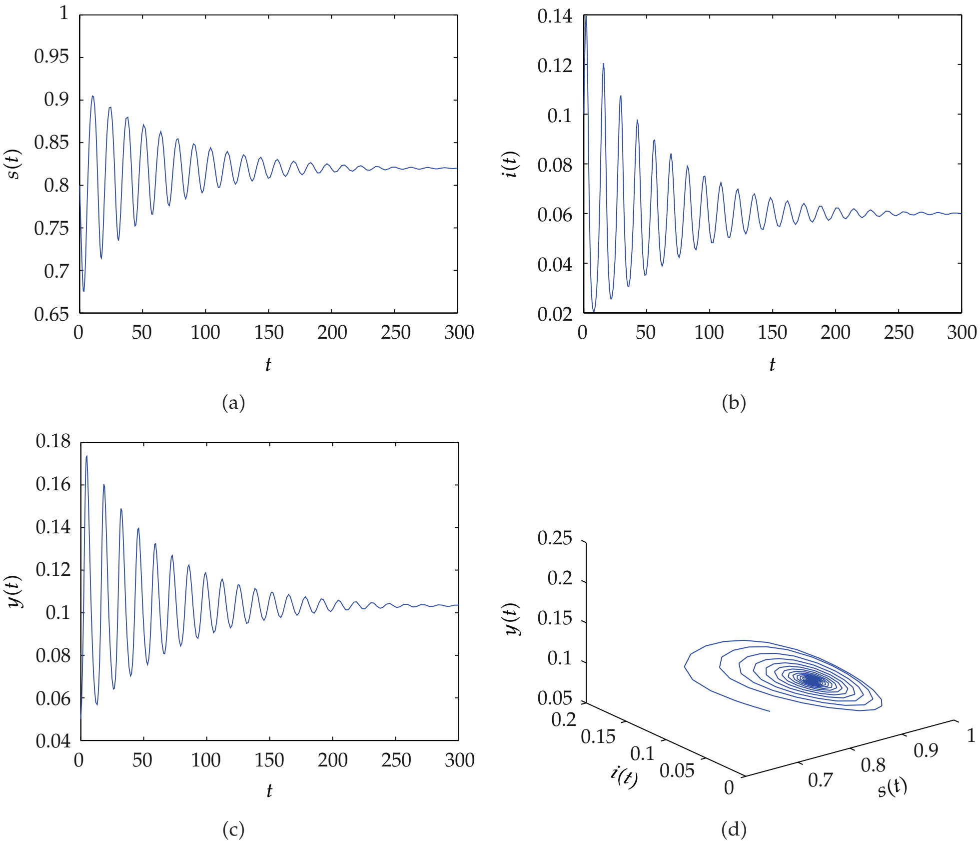

Figure 3: Trajectory portrait and phase portrait of system (3.1) with $\tau_2 = 0.25$, $\tau_1 = 0.13 < \tau_{1_0} \approx 0.16$. The positive equilibrium $E_0(0.82, 0.06, 0.1033)$ is asymptotically stable. The initial value is $(0.8, 0.1, 0.05)$.

In the sequel, we assume that

$$\text{(H4)}\ \ h_*'(z_{*k}) \neq 0. \tag{2.39}$$

Based on above analysis and in view of Kuang [25] and Hale [26], we get the following result.

Theorem 2.7. *For $\tau_2 = 0$, if $(H1)$ and $(H2)$ hold, then the positive equilibrium $E_0(s^*, i^*, y^*)$ of system* (1.2) *is asymptotically stable for $\tau_1 \in [0, \tau_{1_0})$. In addition to the condition $(H1)$ and $(H2)$, one further assumes that $(H4)$ holds, then system* (1.2) *undergoes a Hopf bifurcation at the positive equilibrium $E_0(s^*, i^*, y^*)$ when $\tau_1 = \tau_{1_k}^{(j)}$, $k = 1, 2, 3$; $j = 0, 1, 2, \ldots$.*

