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## Research Article

# Coupled Fixed-Point Theorems for Contractions in Partially Ordered Metric Spaces and Applications

**M. Eshaghi Gordji,<sup>1</sup> Y. J. Cho,<sup>2</sup> S. Ghods,<sup>3</sup>  
M. Ghods,<sup>4</sup> and M. Hadian Dehkordi<sup>4</sup>**

<sup>1</sup> Department of Mathematics, Semnan University, P.O. Box 35195-363, Semnan, Iran

<sup>2</sup> Department of Mathematics Education and the RINS, Gyeongsang National University,  
Chinju 660-701, Republic of Korea

<sup>3</sup> Department of Mathematics, Islamic Azad University, Semnan Branch, Semnan, Iran

<sup>4</sup> Department of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran

Correspondence should be addressed to M. Eshaghi Gordji, [madjid.eshaghi@gmail.com](mailto:madjid.eshaghi@gmail.com) and Y. J. Cho, [yjcho@gnu.ac.kr](mailto:yjcho@gnu.ac.kr)

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Bhaskar and Lakshmikantham (2006) showed the existence of coupled coincidence points of a mapping  $F$  from  $X \times X$  into  $X$  and a mapping  $g$  from  $X$  into  $X$  with some applications. The aim of this paper is to extend the results of Bhaskar and Lakshmikantham and improve the recent fixed-point theorems due to Bessem Samet (2010). Indeed, we introduce the definition of generalized  $g$ -Meir-Keeler type contractions and prove some coupled fixed point theorems under a generalized  $g$ -Meir-Keeler-contractive condition. Also, some applications of the main results in this paper are given.

## 1. Introduction

The Banach contraction principle [1] is a classical and powerful tool in nonlinear analysis and has been generalized by many authors (see [2–15] and others).

Recently, Bhaskar and Lakshmikantham [16] introduced the notion of a coupled fixed-point of the given two variables mapping. More precisely, let  $X$  be a nonempty set and  $F : X \times X \rightarrow X$  be a given mapping. An element  $(x, y) \in X \times X$  is called a *coupled fixed-point* of the mapping  $F$  if

$$F(x, y) = x, \quad F(y, x) = y. \quad (1.1)$$

They also showed the uniqueness of a coupled fixed-point of the mapping  $F$  and applied their theorems to the problems of the existence and uniqueness of a solution for a periodic boundary value problem.

**Theorem 1.1** (see Zeidler [15]). *Let  $(X, \leq)$  be a partially ordered set and suppose that there is a metric  $d$  on  $X$  such that  $(X, d)$  is a complete metric space. Let  $F : X \times X \rightarrow X$  be a continuous mapping having the mixed monotone property on  $X$ . Assume that there exists  $k \in [0, 1)$  such that*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)] \quad (1.2)$$

for all  $x \geq u$  and  $y \leq v$ . Moreover, if there exist  $x_0, y_0 \in X$  such that

$$x_0 \leq F(x_0, y_0), \quad y_0 \geq F(y_0, x_0), \quad (1.3)$$

then there exist  $x, y \in X$  such that  $x = F(x, y)$  and  $y = F(y, x)$ .

Later, in [17], Lakshmikantham and Ćirić investigated some more coupled fixed-point theorems in partially ordered sets, and some others obtained many results on coupled fixed-point theorems in cone metric spaces, intuitionistic fuzzy normed spaces, ordered cone metric spaces and topological spaces (see, e.g., [18–25]).

In [9], Meir and Keeler generalized the well-known Banach fixed-point theorem [1] as follows.

**Theorem 1.2** (Meir and Keeler [9]). *Let  $(X, d)$  be a complete metric space and  $T : X \rightarrow X$  be a given mapping. Suppose that, for any  $\epsilon > 0$ , there exists  $\delta(\epsilon) > 0$  such that*

$$\epsilon \leq d(x, y) < \epsilon + \delta(\epsilon) \implies d(T(x), T(y)) < \epsilon \quad (1.4)$$

for all  $x, y \in X$ . Then  $T$  admits a unique fixed-point  $x_0 \in X$  and, for all  $x \in X$ , the sequence  $\{T^n(x)\}$  converges to  $x_0$ .

**Proposition 1.3** (see [17]). *Let  $(X, d)$  be a partially ordered metric space and  $F : X \times X \rightarrow X$  be a given mapping. If the contraction (1.2) is satisfied, then  $F$  is a generalized Meir-Keeler type contraction.*

Motivated by the results of Bhaskar and Lakshmikantham [16], Lakshmikantham and Ćirić [17], and Samet [26], in this paper, we introduce the definition of  $g$ -Meir-Keeler-contractive mappings and prove some coupled fixed-point theorems under a generalized  $g$ -Meir-Keeler contractive condition.

## 2. Main Results

Let  $X$  be a nonempty set. We note that an element  $(x, y) \in X \times X$  is called a *coupled coincidence point* of a mapping  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  if  $F(x, y) = g(x)$  and  $F(y, x) = g(y)$  for all  $x, y \in X$ . Also, we say that  $F$  and  $g$  are *commutative* (or *commuting*) if  $g(F(x, y)) = F(g(x), g(y))$  for all  $x, y \in X$ .

We introduce the following two definitions.

*Definition 2.1.* Let  $(X, \leq)$  be a partially ordered set and  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$ . We say that  $F$  has the *mixed strict  $g$ -monotone property* if, for any  $x, y \in X$ ,

$$\begin{aligned} x_1, x_2 \in X, \quad g(x_1) < g(x_2) &\implies F(x_1, y) < F(x_2, y), \\ y_1, y_2 \in X, \quad g(y_1) < g(y_2) &\implies F(x, y_1) > F(x, y_2). \end{aligned} \quad (2.1)$$

*Definition 2.2.* Let  $(X, \leq)$  be a partially ordered set and  $d$  be a metric on  $X$ . Let  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  be two given mappings. We say that  $F$  is a *generalized  $g$ -Meir-Keeler type contraction* if, for all  $\epsilon > 0$ , there exists  $\delta(\epsilon) > 0$  such that, for all  $x, y, u, v \in X$  with  $g(x) \leq g(u)$  and  $g(y) \geq g(v)$ ,

$$\epsilon \leq \frac{1}{2} [d(g(x), g(u)) + d(g(y), g(v))] < \epsilon + \delta(\epsilon) \implies d(F(x, y), F(u, v)) < \epsilon. \quad (2.2)$$

**Lemma 2.3.** Let  $(X, \leq)$  be a partially ordered set and  $d$  be a metric on  $X$ . Let  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  be two given mappings. If  $F$  is a generalized  $g$ -Meir-Keeler type contraction, then we have

$$d(F(x, y), F(u, v)) < \frac{1}{2} [d(g(x), g(u)) + d(g(y), g(v))] \quad (2.3)$$

for all  $x, y, u, v$  with  $g(x) < g(u), g(y) \geq g(v)$  or  $g(x) \leq g(u), g(y) > g(v)$ .

