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Research Article

The Existence of Periodic Orbits and Invariant Tori for Some 3-Dimensional Quadratic Systems

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We use the normal form theory, averaging method, and integral manifold theorem to study the existence of limit cycles in Lotka-Volterra systems and the existence of invariant tori in quadratic systems in $\mathbb{R}^3$.

1. Introduction

It is well known that n-dimensional generalized Lotka-Volterra systems are widely used as the first approximation for a community of n interacting species, each of which would exhibit logistic growth in the absence of other species in population dynamics. And this system is of wide interest in different branches of science, such as physics, chemistry, biology, evolutionary game theory, and economics. We refer the reader to the book of Hofbauer and Sigmund [1] for its applications. The existence of limit cycles and invariant tori for these models is interesting and significant in both mathematics and applications since the existence of stable limit cycles and invariant tori provided a satisfactory explanation for those species communities in which populations are observed to oscillate in a rather reproducible periodic manner (cf. [2–4] and references therein).

To study the bifurcation of Lotka-Volterra class, we consider three-dimensional generalized Lotka-Volterra systems

$$\frac{dX_i(t)}{dt} = X_i(t)\left(\beta_i + \sum_{j=1}^{3}\alpha_{ij}X_j(t)\right), \quad i = 1, 2, 3, \quad (1)$$

which describes the interaction of three species in a constant and homogeneous environment, where $X_i(t)$ is the number of individuals in the ith population at time t and $X_i(t) \geq 0$, β_i is the intrinsic growth rate of the ith population, the α_{ij} are interaction coefficients measuring the extent to which the jth species affects the growth rate of the ith, β_i and α_{ij} are parameters, and the values of these parameters are not very small usually.

Over the last several decades, many researchers have devoted their effort to study the existence and number of isolated periodic solutions for system (1). There have been a series of achievements and unprecedented challenges on the theme even if system (1) is a competitive system (cf. [5–12]). In [13], Bobieński and Żołądek gave four components of center variety in the three-dimensional Lotka-Volterra class and studied the existence and number of isolated periodic solutions by certain Poincaré-Melnikov integrals of a new type. In [14], Llibre and Xiao used the averaging method to study the existence of limit cycles of three-dimensional Lotka-Volterra systems. In this paper, we will use the normal form theory to study the same question. And furthermore, we will give the existence of invariant tori in a system of the form (2).

This paper is organized as follows. In Section 2, we obtain some preliminary theorems about a normal form system of degree two in $\mathbb{R}^3$ with two small parameters λ_1 and λ_2 and other bounded parameters. In Section 3, we first change the system (1) into a system of the form

$$\frac{dU}{dt} = u\varepsilon U + vV + \sum_{i+j+k=2} a_{ijk}U^iV^jW^k,$$

$$\frac{dV}{dt} = -\nu U + u\varepsilon V + \sum_{i+j+k=2} b_{ijk}U^iV^jW^k,$$
$$\frac{dW}{dt} = \varepsilon W + \sum_{i+j+k=2} c_{ijk}U^iV^jW^k, \tag{2}$$

where a_{ijk}, b_{ijk}, and c_{ijk} for $i, j, k = 0, 1, 2$ are functions of the parameters β_i and α_{ij} in system (1), u and $\nu > 0$ are bounded parameters, and $0 < \varepsilon \ll 1$ is perturbation parameter. And then we get the real normal form of the system (2) after a series of transformations. Two examples are provided to illustrate these results in the last section.

2. Preliminary Theorems

In this section, we first consider a normal form system of degree two in $\mathbb{R}^3$. Then, by a series of transformations we introduce some theorems for the normal form. The reader is referred to [15] for more details about the following content.

Consider the 3-dimensional system

$$\dot{x} = Dx + X_2(x), \tag{3}$$

where $X_2(x) = O(|x|^2)$ is C^∞ in $x \in \mathbb{R}^3$, and

$$D = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4}$$

By adding up the 2-parameter linear part $\operatorname{diag}(\lambda_1, \lambda_1, \lambda_2)x$ we obtain

$$\dot{x} = D(\lambda_1, \lambda_2)x + X_2(x), \tag{5}$$

where $D(\lambda_1, \lambda_2) = \operatorname{diag}(A(\lambda_1), \lambda_2)$, with

$$A(\lambda_1) = \begin{pmatrix} \lambda_1 & 1 \\ -1 & \lambda_1 \end{pmatrix}. \tag{6}$$

It can be verified that (5) has the following real normal form up to order 3 (see [15]):

$$\begin{aligned}
\dot{x}_1 =& \lambda_1 x_1 + x_2 + a_1x_1x_3 + b_1x_2x_3 + (a_2x_1 + b_2x_2) \\
&\times (x_1^2 + x_2^2) + (a_3x_1 + b_3x_2)x_3^2 + O(|x_1, x_2, x_3|^4), \\
\dot{x}_2 =& -x_1 + \lambda_1 x_2 - b_1x_1x_3 + a_1x_2x_3 + (-b_2x_1 + a_2x_2) \\
&\times (x_1^2 + x_2^2) + (-b_3x_1 + a_3x_2)x_3^2 + O(|x_1, x_2, x_3|^4), \\
\dot{x}_3 =& \lambda_2 x_3 + c_1(x_1^2 + x_2^2) + d_1x_3^2 + c_2(x_1^2 + x_2^2)x_3 \\
&+ d_2x_3^3 + O(|x_1, x_2, x_3|^4).
\end{aligned} \tag{7}$$

For convenience, we assume that $a_1b_1c_1 \neq 0$ as in [15]. By the scaling

$$x_1 \longrightarrow \frac{\sqrt{|d_1|}}{\sqrt{|c_1|}a_1}x_1, \qquad x_2 \longrightarrow \frac{\sqrt{|d_1|}}{\sqrt{|c_1|}a_1}x_2, \qquad x_3 \longrightarrow \frac{1}{a_1}x_3, \tag{8}$$