Case 4. $\tau_1 > 0$, $\tau_2 > 0$. We consider (2.5) with τ_2 in its stable interval. Regarding τ_1 as a parameter. Without loss of generality, we consider system (1.2) under the assumptions (H1) and (H2). Let $i\omega$ $(\omega > 0)$ be a root of (2.5), then we can obtain

$$k_1\omega^6 + k_1\omega^4 + k_2\omega^2 + k_3 = 0, \tag{2.40}$$

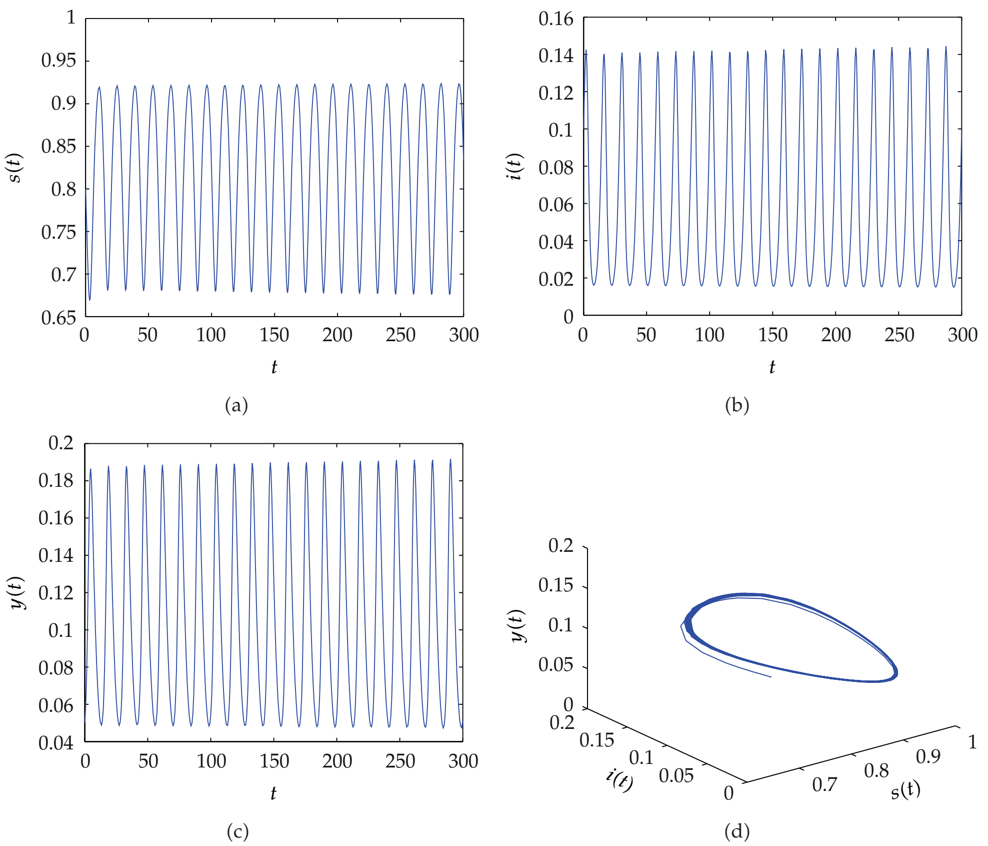

Figure 4: Trajectory portrait and phase portrait of system (3.1) with $\tau_2 = 0.25$, $\tau_1 = 0.21 > \tau_{1_0} \approx 0.16$. Hopf bifurcation occurs from the positive equilibrium $E_0(0.82, 0.06, 0.1033)$. The initial value is $(0.8, 0.1, 0.05)$.

where

$$\begin{aligned}
k_1 &= 3\gamma_0^2, \\
k_2 &= \gamma_1(2\gamma_0\theta_2 - \gamma_1\theta_1) + 3\gamma_1(\gamma_0\theta_2 - \theta_1\gamma_1) + (2\theta_2\gamma_1 - 3\gamma_0)(\theta_2\gamma_1 - \gamma_0), \\
k_3 &= (2\gamma_0\theta_2 - \gamma_1\theta_1)(\gamma_0\theta_2 - \theta_1\gamma_1) - 3\gamma_0\gamma_1\theta_1 + \gamma_0\theta_1(\theta_2\gamma_1 - \gamma_0) \\
&\quad + (2\theta_2\gamma_1 - 3\gamma_0)(\theta_1\gamma_0 - \theta_1\gamma_1), \\
k_4 &= \gamma_0\theta_1(\theta_1\gamma_0 - \theta_1\gamma_1).
\end{aligned} \tag{2.41}$$

Denote

$$H(\omega) = 3\gamma_0^2\omega^4 + k_1\omega^3 + k_2\omega^2 + k_3\omega + k_4. \tag{2.42}$$