*Proof.* Let  $x, y, u, v \in X$  such that  $g(x) < g(u)$  and  $g(y) \geq g(v)$  or  $g(x) \leq g(u)$  and  $g(y) > g(v)$ . Then  $d(g(x), g(u)) + d(g(y), g(v)) > 0$ . Since  $F$  is a generalized  $g$ -Meir-Keeler type contraction, for  $\epsilon = (1/2)[d(g(x), g(u)) + d(g(y), g(v))]$ , there exists  $\delta(\epsilon) > 0$  such that, for all  $x_0, y_0, u_0, v_0 \in X$  with  $g(x_0) \leq g(u_0)$  and  $g(y_0) \geq g(v_0)$ ,

$$\epsilon \leq \frac{1}{2} [d(g(x_0), g(u_0)) + d(g(y_0), g(v_0))] < \epsilon + \delta(\epsilon) \implies d(F(x_0, y_0), F(u_0, v_0)) < \epsilon. \quad (2.4)$$

Therefore, putting  $x_0 = x, y_0 = y, u_0 = u$  and  $v_0 = v$ , we have

$$d(F(x, y), F(u, v)) < \frac{1}{2} [d(g(x), g(u)) + d(g(y), g(v))]. \quad (2.5)$$

This completes the proof.  $\square$

From now on, we suppose that  $(X, \leq)$  is a partially ordered set, and there exists a metric  $d$  on  $X$  such that  $(X, d)$  is a complete metric space.

**Theorem 2.4.** Let  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  be such that  $F(X \times X) \subseteq g(X)$ ,  $g$  is continuous and commutative with  $F$ . Also, suppose that

- (a)  $F$  has the mixed strict  $g$ -monotone property;
- (b)  $F$  is a generalized  $g$ -Meir-keeler type contraction;
- (c) there exist  $x_0, y_0 \in X$  such that  $g(x_0) < F(x_0, y_0)$  and  $g(y_0) > F(y_0, x_0)$ .

Then there exist  $x, y \in X$  such that  $g(x) = F(x, y)$  and  $g(y) = F(y, x)$ ; that is,  $F$  and  $g$  have a coupled coincidence in  $X \times X$ .

*Proof.* Let  $x_0, y_0 \in X$  be such that  $g(x_0) < F(x_0, y_0)$  and  $g(y_0) > F(y_0, x_0)$ . Since  $F(X \times X) \subseteq g(X)$ , we can choose  $x_1, y_1 \in X$  such that  $g(x_1) = F(x_0, y_0)$  and  $g(y_1) = F(y_0, x_0)$ . Again, from  $F(X \times X) \subseteq g(X)$ , we can choose  $x_2, y_2 \in X$  such that  $g(x_2) = F(x_1, y_1)$  and  $g(y_2) = F(y_1, x_1)$ .

Continuing this process, we can construct the sequences  $\{x_n\}$  and  $\{y_n\}$  in  $X$  such that

$$g(x_{n+1}) = F(x_n, y_n), \quad g(y_{n+1}) = F(y_n, x_n) \quad (2.6)$$

for all  $n \geq 0$ .

Now, we show that

$$g(x_n) < g(x_{n+1}), \quad g(y_n) > g(y_{n+1}) \quad (2.7)$$

for all  $n \geq 0$ . For  $n = 0$ , we have

$$g(x_0) < F(x_0, y_0) = g(x_1), \quad g(y_0) > F(y_0, x_0) = g(y_1). \quad (2.8)$$

Since  $F$  has the mixed strict  $g$ -monotone property, then we have

$$\begin{aligned} g(x_0) < g(x_1) &\implies F(x_0, y_1) < F(x_1, y_1), \\ g(y_0) > g(y_1) &\implies F(x_0, y_0) < F(x_0, y_1). \end{aligned} \quad (2.9)$$

It follows that  $F(x_0, y_0) < F(x_1, y_1)$ , that is,  $g(x_1) < g(x_2)$ .

Similarly, we have

$$\begin{aligned} g(y_1) < g(y_0) &\implies F(y_1, x_0) < F(y_0, x_0), \\ g(x_1) > g(x_0) &\implies F(y_1, x_1) < F(y_1, x_0). \end{aligned} \quad (2.10)$$

Thus it follows that  $F(y_1, x_1) < F(y_0, x_0)$ , that is,  $g(y_2) < g(y_1)$ .

Again, we have

$$\begin{aligned} g(x_1) < g(x_2) &\implies F(x_1, y_2) < F(x_2, y_2), \\ g(y_1) > g(y_2) &\implies F(x_1, y_1) < F(x_1, y_2). \end{aligned} \quad (2.11)$$

Thus it follows that  $F(x_1, y_1) < F(x_2, y_2)$ , that is,  $g(x_2) < g(x_3)$ .

Similarly, we have

$$\begin{aligned} g(y_2) < g(y_1) &\implies F(y_2, x_1) < F(y_1, x_1), \\ g(x_2) > g(x_1) &\implies F(y_2, x_2) < F(y_2, x_1). \end{aligned} \quad (2.12)$$

Thus it follows that  $F(y_2, x_2) < F(y_1, x_1)$ , that is,  $g(y_3) < g(y_2)$ .