(7) becomes

$$\begin{aligned}
\dot{x}_1 =& \lambda_1 x_1 + x_2 + x_1x_3 + \tilde{b}_1x_2x_3 + (\tilde{a}_2x_1 + \tilde{b}_2x_2)(x_1^2 + x_2^2) \\
&+ (\tilde{a}_3x_1 + \tilde{b}_3x_2)x_3^2 + O(|x_1, x_2, x_3|^4), \\
\dot{x}_2 =& -x_1 + \lambda_1 x_2 - \tilde{b}_1x_1x_3 + x_2x_3 + (-\tilde{b}_2x_1 + \tilde{a}_2x_2) \\
&\times (x_1^2 + x_2^2) + (-\tilde{b}_3x_1 + \tilde{a}_3x_2)x_3^2 + O(|x_1, x_2, x_3|^4), \\
\dot{x}_3 =& \lambda_2 x_3 + \tilde{c}_1(x_1^2 + x_2^2) + \tilde{d}_1x_3^2 + \tilde{c}_2(x_1^2 + x_2^2)x_3 \\
&+ \tilde{d}_2x_3^3 + O(|x_1, x_2, x_3|^4),
\end{aligned} \tag{9}$$

where

$$\begin{gathered}
\tilde{b}_1 = \frac{b_1}{a_1}, \qquad \tilde{a}_2 = \frac{|d_1|a_2}{|c_1|a_1^2}, \qquad \tilde{b}_2 = \frac{|d_1|b_2}{|c_1|a_1^2}, \\
\tilde{a}_3 = \frac{a_3}{a_1^2}, \qquad \tilde{b}_3 = \frac{b_3}{a_1^2}, \qquad \tilde{c}_1 = \frac{|d_1|}{a_1}\operatorname{sgn}(c_1), \\
\tilde{d}_1 = \frac{d_1}{a_1}, \qquad \tilde{c}_2 = \frac{|d_1|c_2}{|c_1|a_1^2}, \qquad \tilde{d}_2 = \frac{d_2}{a_1^2}.
\end{gathered} \tag{10}$$

Then, by introducing polar coordinates

$$x_1 = p\cos\theta, \qquad x_2 = -p\sin\theta, \tag{11}$$

(9) further becomes

$$\begin{aligned}
\dot{\theta} =& 1 + \tilde{b}_1x_3 + \tilde{b}_2p^2 + \tilde{b}_3x_3^2 + p^{-1}S_1(\theta, p, x_3), \\
\dot{p} =& \lambda_1 p + px_3 + \tilde{a}_2p^3 + \tilde{a}_3px_3^2 + S_2(\theta, p, x_3), \\
\dot{x}_3 =& \lambda_2 x_3 + \tilde{c}_1p^2 + \tilde{d}_1x_3^2 + \tilde{c}_2p^2x_3 + \tilde{d}_2x_3^3 + S_3(\theta, p, x_3),
\end{aligned} \tag{12}$$

where S_1, S_2, and S_3 are 2π periodic in θ, and S_1, S_2, and $S_3 = O(|p, x_3|^4)$. By a further scaling of the form

$$\begin{gathered}
p \longrightarrow \varepsilon p, \qquad x_3 \longrightarrow \varepsilon x_3, \\
\lambda_1 \longrightarrow \varepsilon\delta_1, \qquad \lambda_2 \longrightarrow \varepsilon\delta_2, \\
\varepsilon > 0, \ |\delta_1| = 1,
\end{gathered} \tag{13}$$

(12) becomes

$$\begin{aligned}
\dot{\theta} =& 1 + \varepsilon[\tilde{b}_1x_3 + \varepsilon(\tilde{b}_2p^2 + \tilde{b}_3x_3^2)] + p^{-1}\varepsilon^3O(|p, x_3|^4), \\
\dot{p} =& \varepsilon p[\delta_1 + x_3 + \varepsilon(\tilde{a}_2p^2 + \tilde{a}_3x_3^2)] + O(\varepsilon^3), \\
\dot{x}_3 =& \varepsilon[\delta_2x_3 + \tilde{c}_1p^2 + \tilde{d}_1x_3^2 + \varepsilon(\tilde{c}_2p^2x_3 + \tilde{d}_2x_3^3)] + O(\varepsilon^3).
\end{aligned} \tag{14}$$

We obtain from (14)

$$\begin{aligned}
\frac{dp}{d\theta} =& \varepsilon p[f_0(p, x_3) + \varepsilon f_1(p, x_3) + \overline{f}(\theta, p, x_3, \varepsilon)], \\
\frac{dx_3}{d\theta} =& \varepsilon[g_0(p, x_3) + \varepsilon g_1(p, x_3) + \overline{g}(\theta, p, x_3, \varepsilon)],
\end{aligned} \tag{15}$$

where

$$\begin{aligned} f_0(p, x_3) &= \delta_1 + x_3, \\ g_0(p, x_3) &= \delta_2 x_3 + \tilde{c}_1 p^2 + \tilde{d}_1 x_3^2, \\ f_1(p, x_3) &= -\delta_1 \tilde{b}_1 x_3 + \tilde{a}_2 p^2 + (\tilde{a}_3 - \tilde{b}_1) x_3^2, \\ g_1(p, x_3) &= -\tilde{b}_1 \delta_2 x_3^2 + (\tilde{c}_2 - \tilde{c}_1 \tilde{b}_1) p^2 x_3 + (\tilde{d}_2 - \tilde{b}_1 \tilde{d}_1) x_3^3, \\ \overline{f}, \overline{g} &= O(\varepsilon^2). \end{aligned} \tag{16}$$

Note that the functions $\overline{f}$ and $\overline{g}$ in (15) are 2π periodic in θ but may not be well defined at $p = 0$. Thus, we suppose $p \gg \varepsilon > 0$ for (15).