Assume that

$$\text{(H5)}\ \theta_1\gamma_0 < \theta_1\gamma_1. \tag{2.43}$$

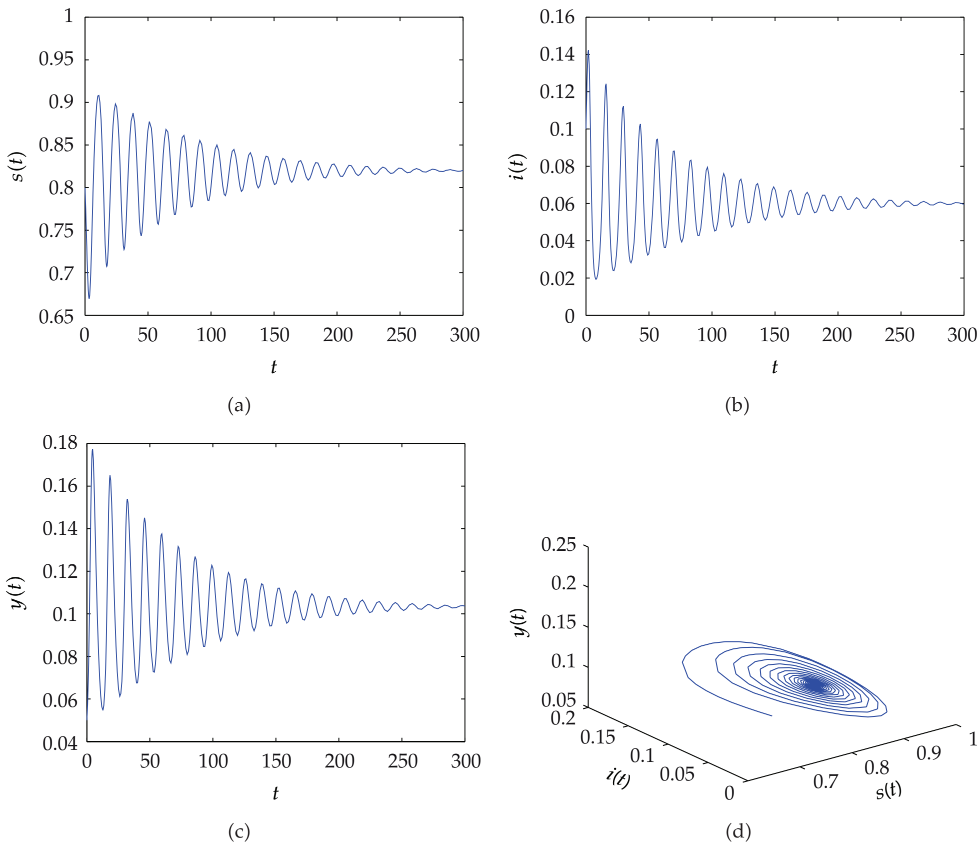

Figure 5: Trajectory portrait and phase portrait of system (3.1) with $\tau_2 = 0$, $\tau_1 = 0.28 < \tau_{1_0} \approx 0.33$. The positive equilibrium $E_0(0.82, 0.06, 0.1033)$ is asymptotically stable. The initial value is $(0.8, 0.1, 0.05)$.

It is easy to check that $H(0) < 0$ if (H5) holds and $\lim_{\omega\to+\infty} H(\omega) = +\infty$. We can obtain that (2.42) has finite positive roots $\omega_1, \omega_2, \ldots, \omega_n$. For every fixed ω_i, $i = 1, 2, 3, \ldots, k$, there exists a sequence $\{\tau_{1_i}^j \mid j = 1, 2, 3, \ldots\}$, such that (2.42) holds. Let

$$\tau_{1_0} = \min\left\{\tau_{1_i}^j \mid i = 1, 2, \ldots, k;\ j = 1, 2, \ldots\right\}. \tag{2.44}$$

When $\tau_1 = \tau_{1_0}$, (2.5) has a pair of purely imaginary roots $\pm i\tilde{\omega}^*$ for $\tau_2 \in [0, \tau_{2_0})$.

In the following, we assume that

$$(\text{H6}) \quad \left[\frac{d(\operatorname{Re}\lambda)}{d\tau_1}\right]_{\lambda = i\tilde{\omega}^*} \neq 0. \tag{2.45}$$

Thus, by the general Hopf bifurcation theorem for FDEs in Hale [26], we have the following result on the stability and Hopf bifurcation in system (1.2).