Continuing this process for each  $n \geq 1$ , we get the following:

$$\begin{aligned} g(x_0) < g(x_1) < g(x_2) < \cdots < g(x_n) < g(x_{n+1}) < \cdots, \\ g(y_0) > g(y_1) > g(y_2) > \cdots > g(y_n) > g(y_{n+1}) > \cdots. \end{aligned} \quad (2.13)$$

Denote that

$$\delta_n := d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})). \quad (2.14)$$

Since  $g(x_{n-1}) < g(x_n)$  and  $g(y_{n-1}) > g(y_n)$ , it follows from (2.6) and Lemma 2.3 that

$$\begin{aligned} d(g(x_n), g(x_{n+1})) &= d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &< \frac{1}{2} [d(g(x_{n-1}), g(x_n)) + d(g(y_{n-1}), g(y_n))]. \end{aligned} \quad (2.15)$$

Since  $g(y_n) < g(y_{n-1})$  and  $g(x_n) > g(x_{n-1})$ , it follows from (2.6) and Lemma 2.3 that

$$\begin{aligned} d(g(y_{n+1}), g(y_n)) &= d(F(y_n, x_n), F(y_{n-1}, x_{n-1})) \\ &< \frac{1}{2} [d(g(y_n), g(y_{n-1})) + d(g(x_n), g(x_{n-1}))]. \end{aligned} \quad (2.16)$$

Thus it follows from (2.14)–(2.16) that  $\delta_n < \delta_{n-1}$ . This means that the sequence  $\{\delta_n/2\}$  is monotone decreasing. Therefore, there exists  $\delta^* \geq 0$  such that  $\lim_{n \rightarrow \infty} \delta_n/2 = \delta^*$ , that is,

$$\lim_{n \rightarrow \infty} \frac{1}{2} [d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1}))] = \delta^*. \quad (2.17)$$

Now, we show that  $\delta^* = 0$ . Suppose that  $\delta^* > 0$  hold. Let  $\delta^* = \epsilon$ . Then there exists a positive integer  $m$  such that

$$\epsilon \leq \frac{1}{2} [d(g(x_m), g(x_{m+1})) + d(g(y_m), g(y_{m+1}))] < \epsilon + \delta(\epsilon). \quad (2.18)$$

Then, by using (2.7) and the condition (b), we have

$$d(F(x_m, y_m), F(x_{m+1}, y_{m+1})) < \epsilon, \quad (2.19)$$

and so, by (2.6), we have

$$d(g(x_{m+1}), g(x_{m+2})) < \epsilon. \quad (2.20)$$

On the other hand, by (2.15), we have

$$\frac{1}{2} [d(g(x_m), g(x_{m+1})) + d(g(y_m), g(y_{m+1}))] < \epsilon, \quad (2.21)$$

which is a contradiction with (2.18). Thus we have  $\epsilon = \delta^* = 0$ , that is,

$$\lim_{n \rightarrow \infty} \frac{1}{2} [d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1}))] = 0, \quad (2.22)$$

that is,

$$\lim_{n \rightarrow \infty} \delta_n = 0. \quad (2.23)$$

Now, we prove that  $\{g(x_n)\}$  and  $\{g(y_n)\}$  are Cauchy sequences in  $X$ . Suppose that at least one of  $\{g(x_n)\}$  or  $\{g(y_n)\}$  is not a Cauchy sequence. Then there exist  $\epsilon > 0$  and two subsequences  $\{l_k\}$ ,  $\{m_k\}$  of integers such that  $m_k > l_k \geq k$  and

$$d(g(x_{l_k}), g(x_{m_k})) \geq \frac{\epsilon}{2}, \quad d(g(y_{l_k}), g(y_{m_k})) \geq \frac{\epsilon}{2} \quad (2.24)$$

for all  $k \geq 1$ . Thus we have

$$r_k = d(g(x_{l_k}), g(x_{m_k})) + d(g(y_{l_k}), g(y_{m_k})) \geq \epsilon \quad (2.25)$$

for all  $k \geq 1$ . Let  $m_k$  be the smallest number exceeding  $l_k$  such that (2.25) holds. Then we have

$$d(g(x_{l_k}), g(x_{m_k-1})) + d(g(y_{l_k}), g(y_{m_k-1})) < \epsilon. \quad (2.26)$$

Thus, from (2.14), (2.25), (2.26) and the triangle inequality, it follows that

$$\begin{aligned} \epsilon &\leq r_k \\ &\leq d(g(x_{l_k}), g(x_{m_k-1})) + d(g(x_{m_k-1}), g(x_{m_k})) \\ &\quad + d(g(y_{l_k}), g(y_{m_k-1})) + d(g(y_{m_k-1}), g(y_{m_k})) \\ &< \epsilon + \delta_{m_k-1} \end{aligned} \quad (2.27)$$

and so

$$\epsilon \leq \lim_{k \rightarrow \infty} r_k \leq \lim_{k \rightarrow \infty} (\epsilon + \delta_{m_k-1}). \quad (2.28)$$

Hence, by (2.23), we have

$$\lim_{k \rightarrow \infty} r_k = \epsilon^+. \quad (2.29)$$

It follows from (2.6), (2.14), and the triangle inequality that

$$\begin{aligned} r_k &= d(g(x_{l_k}), g(x_{m_k})) + d(g(y_{l_k}), g(y_{m_k})) \\ &\leq d(g(x_{l_k}), g(x_{l_{k+1}})) + d(g(x_{l_{k+1}}), g(x_{m_{k+1}})) + d(g(x_{m_{k+1}}), g(x_{m_k})) \\ &\quad + d(g(y_{l_k}), g(y_{l_{k+1}})) + d(g(y_{l_{k+1}}), g(y_{m_{k+1}})) + d(g(y_{m_{k+1}}), g(y_{m_k})) \\ &= \delta_{l_k} + \delta_{m_k} + d(g(x_{l_{k+1}}), g(x_{m_{k+1}})) + d(g(y_{l_{k+1}}), g(y_{m_{k+1}})) \\ &= \delta_{l_k} + \delta_{m_k} + d(F(x_{l_k}, y_{l_k}), F(x_{m_k}, y_{m_k})) + d(F(y_{l_k}, x_{l_k}), F(y_{m_k}, x_{m_k})). \end{aligned} \quad (2.30)$$

Form (2.13) we have  $g(x_{l_k}) < g(x_{m_k})$  and  $g(y_{l_k}) > g(y_{m_k})$ . Now, it follows from Lemma 2.3 and (2.30) that

$$r_k < \delta_{l_k} + \delta_{m_k} + d(g(x_{l_k}), g(x_{m_k})) + d(g(y_{l_k}), g(y_{m_k})), \quad (2.31)$$

that is,

$$r_k < \delta_{l_k} + \delta_{m_k} + r_k. \quad (2.32)$$

This is a contradiction. Therefore,  $\{g(x_n)\}$  and  $\{g(y_n)\}$  are Cauchy sequences. Since  $X$  is complete, there exist  $x, y \in X$  such that

$$\lim_{n \rightarrow \infty} g(x_n) = x, \quad \lim_{n \rightarrow \infty} g(y_n) = y. \quad (2.33)$$