The averaging system

$$\begin{aligned} \frac{dp}{d\theta} &= p f_0(p, x_3), \\ \frac{dx_3}{d\theta} &= g_0(p, x_3) \end{aligned} \tag{17}$$

has a singular point (p_0, s_0) on the half plane $p > 0$ if

$$\tilde{c}_1 (\tilde{d}_1 - \delta_1 \delta_2) < 0, \tag{18}$$

where $p_0 = \sqrt{(\delta_1 \delta_2 - \tilde{d}_1)/\tilde{c}_1}$, $s_0 = -\delta_1$. By denoting

$$B = \frac{\partial (p f_0, g_0)}{\partial (p, x_3)} |_{(p_0, s_0)}, \tag{19}$$

we obtain $|B| = -2\tilde{c}_1 p_0^2 \neq 0$, and the characteristic polynomial of B is $f_\lambda(B) = \lambda^2 - (\delta_2 - 2\tilde{d}_1 \delta_1)\lambda - 2(\tilde{d}_1 - \delta_1 \delta_2)$. We define

$$\Delta = (\delta_2 - 2\tilde{d}_1 \delta_1)^2 + 8(\tilde{d}_1 - \delta_1 \delta_2). \tag{20}$$

According to Theorem 4.1.3 in [15], we can obtain the following theorem.

Theorem 1. *Suppose that (18) holds. Then, (7) has a periodic orbit near the origin for $0 < \varepsilon \ll 1$. Further, the periodic orbit is stable (resp. unstable) if one (resp. none) of the following conditions holds:*

(a) $\Delta = 0$ *and* $\delta_2 - 2\tilde{d}_1 \delta_1 < 0$,

(b) $\Delta < 0$ *and* $\delta_2 - 2\tilde{d}_1 \delta_1 \leq 0$,

(c) $\Delta > 0$, $\delta_2 - 2\tilde{d}_1 \delta_1 < 0$, *and* $\tilde{d}_1 - \delta_1 \delta_2 < 0$,

where Δ is given by (20).

Then, by letting $s = x_3 + \delta_1$ and $\theta \to \varepsilon^{-1}\theta$ and truncating the terms of order ε^2, we have from (15)

$$\begin{aligned} \frac{dp}{d\theta} &= ps + \varepsilon p [\tilde{a}_2 p^2 + f_2(s)], \\ \frac{ds}{d\theta} &= \tilde{c}_1 p^2 + g_2(s) + \varepsilon [g_3(s) p^2 + g_4(s)], \end{aligned} \tag{21}$$

where

$$\begin{aligned} f_2(s) &= (\tilde{a}_3 - \tilde{b}_1) s^2 + \delta_1 (\tilde{b}_1 - 2\tilde{a}_3) s + \tilde{a}_3, \\ g_2(s) &= \tilde{d}_1 s^2 + (\delta_2 - 2\delta_1 \tilde{d}_1) s + \delta_1 (\delta_1 \tilde{d}_1 - \delta_2), \\ g_3(s) &= (\tilde{c}_2 - \tilde{c}_1 \tilde{b}_1)(s - \delta_1), \\ g_4(s) &= (\tilde{d}_2 - \tilde{b}_1 \tilde{d}_1)(s - \delta_1)^3 - \delta_2 \tilde{b}_1 (s - \delta_1)^2. \end{aligned} \tag{22}$$

Thus, in order that (21) has a limit cycle, we necessarily suppose

$$\delta_2 - 2\delta_1 \tilde{d}_1 = \delta\varepsilon, \quad \delta \in \mathbb{R}, \tag{23}$$

and $\tilde{c}_1 < 0$, $\tilde{c}_1 \tilde{d}_1 > 0$, that is,

$$a_1 c_1 < 0, \qquad c_1 d_1 > 0. \tag{24}$$

This yields $\tilde{c}_1 = \tilde{d}_1 < 0$, and hence (21) becomes

$$\begin{aligned} \frac{dp}{d\theta} &= ps + \varepsilon p [\tilde{a}_2 p^2 + f_2(s)], \\ \frac{ds}{d\theta} &= \tilde{c}_1 (p^2 + s^2 - 1) + \varepsilon [g_3(s) p^2 + \overline{g}_4(s) + O(\varepsilon)], \end{aligned} \tag{25}$$

where

$$\begin{aligned} \overline{g}_4(s) = {} & (\tilde{d}_2 - \tilde{c}_1 \tilde{b}_1) s^3 + \delta_1 (\tilde{c}_1 \tilde{b}_1 - 3\tilde{d}_2) s^2 \\ & + (\delta + 3\tilde{d}_2 + \tilde{c}_1 \tilde{b}_1) s - (\delta + \tilde{d}_2 + \tilde{c}_1 \tilde{b}_1) \delta_1. \end{aligned} \tag{26}$$

For small $\varepsilon > 0$, (25) has a focus $A_\varepsilon(p(\varepsilon), s(\varepsilon))$ with

$$p(0) = 1, \qquad s(0) = 0, \qquad s'(0) = -(\tilde{a}_3 + \tilde{a}_2). \tag{27}$$

We define

$$\begin{aligned} \delta_0 &= 2\tilde{c}_1 \tilde{a}_3 - 3\tilde{d}_2 - \tilde{c}_2 - 2\tilde{a}_2 (1 - \tilde{c}_1), \\ \Delta_0 &= \frac{1}{8} [3\tilde{d}_2 - \tilde{c}_2 - 2\tilde{a}_2 (1 - \tilde{c}_1) - 2\tilde{c}_1 \tilde{a}_3], \\ \delta_0' &= \frac{2}{3 - 2\tilde{c}_1} [(2 - \tilde{c}_1)(2\tilde{c}_1 \tilde{a}_3 - 3\tilde{d}_2) - (1 - \tilde{c}_1) \\ & \quad \times (\tilde{c}_2 + 2\tilde{a}_2 (1 - \tilde{c}_1))]. \end{aligned} \tag{28}$$