Theorem 2.8. *For system* (1.2), *suppose* $(H1)$, $(H2)$, $(H3)$, $(H5)$, *and* $(H6)$ *are satisfied, and* $\tau_2 \in [0, \tau_{2_0})$, *then the positive equilibrium* $E_0(s^*, i^*, y^*)$ *is asymptotically stable when* $\tau_1 \in [0, \tau_{1_0})$, *and system* (1.2) *undergoes a Hopf bifurcation at the positive equilibrium* $E_0(s^*, i^*, y^*)$ *when* $\tau_1 = \tau_{1_0}$.

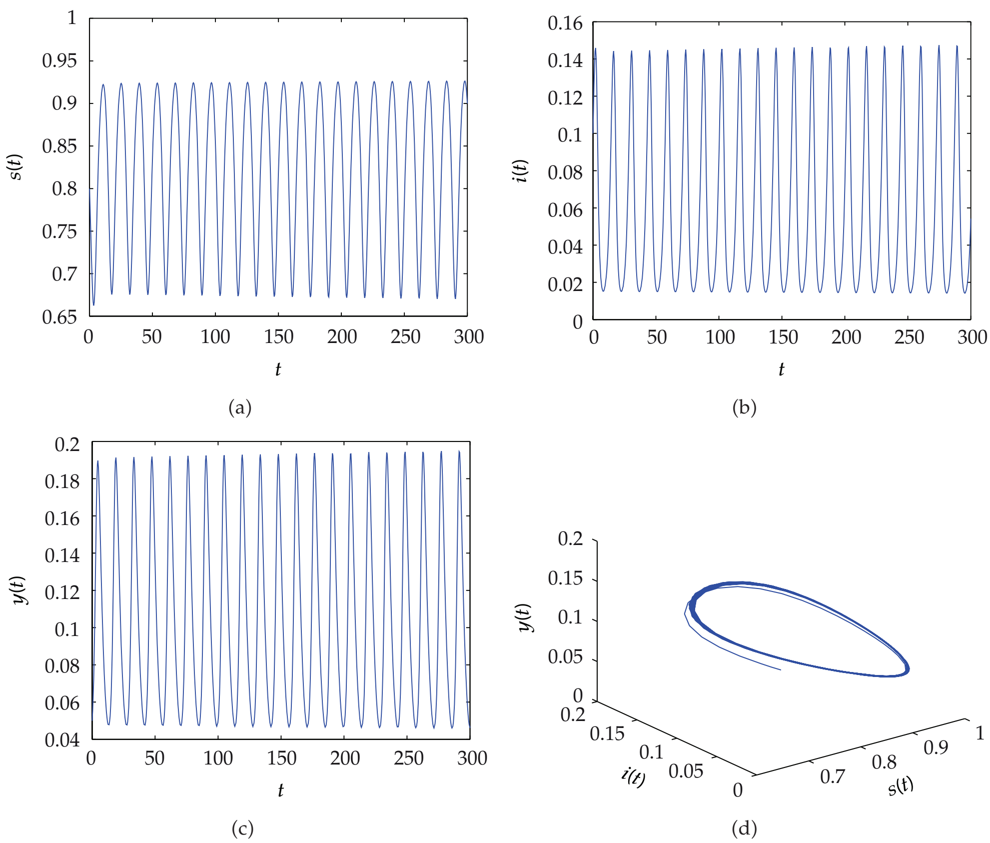

Figure 6: Trajectory portrait and phase portrait of system (3.1) with $\tau_2 = 0$, $\tau_1 = 0.42 > \tau_{1_0} \approx 0.33$. Hopf bifurcation occurs from the positive equilibrium $E_0(0.82, 0.06, 0.1033)$. The initial value is $(0.8, 0.1, 0.05)$.