Since  $\{g(x_n)\}$  is monotone increasing and  $\{g(y_n)\}$  is monotone decreasing, we have

$$g(x_n) < x, \quad g(y_n) > y \quad (2.34)$$

for all  $n \geq 1$ . Thus it follows from (2.33) and the continuity of  $g$  that

$$\lim_{n \rightarrow \infty} g(g(x_n)) = g(x), \quad \lim_{n \rightarrow \infty} g(g(y_n)) = g(y). \quad (2.35)$$

Thus, for all  $m \geq 1$ , there exists a positive integer  $n_0$  such that, for all  $n \geq n_0$ ,

$$d(g(g(x_n)), g(x)) < \frac{1}{4m}, \quad d(g(g(y_n)), g(y)) < \frac{1}{4m}. \quad (2.36)$$

Hence, from (2.6), the commutativity of  $F$  and  $g$  and the triangle inequality, we have

$$\begin{aligned} d(F(x, y), g(x)) &\leq d(F(x, y), g(g(x_n))) + d(g(g(x_n)), g(x)) \\ &= d(F(x, y), g(F(x_{n-1}, y_{n-1}))) + d(g(g(x_n)), g(x)) \\ &= d(F(x, y), F(g(x_{n-1}), g(y_{n-1}))) + d(g(g(x_n)), g(x)). \end{aligned} \quad (2.37)$$

Thus, it follows from (2.34), (2.36), and Lemma 2.3 that

$$\begin{aligned} d(F(x, y), g(x)) &< \frac{1}{2} [d(g(g(x_{n-1})), g(x)) + d(g(g(y_{n-1})), g(y))] + d(g(g(x_n)), g(x)) \\ &< \frac{1}{8m} + \frac{1}{8m} + \frac{1}{4m} \\ &= \frac{1}{2m} \rightarrow 0 \end{aligned} \quad (2.38)$$

as  $m \rightarrow \infty$ . Therefore, we have  $F(x, y) = g(x)$ . Similarly, we can show that  $F(y, x) = g(y)$ . This means that  $F$  and  $g$  have a coupled coincidence point in  $X \times X$ . This completes the proof.  $\square$

**Corollary 2.5.** *Let  $F : X \times X \rightarrow X$  be a mapping satisfying the following conditions:*

- (a)  $F$  has the mixed strict monotone property;
- (b)  $F$  is a generalized Meir-Keeler type contraction;
- (c) there exists  $x_0, y_0 \in X$  such that  $x_0 < F(x_0, y_0)$  and  $y_0 > F(y_0, x_0)$ .

Then there exist  $x, y \in X$  such that  $x = F(x, y)$  and  $y = F(y, x)$ .

*Proof.* The conclusion follows from Theorem 2.4 by putting  $g = I$  (: the identity mapping) on  $X$ .  $\square$

Now, we introduce the product space  $X \times X$  with the following partial order: for all  $(x, y), (u, v) \in X \times X$ ,

$$(u, v) \leq (x, y) \iff u < x, \quad v \geq y. \quad (2.39)$$

**Theorem 2.6.** *Suppose that all the hypotheses of Theorem 2.4 hold and, further, for all  $(x, y), (x^*, y^*) \in X \times X$ , there exists  $(u, v) \in X \times X$  such that  $(F(u, v), F(v, u))$  is comparable with  $(F(x, y), F(y, x))$  and  $(F(x^*, y^*), F(y^*, x^*))$ . Then  $F$  and  $g$  have a unique coupled common fixed-point, that is, there exists a unique  $(x, y) \in X \times X$  such that*

$$x = g(x) = F(x, y), \quad y = g(y) = F(y, x). \quad (2.40)$$



*Proof.* By Theorem 2.4, the set of coupled coincidences of the mapping  $F$  and  $g$  is nonempty.

First, we show that, if  $(x, y)$  and  $(x^*, y^*)$  are coupled coincidence points of  $F$  and  $g$ , that is, if

$$g(x) = F(x, y), \quad g(y) = F(y, x), \quad g(x^*) = F(x^*, y^*), \quad g(y^*) = F(y^*, x^*), \quad (2.41)$$

then we have

$$g(x) = g(x^*), \quad g(y) = g(y^*). \quad (2.42)$$

Put  $u_0 = u, v_0 = v$  and choose  $u_1, v_1 \in X$  such that  $g(u_1) = F(u_0, v_0)$  and  $g(v_1) = F(v_0, u_0)$ . Then, similarly as in the proof of Theorem 2.4, we can inductively define the sequences  $\{g(u_n)\}$  and  $\{g(v_n)\}$  such that

$$g(u_{n+1}) = F(u_n, v_n), \quad g(v_{n+1}) = F(v_n, u_n) \quad (2.43)$$

for all  $n \geq 0$ . Also, if we set  $x_0 = x, y_0 = y, x_0^* = x^*,$  and  $y_0^* = y^*$ , then we can define the sequences  $\{g(x_n)\}, \{g(y_n)\}, \{g(x_n^*)\},$  and  $\{g(y_n^*)\}$  as follows:

$$\begin{aligned} g(x_{n+1}) &= F(x_n, y_n), & g(y_{n+1}) &= F(y_n, x_n), \\ g(x_{n+1}^*) &= F(x_n^*, y_n^*), & g(y_{n+1}^*) &= F(y_n^*, x_n^*) \end{aligned} \quad (2.44)$$

for all  $n \geq 0$ . Since

$$\begin{aligned} (F(x, y), F(y, x)) &= (g(x_1), g(y_1)) = (g(x), g(y)), \\ (F(u, v), F(v, u)) &= (g(u_1), g(v_1)) \end{aligned} \quad (2.45)$$

are comparable each other, then  $g(x) < g(u_1)$  and  $g(y) \geq g(v_1)$ . It is easy to show that  $(g(x), g(y))$ , and  $(g(u_n), g(v_n))$  are comparable each other, that is,  $g(x) < g(u_n)$  and  $g(y) \geq g(v_n)$  for all  $n \geq 1$ . Thus it follows from Lemma 2.3 that

$$\begin{aligned} &d(g(x), g(u_{n+1})) + d(g(y), g(v_{n+1})) \\ &= d(F(x, y), F(u_n, v_n)) + d(F(y, x), F(v_n, u_n)) \\ &< \frac{1}{2} [d(g(x), g(u_n)) + d(g(y), g(v_n))] + \frac{1}{2} [d(g(y), g(v_n)) + d(g(x), g(u_n))] \\ &= d(g(x), g(u_n)) + d(g(y), g(v_n)) \end{aligned} \quad (2.46)$$

and so

$$\frac{1}{2} [d(g(x), g(u_{n+1})) + d(g(y), g(v_{n+1}))] < \frac{1}{2^n} [d(g(x), g(u_1)) + d(g(y), g(v_1))] \longrightarrow 0 \quad (2.47)$$

as  $n \rightarrow \infty$ . Therefore, we have

$$\lim_{n \rightarrow \infty} d(g(x), g(u_{n+1})) = 0, \quad \lim_{n \rightarrow \infty} d(g(y), g(v_{n+1})) = 0. \quad (2.48)$$