By using the coefficients in (7), we have

$$\begin{aligned} \delta_0 &= \frac{2d_1 a_3}{a_1^3} - \frac{3d_2}{a_1^2} - \frac{d_1}{c_1 a_1^2} \left[c_2 + 2a_2 \left(1 - \frac{d_1}{a_1}\right)\right], \\ \Delta_0 &= \frac{1}{8} \left[\frac{1}{a_1^3} (3a_1 d_2 - 2a_3 d_1) \right. \\ & \qquad \left. - \frac{d_1}{c_1 a_1^2} \left(c_2 + 2a_2 \left(1 - \frac{d_1}{a_1}\right)\right)\right], \\ \delta_0' &= \frac{2}{3a_1 - 2d_1} \left[\left(2 - \frac{d_1}{a_1}\right)\left(\frac{2d_1 a_3}{a_1^2} - \frac{3d_2}{a_1}\right) \right. \\ & \qquad \left. - \frac{d_1}{c_1 a_1}\left(1 - \frac{d_1}{a_1}\right)\left(c_2 + 2a_2 \left(1 - \frac{d_1}{a_1}\right)\right)\right]. \end{aligned} \tag{29}$$

Then, in 1997, the following result was obtained in [15].

Theorem 2. *Suppose that* (24) *holds and* $\Delta_0 \neq 0$. *Then, for any given* $\varepsilon_1 > 0$ *there exist an* $\varepsilon_0 > 0$ *and a* C^1 *function* $\phi_0(\lambda_1) = (2d_1\lambda_1/a_1) + \delta_0\lambda_1^2 + O(\lambda_1^3)$ *and* $\phi_1(\lambda_1) = (2d_1\lambda_1/a_1) + \delta_0'\lambda_1^2 + O(\lambda_1^3)$ *such that for* $0 < \lambda_1^2 + \lambda_2^2 < \varepsilon_0$, (7) *has a unique invariant torus near the origin if* $\Delta_0\phi_1(\lambda_1) - \varepsilon_1\lambda_1^2 < \Delta_0\lambda_2 < \Delta_0\phi_0(\lambda_1)$ *and has no invariant torus if* $\Delta_0\lambda_2 > \Delta_0\phi_0(\lambda_1)$. *Moreover, the torus, if it exists, is stable (resp. unstable) when* $\Delta_0 < 0$ *(resp.* > 0*).*

3. Normal Form of System (2)

In this section, we consider system (1) in the first octant $\mathbb{R}_+^3$, where $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$. We now look for the conditions for the existence of positive equilibria of system (1), which is equivalent to find the positive solutions of the following system:

$$\beta_i + \sum_{j=1}^{3} \alpha_{ij} X_j = 0, \quad i = 1, 2, 3. \tag{30}$$

We suppose that there exists at least one positive solution of (30). Without loss of generality, we assume that the positive equilibrium is (1, 1, 1). Then, we move it to the origin by doing the change of variables $Y_i = X_i - 1$, $i = 1, 2, 3$. Then, system (1) can be written as

$$\frac{dY_i}{dt} = (Y_i + 1) \sum_{j=1}^{3} \alpha_{ij} Y_j, \quad i = 1, 2, 3. \tag{31}$$

Now, we shall investigate a special form of system (31) with a small parameter; we write the perturbed system as

$$\frac{dY_i}{dt} = (Y_i + 1) \sum_{j=1}^{3} \alpha_{ij}(\varepsilon) Y_j, \quad i = 1, 2, 3. \tag{32}$$

Denote $M(\varepsilon) = (\alpha_{ij}(\varepsilon))_{3\times 3}$, and we suppose $M(\varepsilon)$ is similar to

$$\Psi = \begin{pmatrix} u\varepsilon & v & 0 \\ -v & u\varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}, \quad (u, v) = (u(\varepsilon), v(\varepsilon)). \tag{33}$$

Then, system (32) can be changed into the system (2) by a linear transformation.

In this section, our task is to change system (2) into the normal form of (7). Making the transformation

$$x_1 = U, \qquad x_2 = V, \qquad x_3 = W, \qquad t \longrightarrow \frac{1}{v} t, \tag{34}$$

system (2) becomes

$$\begin{aligned} \dot{x}_1 &= \lambda_1 x_1 + x_2 + \sum_{i+j+k=2} \tilde{a}_{ijk} x_1^i x_2^j x_3^k, \\ \dot{x}_2 &= -x_1 + \lambda_1 x_2 + \sum_{i+j+k=2} \tilde{b}_{ijk} x_1^i x_2^j x_3^k, \\ \dot{x}_3 &= \lambda_2 x_3 + \sum_{i+j+k=2} \tilde{c}_{ijk} x_1^i x_2^j x_3^k, \end{aligned} \tag{35}$$

where

$$\begin{gathered} \tilde{a}_{ijk} = \frac{1}{v} a_{ijk}, \qquad \tilde{b}_{ijk} = \frac{1}{v} b_{ijk}, \qquad \tilde{c}_{ijk} = \frac{1}{v} c_{ijk}, \\ \lambda_1 = \frac{u}{v}\varepsilon, \qquad \lambda_2 = \frac{1}{v}\varepsilon. \end{gathered} \tag{36}$$

Let

$$T = \begin{pmatrix} 1 & i & 0 \\ i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{37}$$

by changing $y = Tx$, where $y = (y_1, y_2, y_3)^T$, $x = (x_1, x_2, x_3)^T$, and system (35) becomes a complex system of the form

$$\begin{aligned} \dot{y}_1 &= (\lambda_1 + i) y_1 + \sum_{i+j+k=2} a_{ijk}^* y_1^i y_2^j y_3^k, \\ \dot{y}_2 &= (\lambda_1 - i) y_2 + \sum_{i+j+k=2} b_{ijk}^* y_1^i y_2^j y_3^k, \\ \dot{y}_3 &= \lambda_2 y_3 + \sum_{i+j+k=2} c_{ijk}^* y_1^i y_2^j y_3^k, \end{aligned} \tag{38}$$