Case 5. $\tau_1 > 0$, $\tau_2 > 0$. We consider (2.5) with τ_1 in its stable interval. Regarding τ_2 as a parameter. Without loss of generality, we consider system (1.2) under the assumptions (H1) and (H2). Let $i\omega^*$ $(\omega^* > 0)$ be a root of (2.5), then we can obtain

$$k_1\omega^{*6} + k_1\omega^{*4} + k_2\omega^{*2} + k_3 = 0, \tag{2.46}$$

where k_1, k_2, k_3, and k_4 are defined by (2.41). Denote

$$H_*(\omega^*) = 3\gamma_0^2\omega^{*4} + k_1\omega^{*3} + k_2\omega^{*2} + k_3\omega^* + k_4. \tag{2.47}$$

Obviously, $H(0) < 0$ if (H5) holds and $\lim_{\omega\to+\infty} H_*(\omega^*) = +\infty$. We can obtain that (2.47) has finite positive roots $\omega_1^*, \omega_2^*, \ldots, \omega_n^*$. For every fixed ω_i^*, $i = 1, 2, 3, \ldots, k$, there exists a sequence $\{\tau_{2_i}^j \mid j = 1, 2, 3, \ldots\}$, such that (2.47) holds. Let

$$\tau_{2_0} = \min\left\{\tau_{2_i}^j \mid i = 1, 2, \ldots, k;\ j = 1, 2, \ldots\right\}. \tag{2.48}$$

When $\tau_2 = \tau_{2_0}$, (2.5) has a pair of purely imaginary roots $\pm i\overline{\omega}^*$ for $\tau_1 \in [0, \tau_{1_0})$.

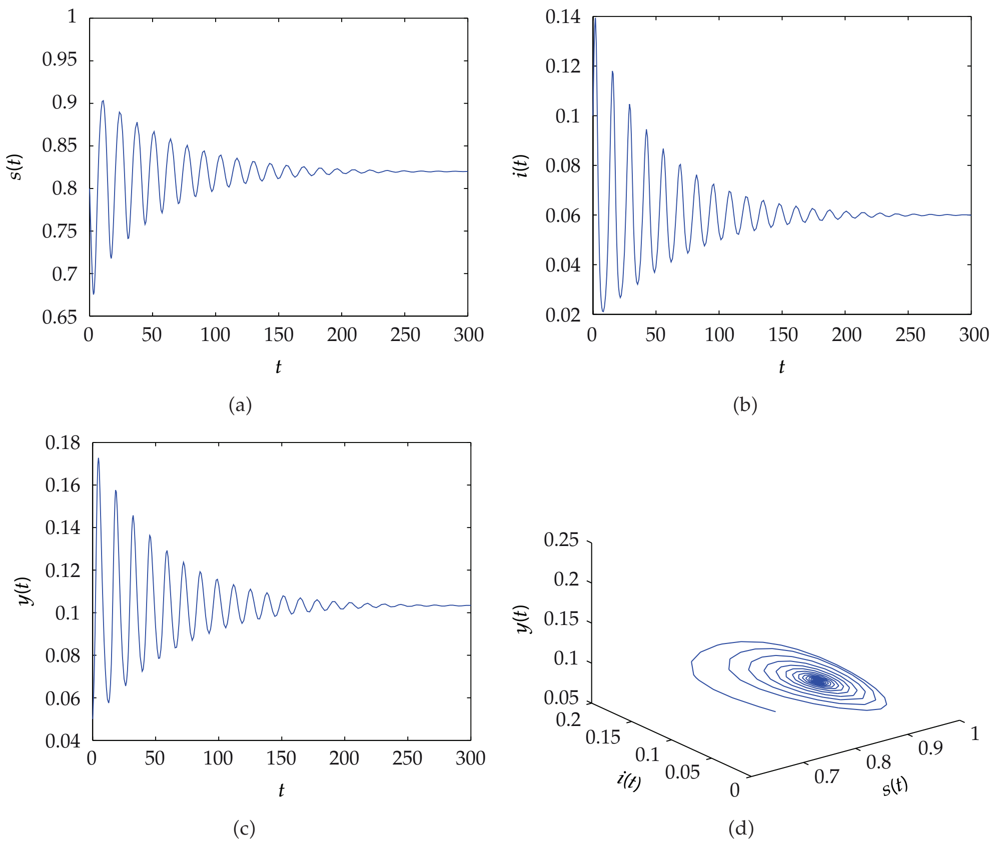

Figure 7: Trajectory portrait and phase portrait of system (3.1) with $\tau_1 = 0.12$, $\tau_2 = 1.8 < \tau_{2_0} \approx 2.1$. The positive equilibrium $E_0(0.82, 0.06, 0.1033)$ is asymptotically stable. The initial value is $(0.8, 0.1, 0.05)$.

In the following, we assume that

$$\text{(H7)} \quad \left[\frac{d(\operatorname{Re}\lambda)}{d\tau_2}\right]_{\lambda=i\overline{\omega}^*} \neq 0. \tag{2.49}$$

In view of the general Hopf bifurcation theorem for FDEs in Hale [26], we have the following result on the stability and Hopf bifurcation in system (1.2).