Similarly, we can prove that

$$\lim_{n \rightarrow \infty} d(g(x^*), g(u_{n+1})) = 0, \quad \lim_{n \rightarrow \infty} d(g(y^*), g(v_{n+1})) = 0. \quad (2.49)$$

Thus, by the triangle inequality, (2.48) and (2.49), we have

$$\begin{aligned} d(g(x), g(x^*)) &\leq d(g(x), g(u_{n+1})) + d(g(x^*), g(u_{n+1})) \rightarrow 0, \\ d(g(y), g(y^*)) &\leq d(g(y), g(v_{n+1})) + d(g(y^*), g(v_{n+1})) \rightarrow 0 \end{aligned} \quad (2.50)$$

as  $n \rightarrow \infty$ , which imply that  $g(x) = g(x^*)$  and  $g(y) = g(y^*)$ .

Now, we prove that  $g(x) = x$  and  $g(y) = y$ . Denote that  $g(x) = z$  and  $g(y) = w$ . Since  $g(x) = F(x, y)$  and  $g(y) = F(y, x)$ , by the commutativity of  $F$  and  $g$ , we have

$$g(z) = g(g(x)) = g(F(x, y)) = F(g(x), g(y)) = F(z, w), \quad (2.51)$$

$$g(w) = g(g(y)) = g(F(y, x)) = F(g(y), g(x)) = F(w, z). \quad (2.52)$$

Thus,  $(z, w)$  is a coupled coincidence point of  $F$  and  $g$ .

Putting  $x^* = z$  and  $y^* = w$  in (2.52), it follows from (2.42) that

$$z = g(x) = g(x^*) = g(z), \quad w = g(y) = g(y^*) = g(w) \quad (2.53)$$

and so, from (2.51) and (2.52),

$$z = g(z) = F(z, w), \quad w = g(w) = F(w, z). \quad (2.54)$$

Therefore,  $(z, w)$  is a coupled common fixed-point of  $F$  and  $g$ .

Finally, to prove the uniqueness of the coupled common fixed-point of  $F$  and  $g$ , assume that  $(p, q)$  is another coupled common fixed-point of  $F$  and  $g$ . Then, by (2.42), we have  $p = g(p) = g(z) = z$  and  $q = g(q) = g(w) = w$ . This completes the proof.  $\square$

**Corollary 2.7.** *Suppose that all the hypotheses of Corollary 2.5 hold and, further, for all  $(x, y)$  and  $(x^*, y^*) \in X \times X$ , there exists  $(u, v) \in X \times X$  that is comparable with  $(x, y)$  and  $(x^*, y^*)$ . Then there exists a unique  $x \in X$  such that  $x = F(x, x)$ .*

### 3. Applications

Now, we give some applications of the main results in Section 2.

**Theorem 3.1.** *Let  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  be two given mappings. Assume that there exists a function  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  satisfying the following conditions:*

- (a)  $\varphi(0) = 0$  and  $\varphi(t) > 0$  for any  $t > 0$ ;
- (b)  $\varphi$  is nondecreasing and right continuous;
- (c) for any  $\epsilon > 0$ , there exists  $\delta(\epsilon) > 0$  such that, for all  $x, y, u, v \in X$  with  $g(x) \leq g(u)$  and  $g(y) \geq g(v)$ ,

$$\epsilon \leq \varphi\left(\frac{1}{2}[d(g(x), g(u)) + d(g(y), g(v))]\right) < \epsilon + \delta(\epsilon) \implies \varphi[d(F(x, y), F(u, v))] < \epsilon. \quad (3.1)$$

Then  $F$  is a generalized  $g$ -Meir-Keeler type contraction.

*Proof.* For any  $\epsilon > 0$ , it follows from (a) that  $\varphi(\epsilon) > 0$  and so there exists  $\alpha > 0$  such that, for all  $u, v, u^*, v^* \in X$  with  $g(u) \leq g(u^*)$  and  $g(v) \geq g(v^*)$ ,

$$\begin{aligned} \varphi(\epsilon) &\leq \varphi\left(\frac{1}{2}[d(g(u), g(u^*)) + d(g(v), g(v^*))]\right) < \varphi(\epsilon) + \alpha \\ &\implies \varphi[d(F(u, v), F(u^*, v^*))] < \varphi(\epsilon). \end{aligned} \quad (3.2)$$

From the right continuity of  $\varphi$ , there exists  $\delta > 0$  such that  $\varphi(\epsilon + \delta) < \varphi(\epsilon) + \alpha$ . For any  $x, y, u, v \in X$  such that  $g(x) \leq g(u)$ ,  $g(y) \geq g(v)$  and

$$\epsilon \leq \frac{1}{2}[d(g(x), g(u)) + d(g(y), g(v))] < \epsilon + \delta, \quad (3.3)$$

since  $\varphi$  is nondecreasing function, we get the following:

$$\varphi(\epsilon) \leq \varphi\left(\frac{1}{2}[d(g(x), g(u)) + d(g(y), g(v))]\right) < \varphi(\epsilon + \delta) < \varphi(\epsilon) + \alpha. \quad (3.4)$$

By (3.2), we have  $\varphi[d(F(x, y), F(u, v))] < \varphi(\epsilon)$  and so  $d(F(x, y), F(u, v)) < \epsilon$ . Therefore, it follows that  $F$  is a generalized  $g$ -Meir-Keeler type contraction. This completes the proof.  $\square$

**Corollary 3.2** (see [26, Theorem 3.1]). *Let  $F : X \times X \rightarrow X$  be a given mapping. Assume that there exists a function  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  satisfying the following conditions:*

- (a)  $\varphi(0) = 0$  and  $\varphi(t) > 0$  for any  $t > 0$ ;
- (b)  $\varphi$  is nondecreasing and right continuous;

(c) for any  $\epsilon > 0$ , there exists  $\delta(\epsilon) > 0$  such that  $x \leq u$ ,  $y \geq v$  and

$$\epsilon \leq \varphi\left(\frac{1}{2}[d(x, u) + d(y, v)]\right) < \epsilon + \delta(\epsilon) \implies \varphi[d(F(x, y), F(u, v))] < \epsilon. \quad (3.5)$$

Then  $F$  is a generalized Meir-Keeler type contraction.