where

$$\begin{aligned}
a_{200}^* &= \frac{1}{4}\left(\tilde{a}_{200} - \tilde{a}_{020} + \tilde{b}_{110}\right) + \frac{1}{4}\left(-\tilde{a}_{110} + \tilde{b}_{200} - \tilde{b}_{020}\right) i, \\
a_{020}^* &= \frac{1}{4}\left(\tilde{a}_{020} - \tilde{a}_{200} + \tilde{b}_{110}\right) + \frac{1}{4}\left(-\tilde{a}_{110} + \tilde{b}_{020} - \tilde{b}_{200}\right) i, \\
a_{002}^* &= \tilde{a}_{002} + \tilde{b}_{002} i, \\
a_{110}^* &= \frac{1}{2}\left(\tilde{b}_{200} + \tilde{b}_{020}\right) - \frac{1}{2}\left(\tilde{a}_{200} + \tilde{a}_{020}\right) i, \\
a_{101}^* &= \frac{1}{2}\left(\tilde{a}_{101} - \tilde{a}_{011}\right) + \frac{1}{2}\left(\tilde{b}_{101} - \tilde{b}_{011}\right) i, \\
a_{011}^* &= \frac{1}{2}\left(\tilde{a}_{011} + \tilde{b}_{101}\right) + \frac{1}{2}\left(-\tilde{a}_{101} + \tilde{b}_{011}\right) i, \\
b_{200}^* &= \frac{1}{4}\left(\tilde{b}_{200} - \tilde{b}_{020} + \tilde{a}_{110}\right) + \frac{1}{4}\left(-\tilde{b}_{110} + \tilde{a}_{200} - \tilde{a}_{020}\right) i, \\
b_{020}^* &= \frac{1}{4}\left(\tilde{b}_{020} - \tilde{b}_{200} + \tilde{a}_{110}\right) + \frac{1}{4}\left(-\tilde{b}_{110} + \tilde{a}_{020} - \tilde{a}_{200}\right) i, \\
b_{002}^* &= \tilde{b}_{002} + \tilde{a}_{002} i, \\
b_{110}^* &= \frac{1}{2}\left(\tilde{a}_{200} + \tilde{a}_{020}\right) - \frac{1}{2}\left(\tilde{b}_{200} + \tilde{b}_{020}\right) i, \\
b_{101}^* &= \frac{1}{2}\left(\tilde{b}_{101} - \tilde{b}_{011}\right) + \frac{1}{2}\left(\tilde{a}_{101} - \tilde{a}_{011}\right) i, \\
b_{011}^* &= \frac{1}{2}\left(\tilde{b}_{011} + \tilde{a}_{101}\right) + \frac{1}{2}\left(-\tilde{b}_{101} + \tilde{a}_{011}\right) i, \\
c_{200}^* &= \frac{1}{4}\left(\tilde{a}_{200} - \tilde{a}_{020}\right) - \frac{1}{4}\tilde{a}_{110} i, \\
c_{020}^* &= \frac{1}{4}\left(\tilde{a}_{020} - \tilde{a}_{200}\right) - \frac{1}{4}\tilde{a}_{110} i,
\end{aligned}$$

$$\begin{aligned} c_{002}^{*} &= \widetilde{a}_{002}, \\ c_{110}^{*} &= -\frac{1}{2}\left(\widetilde{a}_{200}+\widetilde{a}_{020}\right) i, \\ c_{101}^{*} &= \frac{1}{2}\left(\widetilde{a}_{101}-\widetilde{a}_{011}\right), \\ c_{011}^{*} &= \frac{1}{2}\widetilde{a}_{011}-\frac{1}{2}\widetilde{a}_{101} i. \end{aligned} \tag{39}$$

By the fundamental theory of normal form [16], we know that system (38) can be converted to the normal form by some transformations. So our following task is to find the transformations and work out the normal form of system (38).

We denote (38) as $\dot{y} = F(y)$, where $F(0) = 0$, and for simplicity, we write the nonlinear part of (38) as $\Theta(y)$. By doing the following transformation:

$$y = z + P(z) \equiv h_1(z), \quad z \in \mathbb{R}^3, \tag{40}$$

where $P(z) = (P_1(z), P_2(z), P_3(z))^T$, which is to be determined, (38) becomes

$$\dot{z} = \left[Dh_1(z)\right]^{-1} F\left(h_1(z)\right). \tag{41}$$

Then, by noting

$$\left(Dh_1\right)^{-1} = I - DP + (DP)^2 + O\left(|DP|^3\right), \tag{42}$$

we can get from (41)

$$\begin{aligned} \dot{z}_i = {} & \gamma_i z_i + \gamma_i P_i - \sum_{j=1}^{3}\frac{\partial P_i}{\partial z_j}\gamma_j z_j \\ & + \Theta_i(z) + O\left(|z|^3\right), \quad i = 1,2,3, \end{aligned} \tag{43}$$

where $\gamma_1 = \lambda_1 + i$ and $\gamma_2 = \lambda_1 - i$, $\gamma_3 = \lambda_2$. In order to eliminate the quadratic homogeneous polynomial, we need

$$\gamma_i P_i - \sum_{j=1}^{3}\frac{\partial P_i}{\partial z_j}\gamma_j z_j = -\Theta_i(z) + O\left(|z|^3\right), \quad i = 1,2,3. \tag{44}$$