Theorem 2.9. *For system* (1.2), *assume that* $(H1)$, $(H2)$, $(H3)$, $(H4)$, *and* $(H7)$ *are satisfied and* $\tau_1 \in [0, \tau_{1_0})$, *then the positive equilibrium* $E_0(s^*, i^*, y^*)$ *is asymptotically stable when* $\tau_2 \in [0, \tau_{2_0})$, *and system* (1.2) *undergoes a Hopf bifurcation at the positive equilibrium* $E_0(s^*, i^*, y^*)$ *when* $\tau_2 = \tau_{2_0}$.

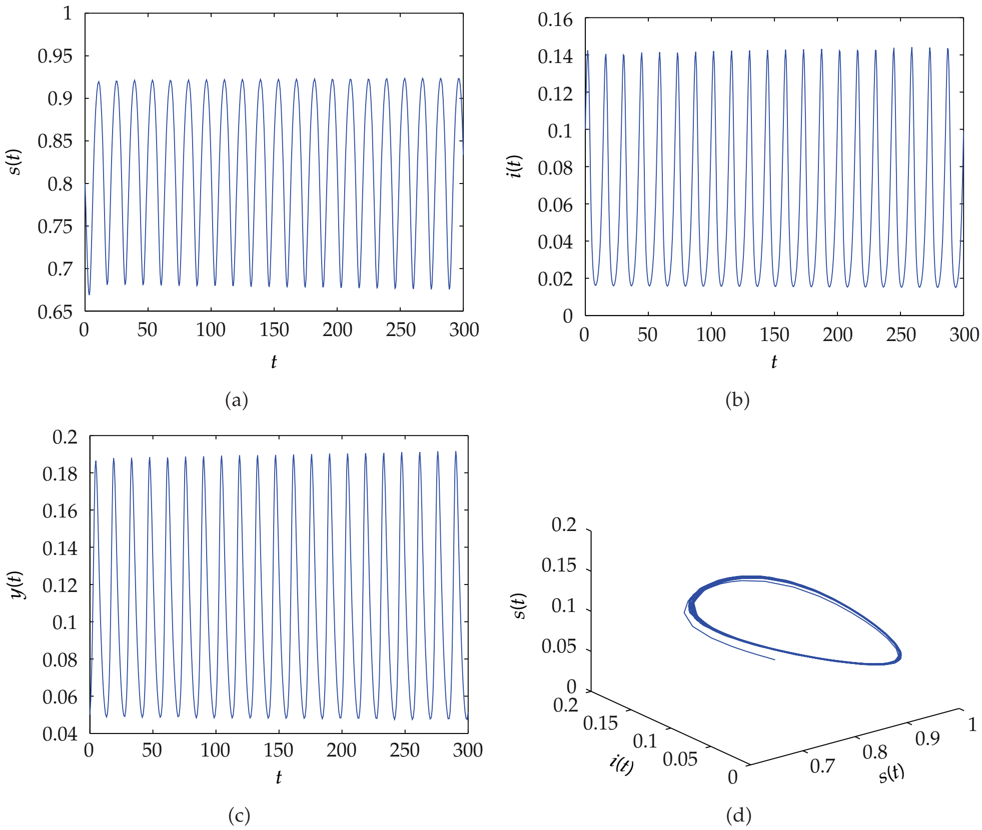

Figure 8: Trajectory portrait and phase portrait of system (3.1) with $\tau_1 = 0.12$, $\tau_2 = 2.5 > \tau_{2_0} \approx 2.1$. Hopf bifurcation occurs from the positive equilibrium $E_0(0.82, 0.06, 0.1033)$. The initial value is $(0.8, 0.1, 0.05)$.