The following result is an immediate consequence of Theorems 2.4 and 3.1.

**Corollary 3.3.** Let  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  be two given mappings such that  $F(X \times X) \subseteq g(X)$ ,  $g$  is continuous and commutative with  $F$ . Also, suppose that

- (a)  $F$  has the mixed strict  $g$ -monotone property;
- (b) for any  $\epsilon > 0$ , there exists  $\delta(\epsilon) > 0$  such that, for all  $x, y, u, v \in X$  with  $g(x) \leq g(u)$  and  $g(y) \geq g(v)$ ,

$$\epsilon \leq \int_0^{(1/2)[d(g(x), g(u)) + d(g(y), g(v))]} \varphi(t) dt < \epsilon + \delta(\epsilon) \implies \int_0^{d(F(x, y), F(u, v))} \varphi(t) dt < \epsilon, \quad (3.6)$$

where  $\varphi$  is a locally integrable function from  $[0, +\infty)$  into itself satisfying the following condition:

$$\int_0^s \varphi(t) dt > 0 \quad (3.7)$$

for all  $s > 0$ ;

- (c) there exist  $x_0, y_0 \in X$  such that  $g(x_0) < F(x_0, y_0)$  and  $g(y_0) > F(y_0, x_0)$ .

Then there exists  $(x, y) \in X \times X$  such that  $g(x) = F(x, y)$  and  $g(y) = F(y, x)$ . Moreover, if  $g(x_0)$  and  $g(y_0)$  are comparable to each other, then  $F$  and  $g$  have a unique coupled common fixed-point in  $X \times X$ .

**Corollary 3.4.** Let  $F : X \times X \rightarrow X$  be a mapping satisfying the following conditions:

- (a)  $F$  has the mixed strict monotone property;
- (b) for any  $\epsilon > 0$ , there exists  $\delta(\epsilon) > 0$  such that  $x \leq u$ ,  $y \geq v$  and

$$\epsilon \leq \int_0^{(1/2)[d(x, u) + d(y, v)]} \varphi(t) dt < \epsilon + \delta(\epsilon) \implies \int_0^{[d(F(x, y), F(u, v))]} \varphi(t) dt < \epsilon, \quad (3.8)$$

where  $\varphi$  is a locally integrable function from  $[0, +\infty)$  into itself satisfying

$$\int_0^s \varphi(t) dt > 0 \quad (3.9)$$

for all  $s > 0$ ;

(c) there exist  $x_0, y_0 \in X$  such that  $x_0 < F(x_0, y_0)$  and  $y_0 > F(y_0, x_0)$ .

Then there exists  $(x, y) \in X \times X$  such that  $x = F(x, y)$  and  $y = F(y, x)$ . Moreover, if  $x_0$  and  $y_0$  are comparable to each other, then  $F$  has a unique coupled common fixed-point in  $X \times X$ .

**Corollary 3.5.** Let  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  be two given mappings such that  $F(X \times X) \subseteq g(X)$ ,  $g$  is continuous and commutes with  $F$ . Also, suppose that

(a)  $F$  has the mixed strict  $g$ -monotone property;

(b) for any  $x, y, u, v \in X$  with  $g(x) \leq g(u)$  and  $g(y) \geq g(v)$ ,

$$\int_0^{[d(F(x,y), F(u,v))]} \varphi(t) dt \leq k \int_0^{(1/2)[d(g(x), g(u)) + d(g(y), g(v))]} \varphi(t) dt, \quad (3.10)$$

where  $k \in (0, 1)$  and  $\varphi$  is a locally integrable function from  $[0, +\infty)$  into itself satisfying

$$\int_0^s \varphi(t) dt > 0 \quad (3.11)$$

for all  $s > 0$ ;

(c) there exist  $x_0, y_0 \in X$  such that  $g(x_0) < F(x_0, y_0)$  and  $g(y_0) > F(y_0, x_0)$ .

Then there exists  $(x, y) \in X \times X$  such that  $g(x) = F(x, y)$  and  $g(y) = F(y, x)$ . Moreover, if  $g(x_0)$  and  $g(y_0)$  are comparable to each other, then  $F$  and  $g$  have a unique coupled common fixed-point in  $X \times X$ .

*Proof.* For any  $\epsilon > 0$ , if we take  $\delta(\epsilon) = (1/k - 1)\epsilon$  and apply Corollary 3.3, then we can get the conclusion.  $\square$

**Corollary 3.6.** Let  $F : X \times X \rightarrow X$  be a mapping satisfying the following conditions:

(a)  $F$  has the mixed strict monotone property,

(b) for any  $x, y, u, v \in X$  with  $x \leq u$  and  $y \geq v$ ,

$$\int_0^{d(F(x,y), F(u,v))} \varphi(t) dt \leq k \int_0^{(1/2)[d(x,u) + d(y,v)]} \varphi(t) dt, \quad (3.12)$$

where  $k \in (0, 1)$  and  $\varphi$  is a locally integrable function from  $[0, +\infty)$  into itself satisfying

$$\int_0^s \varphi(t) dt > 0 \quad (3.13)$$

for all  $s > 0$ ;

(c) there exist  $x_0, y_0 \in X$  such that  $x_0 < F(x_0, y_0)$  and  $y_0 > F(y_0, x_0)$ .

Then there exist  $x, y \in X$  such that  $x = F(x, y)$  and  $y = F(y, x)$ . Moreover, if  $x_0$  and  $y_0$  are comparable to each other, then  $F$  has a unique coupled common fixed-point in  $X \times X$ .

Finally, by using the above results, we show the existence of solutions for the following integral equation:

$$(x(t), y(t)) = \left( \int_0^T G(t, s) [(f(s, x(s)) + \lambda x(s)) - (f(s, y(s)) + \lambda y(s))] ds, \right. \\ \left. \int_0^T G(t, s) [(f(s, y(s)) + \lambda y(s)) - (f(s, x(s)) + \lambda x(s))] ds \right), \quad (3.14)$$

where  $x, y \in C(I, \mathbb{R})$  (: the set of continuous functions from  $I$  into  $\mathbb{R}$ ),  $T > 0$ ,  $f : I \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function and

$$G(t, s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1}, & \text{if } 0 \leq s < t \leq T; \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1}, & \text{if } 0 \leq t < s \leq T. \end{cases} \quad (3.15)$$

*Definition 3.7.* A lower solution for the integral equation (3.14) is an element  $(\alpha, \beta) \in C^1(I, \mathbb{R}) \times C^1(I, \mathbb{R})$  such that

$$\begin{aligned} \alpha'(t) + \lambda\beta(t) &\leq f(t, \alpha(t)) - f(t, \beta(t)), & \alpha(0) < \alpha(T), \\ \beta'(t) + \lambda\alpha(t) &\geq f(t, \beta(t)) - f(t, \alpha(t)), & \beta(0) \geq \beta(T), \end{aligned} \quad (3.16)$$

where  $C^1(I, \mathbb{R})$  denotes the set of differentiable functions from  $I$  into  $\mathbb{R}$ .