We take P_i, $i = 1, 2, 3$ as quadratic homogeneous polynomial, having the form

$$P_i = l_{i1}z_1^2 + l_{i2}z_2^2 + l_{i3}z_3^2 + l_{i4}z_1z_2 + l_{i5}z_1z_3 + l_{i6}z_2z_3, \tag{45}$$

where l_{ik}, $k = 1, \dots, 6$, are real undermined coefficients. By inserting (45) into (44) and comparing the coefficients of similar items, we can obtain

$$\begin{aligned} & l_{11} = \frac{a_{200}^*}{\gamma_1}, \quad l_{12} = \frac{a_{020}^*}{2\gamma_2-\gamma_1}, \quad l_{13} = \frac{a_{002}^*}{2\gamma_3-\gamma_1}, \\ & l_{14} = \frac{a_{110}^*}{\gamma_2}, \quad l_{15} = \frac{a_{101}^*}{\gamma_3}, \quad l_{16} = \frac{a_{200}^*}{\gamma_2+\gamma_3-\gamma_1}; \\ & l_{21} = \frac{b_{200}^*}{2\gamma_1-\gamma_2}, \quad l_{22} = \frac{b_{020}^*}{\gamma_2}, \quad l_{23} = \frac{b_{002}^*}{2\gamma_3-\gamma_2}, \\ & l_{24} = \frac{b_{110}^*}{\gamma_1}, \quad l_{25} = \frac{b_{101}^*}{\gamma_1+\gamma_3-\gamma_2}, \quad l_{26} = \frac{a_{200}^*}{\gamma_3}; \\ & l_{31} = \frac{c_{200}^*}{2\gamma_1-\gamma_3}, \quad l_{32} = \frac{c_{020}^*}{2\gamma_2-\gamma_3}, \quad l_{33} = \frac{c_{002}^*}{\gamma_3}, \\ & l_{34} = \frac{c_{110}^*}{\gamma_1+\gamma_2-\gamma_3}, \quad l_{35} = \frac{c_{101}^*}{\gamma_1}, \quad l_{36} = \frac{c_{200}^*}{\gamma_2}. \end{aligned} \tag{46}$$

Note that $|\gamma_3| = |\lambda_2| \ll 1$, $|\gamma_1 + \gamma_2 - \gamma_3| = |2\lambda_1 - \lambda_2| \ll 1$. The terms with coefficients l_{15}, l_{26}, l_{33}, and l_{34} that appeared above cannot be removed. Those terms are called the resonance terms. Then, we have

$$\begin{aligned} P_1 &= l_{11}z_1^2 + l_{12}z_2^2 + l_{13}z_3^2 + l_{14}z_1z_2 + l_{16}z_2z_3, \\ P_2 &= l_{21}z_1^2 + l_{22}z_2^2 + l_{23}z_3^2 + l_{24}z_1z_2 + l_{25}z_1z_3, \\ P_3 &= l_{31}z_1^2 + l_{32}z_2^2 + l_{35}z_1z_3 + l_{36}z_2z_3, \end{aligned} \tag{47}$$

and system (43) becomes

$$\begin{aligned} \dot{z}_1 &= \gamma_1 z_1 + a_{101}^* z_1 z_3 + O\left(\left|z_1, z_2, z_3\right|^3\right), \\ \dot{z}_2 &= \gamma_2 z_2 + b_{011}^* z_2 z_3 + O\left(\left|z_1, z_2, z_3\right|^3\right), \\ \dot{z}_3 &= \gamma_3 z_3 + c_{002}^* z_3^2 + c_{110}^* z_1 z_2 + O\left(\left|z_1, z_2, z_3\right|^3\right). \end{aligned} \tag{48}$$

Let $L(z)$ denote the cubic terms in z of (48). Then, from (41) and (42) we have

$$L(z) = \begin{pmatrix} -\sum_{j=1}^{3} P_{1j}\gamma_j P_j \\ -\sum_{j=1}^{3} P_{2j}\gamma_j P_j \\ -\sum_{j=1}^{3} P_{3j}\gamma_j P_j \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} + (DP)^2 \begin{pmatrix} \gamma_1 z_1 \\ \gamma_2 z_2 \\ \gamma_3 z_3 \end{pmatrix}, \tag{49}$$

where $P_{ij} = \partial P_i/\partial z_j$, $i, j = 1, 2, 3$,

$$\begin{aligned}
h_1 = & -P_{11}\left(a_{200}^{*}z_1^2 + a_{020}^{*}z_2^2 + a_{002}^{*}z_3^2\right) \\
& - P_{12}\left(b_{200}^{*}z_1^2 + b_{020}^{*}z_2^2 + b_{002}^{*}z_3^2\right) \\
& - P_{13}\left(c_{200}^{*}z_1^2 + c_{020}^{*}z_2^2 + c_{002}^{*}z_3^2\right) \\
& - P_{11}\left(a_{110}^{*}z_1z_2 + a_{101}^{*}z_1z_3 + a_{011}^{*}z_2z_3\right) \\
& - P_{12}\left(b_{110}^{*}z_1z_2 + b_{101}^{*}z_1z_3 + b_{011}^{*}z_2z_3\right) \\
& - P_{13}\left(c_{110}^{*}z_1z_2 + c_{101}^{*}z_1z_3 + c_{011}^{*}z_2z_3\right) \\
& + 2\left(a_{200}^{*}z_1P_1 + a_{020}^{*}z_2P_2 + a_{002}^{*}z_3P_3\right) \\
& + a_{110}^{*}\left(z_1P_2 + z_2P_1\right) + a_{101}^{*}\left(z_1P_3 + z_3P_1\right) \\
& + a_{011}^{*}\left(z_2P_3 + z_3P_2\right), \\
h_2 = & -P_{21}\left(a_{200}^{*}z_1^2 + a_{020}^{*}z_2^2 + a_{002}^{*}z_3^2\right) \\
& - P_{22}\left(b_{200}^{*}z_1^2 + b_{020}^{*}z_2^2 + b_{002}^{*}z_3^2\right) \\
& - P_{23}\left(c_{200}^{*}z_1^2 + c_{020}^{*}z_2^2 + c_{002}^{*}z_3^2\right) \\
& - P_{21}\left(a_{110}^{*}z_1z_2 + a_{101}^{*}z_1z_3 + a_{011}^{*}z_2z_3\right) \\
& - P_{22}\left(b_{110}^{*}z_1z_2 + b_{101}^{*}z_1z_3 + b_{011}^{*}z_2z_3\right) \\
& - P_{23}\left(c_{110}^{*}z_1z_2 + c_{101}^{*}z_1z_3 + c_{011}^{*}z_2z_3\right) \\
& + 2\left(b_{200}^{*}z_1P_1 + b_{020}^{*}z_2P_2 + b_{002}^{*}z_3P_3\right) \\
& + b_{110}^{*}\left(z_1P_2 + z_2P_1\right) + b_{101}^{*}\left(z_1P_3 + z_3P_1\right) \\
& + b_{011}^{*}\left(z_2P_3 + z_3P_2\right), \\
h_3 = & -P_{31}\left(a_{200}^{*}z_1^2 + a_{020}^{*}z_2^2 + a_{002}^{*}z_3^2\right) \\
& - P_{32}\left(b_{200}^{*}z_1^2 + b_{020}^{*}z_2^2 + b_{002}^{*}z_3^2\right) \\
& - P_{33}\left(c_{200}^{*}z_1^2 + c_{020}^{*}z_2^2 + c_{002}^{*}z_3^2\right) \\
& - P_{31}\left(a_{110}^{*}z_1z_2 + a_{101}^{*}z_1z_3 + a_{011}^{*}z_2z_3\right) \\
& - P_{32}\left(b_{110}^{*}z_1z_2 + b_{101}^{*}z_1z_3 + b_{011}^{*}z_2z_3\right) \\
& - P_{33}\left(c_{110}^{*}z_1z_2 + c_{101}^{*}z_1z_3 + c_{011}^{*}z_2z_3\right) \\
& + 2\left(c_{200}^{*}z_1P_1 + c_{020}^{*}z_2P_2 + c_{002}^{*}z_3P_3\right) \\
& + c_{110}^{*}\left(z_1P_2 + z_2P_1\right) + c_{101}^{*}\left(z_1P_3 + z_3P_1\right) \\
& + c_{011}^{*}\left(z_2P_3 + z_3P_2\right).
\end{aligned} \tag{50}$$