3. Computer Simulations

In this section, we present some numerical results of system (1.2) to verify the analytical predictions obtained in the previous section. Let us consider the following system:

$$\begin{aligned} \dot{s}(t) &= 0.5s[1-(s+i)] - si, \\ \dot{i}(t) &= -0.2i + si - 6iy(t-\tau_1), \\ \dot{y}(t) &= -0.3y + 5yi(t-\tau_1), \end{aligned} \tag{3.1}$$

which has a positive equilibrium $E_0(0.82, 0.06, 0.1033)$. We can easily obtain that (H1)–(H7) are satisfied. When $\tau_1 = 0$, using Matlab 7.0, we obtain $\omega_0 \approx 0.4742$, $\tau_{2_0} \approx 0.32$. The positive equilibrium $E_0(0.82, 0.06, 0.1033)$ is asymptotically stable for $\tau_2 < \tau_{2_0} \approx 0.32$ and unstable for $\tau_2 > \tau_{2_0} \approx 0.32$ which is shown in Figure 1. When $\tau_2 = \tau_{2_0} \approx 0.32$, (3.1) undergoes a Hopf bifurcation at the positive equilibrium $E_0(0.82, 0.06, 0.1033)$, that is, a small amplitude periodic solution occurs around $E_0(0.82, 0.06, 0.1033)$ when $\tau_1 = 0$ and τ_2 is close to $\tau_{2_0} = 0.32$ which is shown in Figure 2.

Let $\tau_2 = 0.25 \in (0, 0.32)$ and choose τ_1 as a parameter. We have $\tau_{1_0} \approx 0.16$, Then the positive equilibrium is asymptotically when $\tau_1 \in [0, \tau_{1_0})$. The Hopf bifurcation value of (3.1) is $\tau_{1_0} \approx 0.16$ (see Figures 3 and 4).

When $\tau_2 = 0$, using Matlab 7.0, we obtain $\omega_{*0} \approx 0.7745$, $\tau_{1_0} \approx 0.16$. The positive equilibrium $E_0(0.82, 0.06, 0.1033)$ is asymptotically stable for $\tau_1 < \tau_{1_0} \approx 0.16$ and unstable for $\tau_1 > \tau_{1_0} \approx 0.16$ which is shown in Figure 5. When $\tau_1 = \tau_{1_0} \approx 0.16$, (3.1) undergoes a Hopf bifurcation at the positive equilibrium $E_0(0.82, 0.06, 0.1033)$, that is, a small amplitude periodic solution occurs around $E_0(0.82, 0.06, 0.1033)$ when $\tau_2 = 0$ and τ_1 is close to $\tau_{1_0} = 0.16$ which is illustrated in Figure 6.

Let $\tau_1 = 0.25 \in (0, 0.32)$ and choose τ_2 as a parameter. We have $\tau_{2_0} \approx 0.33$. Then the positive equilibrium is asymptotically stable when $\tau_2 \in [0, \tau_{2_0})$. The Hopf bifurcation value of (3.1) is $\tau_{2_0} \approx 0.33$ (see Figures 7 and 8).

4. Conclusions

In this paper, we have investigated local stability of the positive equilibrium $E_0(s^*, i^*, y^*)$ and local Hopf bifurcation of an ecoepidemiological model with two delays. It is shown that if some conditions hold true, and $\tau_2 \in [0, \tau_{2_0})$, then the positive equilibrium $E_0(s^*, i^*, y^*)$ is asymptotically stable when $\tau_1 \in (0, \tau_{1_0})$, when the delay τ_1 increases, the positive equilibrium $E_0(s^*, i^*, y^*)$ loses its stability and a sequence of Hopf bifurcations occur at the positive equilibrium $E_0(s^*, i^*, y^*)$, that is, a family of periodic orbits bifurcates from the the positive equilibrium $E_0(s^*, i^*, y^*)$. We also showed if a certain condition is satisfied and $\tau_1 \in [0, \tau_{1_0})$, then the positive equilibrium $E_0(s^*, i^*, y^*)$ is asymptotically stable when $\tau_2 \in (0, \tau_{2_0})$, when the delay τ_2 increases, the positive equilibrium $E_0(s^*, i^*, y^*)$ loses its stability and a sequence of Hopf bifurcations occur at the positive equilibrium $E_0(s^*, i^*, y^*)$. Some numerical simulations verifying our theoretical results is performed. In addition, we must point out that although Song et al. [20] have also investigated the the existence of Hopf bifurcation for system (1.2) with respect to positive equilibrium $E_0(s^*, i^*, y^*)$, it is assumed that $\tau_1 + \tau_2 = \tau$. But what effect different time delay has on the dynamical behavior of system (1.2)? Song et al. [20] did not consider this issue. Thus we think that our work generalizes the known results of Song et al. [20]. In addition, we can investigate the Hopf bifurcation nature of system (1.2) by choosing the delay τ_1 or τ_2 as bifurcation parameter. We will further investigate the topic elsewhere in the near future.

Acknowledgments

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