Now, we prove the existence of solutions for the integral equation (3.14) by using the existence of a lower solution for the integral equation (3.14).

**Theorem 3.8.** Let  $\mathcal{A}$  be the class of the functions  $\varphi : [0, \infty) \rightarrow [0, \infty)$  satisfying the following conditions:

- (a)  $\varphi$  is increasing;
- (b) for any  $x \geq 0$ , there exists  $k \in [0, 1)$  such that  $\varphi(x) < (k/2)x$ .

In the integral equation (3.14), suppose that there exists  $\lambda > 0$  such that, for all  $x, y \in \mathbb{R}$  with  $y > x$ ,

$$0 < f(t, y) + \lambda y - [f(t, x) + \lambda x] \leq \lambda \varphi(y - x), \quad (3.17)$$

where  $\varphi \in \mathcal{A}$ . If a lower solution of the integral equation (3.14) exists, then a solution of the integral equation (3.14) exists.

*Proof.* Define a mapping  $F : C(I, \mathbb{R}) \times C(I, \mathbb{R}) \rightarrow C(I, \mathbb{R})$  by

$$F(x(t), y(t)) = \int_0^T G(t, s) [(f(s, x(s)) + \lambda x(s)) - (f(s, y(s)) + \lambda y(s))] ds. \quad (3.18)$$

Note that, if  $(x(t), y(t)) \in C(I, \mathbb{R}) \times C(I, \mathbb{R})$  is a coupled fixed-point of  $F$ , then  $(x(t), y(t))$  is a solution of the integral equation (3.14).

Now, we check the hypotheses in Corollary 2.5 as follows:

- (1)  $X \times X = C(I, \mathbb{R}) \times C(I, \mathbb{R})$  is a partially ordered set if we define the order relation in  $X \times X$  as follows:

$$(u(t), v(t)) \leq (x(t), y(t)) \quad \text{iff } u(t) < x(t), \quad v(t) \geq y(t) \quad (3.19)$$

for all  $(x(t), y(t)), (u(t), v(t)) \in X \times X$  and  $t \in I$ .

- (2)  $(X, d)$  is a complete metric space if we define a metric  $d$  as follows:

$$d(x(t), y(t)) = \sup_{t \in I} \{|x(t) - y(t)| : x(t), y(t) \in X\}. \quad (3.20)$$

- (3) The mapping  $F$  has the mixed strict monotone property. In fact, by hypothesis, if  $x_2 > x_1$ , then we have

$$f(t, x_2) + \lambda x_2 > f(t, x_1) + \lambda x_1, \quad (3.21)$$

which implies that, for any  $t \in I$ ,

$$\begin{aligned} & \int_0^T [f(s, x_2(s)) + \lambda x_2(s) - f(s, y(s)) - \lambda y(s)] G(t, s) ds \\ & > \int_0^T [f(s, x_1(s)) + \lambda x_1(s) - f(s, y(s)) - \lambda y(s)] G(t, s) ds, \end{aligned} \quad (3.22)$$

that is,

$$F(x_2(t), y(t)) > F(x_1(t), y(t)). \quad (3.23)$$

Similarly, if  $y_1 < y_2$ , then we have

$$f(t, y_2) + \lambda y_2 > f(t, y_1) + \lambda y_1, \quad (3.24)$$

which implies that, for any  $t \in I$ ,

$$\begin{aligned} & \int_0^T [f(s, x(s)) + \lambda x(s) - f(s, y_2(s)) - \lambda y_2(s)] G(t, s) ds \\ & < \int_0^T [f(s, x(s)) + \lambda x(s) - f(s, y_1(s)) - \lambda y_1(s)] G(t, s) ds, \end{aligned} \quad (3.25)$$

that is,

$$F(x(t), y_2(t)) < F(x(t), y_1(t)). \quad (3.26)$$

Now, we show that  $F$  satisfies (1.2). In fact, let  $(x, y) \leq (u, v)$  and  $t \in I$ . Then we have

$$\begin{aligned} & d(F(x(t), y(t)), F(u(t), v(t))) \\ & = \sup\{|F(x(t), y(t)) - F(u(t), v(t))| : t \in I\} \\ & = \sup_{t \in I} \left\{ \left| \int_0^T G(t, s) [f(s, x(s)) + \lambda x(s) - f(s, y(s)) - \lambda y(s)] ds \right. \right. \\ & \quad \left. \left. - \int_0^T G(t, s) [f(s, u(s)) + \lambda u(s) - f(s, v(s)) - \lambda v(s)] ds \right| \right\} \\ & \leq \sup_{t \in I} \int_0^T G(t, s) |f(s, x(s)) + \lambda x(s) - f(s, u(s)) - \lambda u(s) \\ & \quad + f(s, v(s)) + \lambda v(s) - f(s, y(s)) - \lambda y(s)| ds. \end{aligned} \quad (3.27)$$