By substituting (42) into the above, we obtain

$$L(z) = \begin{pmatrix} e_{11}z_1^3 + e_{12}z_2^3 + e_{13}z_3^3 + e_{14}z_1^2z_2 + e_{15}z_1^2z_3 + e_{16}z_1z_2^2 + e_{17}z_1z_3^2 + e_{18}z_2^2z_3 + e_{19}z_2z_3^2 + e_{10}z_1z_2z_3 \\ e_{21}z_1^3 + e_{22}z_2^3 + e_{23}z_3^3 + e_{24}z_1^2z_2 + e_{25}z_1^2z_3 + e_{26}z_1z_2^2 + e_{27}z_1z_3^2 + e_{28}z_2^2z_3 + e_{29}z_2z_3^2 + e_{20}z_1z_2z_3 \\ e_{31}z_1^3 + e_{32}z_2^3 + e_{33}z_3^3 + e_{34}z_1^2z_2 + e_{35}z_1^2z_3 + e_{36}z_1z_2^2 + e_{37}z_1z_3^2 + e_{38}z_2^2z_3 + e_{39}z_2z_3^2 + e_{30}z_1z_2z_3 \end{pmatrix}, \tag{51}$$

where

$$\begin{aligned}
e_{11} &= \frac{a_{101}^{*}c_{200}^{*}}{2\gamma_1 - \gamma_3} + \frac{2a_{200}^{*2}}{\gamma_1} + \frac{a_{110}^{*}b_{200}^{*}}{2\gamma_1 - \gamma_2}, \\
e_{12} &= \frac{2a_{020}^{*}b_{020}^{*}}{\gamma_2} + \frac{a_{020}^{*}a_{110}^{*}}{2\gamma_2 - \gamma_1} + \frac{a_{011}^{*}c_{020}^{*}}{2\gamma_2 - \gamma_3}, \\
e_{13} &= \frac{a_{002}^{*}\left(a_{101}^{*} - 2c_{002}^{*}\right)}{2\gamma_3 - \gamma_1} + \frac{a_{011}^{*}b_{002}^{*}}{2\gamma_3 - \gamma_2}, \\
e_{14} &= \frac{2a_{020}^{*}b_{200}^{*}}{2\gamma_1 - \gamma_2} + \frac{a_{011}^{*}c_{200}^{*}}{2\gamma_1 - \gamma_3} + \frac{a_{110}^{*}\left(a_{200}^{*} + b_{110}^{*}\right)}{\gamma_1} \\
&\quad + \frac{2a_{200}^{*}a_{110}^{*}}{\gamma_2}, \\
e_{15} &= \frac{a_{110}^{*}b_{101}^{*}}{\gamma_1 + \gamma_3 - \gamma_2} + \frac{a_{101}^{*}\left(c_{101}^{*} - a_{200}^{*}\right)}{\gamma_1} \\
&\quad + \frac{2a_{002}^{*}c_{200}^{*}}{2\gamma_1 - \gamma_3} + \frac{a_{011}^{*}b_{200}^{*}}{2\gamma_1 - \gamma_2}, \\
e_{16} &= \frac{2a_{020}^{*}b_{110}^{*}}{\gamma_1} + \frac{a_{110}^{*}\left(a_{110}^{*} + b_{020}^{*}\right)}{\gamma_2} + \frac{a_{101}^{*}c_{020}^{*}}{2\gamma_2 - \gamma_3} \\
&\quad - \frac{a_{011}^{*}c_{110}^{*}}{\gamma_2 + \gamma_3 - \gamma_1} + \frac{2a_{200}^{*}a_{020}^{*}}{2\gamma_2 - \gamma_1}, \\
e_{17} &= \frac{a_{011}^{*}b_{101}^{*}}{\gamma_1 + \gamma_3 - \gamma_2} + \frac{2a_{200}^{*}a_{002}^{*}}{2\gamma_3 - \gamma_1} + \frac{2a_{002}^{*}c_{101}^{*}}{\gamma_1} + \frac{a_{110}^{*}b_{002}^{*}}{2\gamma_3 - \gamma_2}, \\
e_{18} &= \frac{a_{110}^{*}a_{011}^{*}}{\gamma_2 + \gamma_3 - \gamma_1} + \frac{a_{020}^{*}\left(a_{101}^{*} - 2b_{011}^{*}\right)}{2\gamma_2 - \gamma_1} \\
&\quad + \frac{a_{011}^{*}\left(b_{020}^{*} + c_{011}^{*}\right)}{\gamma_2} + \frac{2a_{002}^{*}c_{020}^{*}}{2\gamma_2 - \gamma_3}, \\
e_{19} &= \frac{a_{011}^{*}\left(a_{101}^{*} - b_{011}^{*} - c_{002}^{*}\right)}{\gamma_2 + \gamma_3 - \gamma_1} + \frac{2a_{020}^{*}b_{002}^{*}}{2\gamma_3 - \gamma_2} \\
&\quad + \frac{2a_{002}^{*}c_{011}^{*}}{\gamma_2} + \frac{a_{002}^{*}a_{110}^{*}}{2\gamma_3 - \gamma_1},
\end{aligned}$$