Since the function  $\varphi(x)$  is increasing and  $(x, y) \leq (u, v)$ , we have

$$\varphi(x(s) - u(s)) \leq \varphi(d(x(s), u(s))), \quad \varphi(v(s) - y(s)) \leq \varphi(d(v(s), y(s))), \quad (3.28)$$



we obtain the following:

$$\begin{aligned}
& d(F(x(t), y(t)), F(u(t), v(t))) \\
& \leq \sup_{t \in I} \int_0^T G(t, s) |\lambda \varphi(x(s) - u(s)) + \lambda \varphi(v(s) - y(s))| ds \\
& \leq \lambda \sup_{t \in I} \int_0^T G(t, s) |\varphi(d(x(s), u(s))) + \varphi(d(v(s), y(s)))| ds \\
& = \lambda (\varphi(d(x(s), u(s))) + \varphi(d(v(s), y(s)))) \cdot \sup_{t \in I} \int_0^T G(t, s) ds \\
& = \lambda (\varphi(d(x(s), u(s))) + \varphi(d(v(s), y(s)))) \cdot \sup_{t \in I} \frac{1}{e^{\lambda T} - 1} \left( \left[ \frac{1}{\lambda} e^{\lambda(T+s-t)} \right]_0^t + \left[ \frac{1}{\lambda} e^{\lambda(s-t)} \right]_t^T \right) \\
& = \lambda (\varphi(d(x(s), u(s))) + \varphi(d(v(s), y(s)))) \cdot \frac{1}{\lambda e^{\lambda T} - 1} (e^{\lambda T} - 1) \\
& = \varphi(d(x(s), u(s))) + \varphi(d(v(s), y(s))) \\
& < \frac{k}{2} [d(x(s), u(s)) + d(v(s), y(s))] \\
& \leq \frac{k}{2} \sup\{|x(t) - u(t)| : t \in I\} + \frac{k}{2} \sup\{|v(t) - y(t)| : t \in I\} \\
& = \frac{k}{2} [d(x(t), u(t)) + d(y(t), v(t))].
\end{aligned} \tag{3.29}$$

Then, by Proposition 1.3,  $F$  is a generalized Meir-Keeler type contraction.

Finally, let  $(\alpha(t), \beta(t)) \in C^1(I, \mathbb{R}) \times C^1(I, \mathbb{R})$  be a lower solution for the integral equation (3.14). Then we show that

$$\alpha < F(\alpha, \beta), \quad \beta \geq F(\beta, \alpha). \tag{3.30}$$

Indeed, we have  $\alpha'(t) + \lambda \beta(t) \leq f(t, \alpha(t)) - f(t, \beta(t))$  for any  $t \in I$  and so

$$\alpha'(t) + \lambda \alpha(t) \leq f(t, \alpha(t)) - f(t, \beta(t)) + \lambda \alpha(t) - \lambda \beta(t) \tag{3.31}$$

for any  $t \in I$ . Multiplying by  $e^{\lambda t}$  in (3.31), we get the following:

$$(\alpha(t)e^{\lambda t})' \leq [(f(t, \alpha(t)) + \lambda \alpha(t)) - (f(t, \beta(t)) + \lambda \beta(t))] e^{\lambda t} \tag{3.32}$$

for any  $t \in I$ , which implies that

$$\alpha(t)e^{\lambda t} \leq \alpha(0) + \int_0^t [(f(s, \alpha(s)) + \lambda \alpha(s)) - (f(s, \beta(s)) + \lambda \beta(s))] e^{\lambda s} ds \tag{3.33}$$

for any  $t \in I$ . This implies that

$$\alpha(0)e^{\lambda t} < \alpha(T)e^{\lambda T} \leq \alpha(0) + \int_0^T [f(s, \alpha(s)) + \lambda\alpha(s) - f(s, \beta(s)) - \lambda\beta(s)] e^{\lambda s} ds \quad (3.34)$$

and so

$$\alpha(0) < \int_0^T \frac{e^{\lambda s}}{e^{\lambda T} - 1} [f(s, \alpha(s)) + \lambda\alpha(s) - f(s, \beta(s)) - \lambda\beta(s)] ds. \quad (3.35)$$

Thus it follows from (3.35) and (3.33) that

$$\begin{aligned} \alpha(t)e^{\lambda t} &< \int_t^T \frac{e^{\lambda s}}{e^{\lambda T} - 1} [f(s, \alpha(s)) + \lambda\alpha(s) - f(s, \beta(s)) - \lambda\beta(s)] ds \\ &+ \int_0^t \frac{e^{\lambda(T+s)}}{e^{\lambda T} - 1} [f(s, \alpha(s)) + \lambda\alpha(s) - f(s, \beta(s)) - \lambda\beta(s)] ds, \end{aligned} \quad (3.36)$$

and so

$$\begin{aligned} \alpha(t) &< \int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1} [f(s, \alpha(s)) + \lambda\alpha(s) - f(s, \beta(s)) - \lambda\beta(s)] ds \\ &+ \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1} [f(s, \alpha(s)) + \lambda\alpha(s) - f(s, \beta(s)) - \lambda\beta(s)] ds. \end{aligned} \quad (3.37)$$

Hence we have

$$\alpha(t) < \int_0^T G(t, s) [f(s, \alpha(s)) + \lambda\alpha(s) - f(s, \beta(s)) - \lambda\beta(s)] ds = F(\alpha(t), \beta(t)) \quad (3.38)$$

for any  $t \in I$ .

Similarly, we have  $\beta(t) \geq F(\beta(t), \alpha(t))$ . Therefore, by Corollary 2.5,  $F$  has a coupled fixed-point.  $\square$

*Example 3.9.* In the integral equation (3.14), we put  $\lambda = 1.5$ ,  $f(u, v) = u - v$  for all  $(u, v) \in I \times \mathbb{R}$  and  $T = 0.5$ . Then  $f$  is a continuous function, and we have

$$(x(t), y(t)) = \left( \int_0^{0.5} G(t, s) [0.5x(s) - 0.5y(s)] ds, \int_0^{0.5} G(t, s) [0.5y(s) - 0.5x(s)] ds \right), \quad (3.39)$$

where  $x, y \in C(I, \mathbb{R})$ , and

$$G(t, s) = \begin{cases} \frac{e^{1.5(0.5+s-t)}}{e^{0.75} - 1}, & \text{if } 0 \leq s < t \leq 0.5, \\ \frac{e^{1.5(s-t)}}{e^{0.75} - 1}, & \text{if } 0 \leq t < s \leq 0.5. \end{cases} \quad (3.40)$$

Also,  $(\alpha(t), \beta(t)) = (-2e^{-0.5t}, 3e^{-0.5t})$  is a lower solution of (3.39). Moreover, if we define  $\varphi(x) = x/3$  for all  $x \in [0, \infty)$ , then  $\varphi$  is increasing and, for any  $x > 0$ , there exists  $k = 1/1.1 \in [0, 1)$  such that  $\varphi(x) = x/3 < (k/2)x = x/2.2$ . For all  $x, y \in \mathbb{R}$  with  $y > x$ , we have

$$0 < f(t, y) + \lambda y - [f(t, x) + \lambda x] = 0.5(y - x) \leq \lambda \varphi(y - x) = 1.5 \frac{y - x}{3} = 0.5(y - x). \quad (3.41)$$

Therefore, all the conditions of Theorem 3.8 hold, and a solution of (3.39) exists.

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