$$
\begin{aligned}
e_{10} &= \frac{2a^*_{200}a^*_{011}}{\gamma_2+\gamma_3-\gamma_1} + \frac{a^*_{101}c^*_{011} - a^*_{110}b^*_{011}}{\gamma_2} - \frac{2a^*_{002}c^*_{110}}{2\gamma_3-\gamma_1} \\
&\quad + \frac{a^*_{011}\left(b^*_{110}+c^*_{101}\right)}{\gamma_1} + \frac{2a^*_{020}b^*_{101}}{\gamma_1+\gamma_3-\gamma_2}; \\
e_{21} &= \frac{2a^*_{200}b^*_{200}}{\gamma_1} + \frac{b^*_{200}b^*_{110}}{2\gamma_1-\gamma_2} + \frac{b^*_{101}c^*_{200}}{2\gamma_1-\gamma_3}, \\
e_{22} &= \frac{a^*_{020}b^*_{110}}{2\gamma_2-\gamma_1} + \frac{2b^{*2}_{020}}{\gamma_2} + \frac{b^*_{011}c^*_{020}}{2\gamma_2-\gamma_3}, \\
e_{23} &= \frac{b^*_{002}\left(b^*_{011}-2c^*_{002}\right)}{2\gamma_3-\gamma_2} + \frac{a^*_{002}b^*_{101}}{2\gamma_3-\gamma_1}, \\
e_{24} &= \frac{2b^*_{020}b^*_{200}}{2\gamma_1-\gamma_2} + \frac{b^*_{011}c^*_{200}}{2\gamma_1-\gamma_3} \\
&\quad + \frac{b^*_{110}\left(a^*_{200}+b^*_{110}\right)}{\gamma_1} + \frac{2b^*_{200}a^*_{110}}{\gamma_2}, \\
e_{25} &= \frac{b^*_{110}b^*_{101}}{\gamma_1+\gamma_3-\gamma_2} + \frac{b^*_{101}\left(c^*_{101}+a^*_{200}\right)}{\gamma_1} \\
&\quad + \frac{2b^*_{002}c^*_{200}}{2\gamma_1-\gamma_3} + \frac{b^*_{200}\left(b^*_{011}-2a^*_{101}\right)}{2\gamma_1-\gamma_2}, \\
e_{26} &= \frac{2b^*_{020}b^*_{110}}{\gamma_1} + \frac{b^*_{110}\left(a^*_{110}+b^*_{020}\right)}{\gamma_2} \\
&\quad + \frac{b^*_{101}c^*_{020}}{2\gamma_2-\gamma_3} + \frac{2b^*_{200}a^*_{020}}{2\gamma_2-\gamma_1}, \\
e_{27} &= \frac{b^*_{101}\left(b^*_{011}-a^*_{101}-c^*_{002}\right)}{\gamma_1+\gamma_3-\gamma_2} + \frac{2b^*_{200}a^*_{002}}{2\gamma_3-\gamma_1} \\
&\quad + \frac{2b^*_{002}c^*_{101}}{\gamma_1} + \frac{b^*_{110}b^*_{002}}{2\gamma_3-\gamma_2}, \\
e_{28} &= \frac{b^*_{110}a^*_{011}}{\gamma_2+\gamma_3-\gamma_1} + \frac{a^*_{020}b^*_{101}}{2\gamma_2-\gamma_1} \\
&\quad + \frac{b^*_{011}\left(c^*_{011}-b^*_{020}\right)}{\gamma_2} + \frac{2b^*_{002}c^*_{020}}{2\gamma_2-\gamma_3}, \\
e_{29} &= \frac{a^*_{011}b^*_{101}}{\gamma_2+\gamma_3-\gamma_1} + \frac{2b^*_{020}b^*_{002}}{2\gamma_3-\gamma_2} + \frac{2b^*_{002}c^*_{011}}{\gamma_2} + \frac{a^*_{002}b^*_{110}}{2\gamma_3-\gamma_1}, \\
e_{20} &= \frac{2b^*_{200}a^*_{011}}{\gamma_2+\gamma_3-\gamma_1} + \frac{b^*_{101}\left(c^*_{011}+a^*_{110}\right)}{\gamma_2} - \frac{2b^*_{002}c^*_{110}}{2\gamma_3-\gamma_2} \\
&\quad + \frac{b^*_{011}c^*_{101} - b^*_{110}a^*_{101}}{\gamma_1} + \frac{2b^*_{020}b^*_{101}}{\gamma_1+\gamma_3-\gamma_2}; \\
e_{31} &= \frac{c^*_{101}c^*_{200}}{2\gamma_1-\gamma_3} + \frac{2a^*_{200}c^*_{200}}{\gamma_1} + \frac{c^*_{110}b^*_{200}}{2\gamma_1-\gamma_2}, \\
e_{32} &= \frac{2c^*_{020}b^*_{020}}{\gamma_2} + \frac{a^*_{020}c^*_{110}}{2\gamma_2-\gamma_1} + \frac{c^*_{011}c^*_{020}}{2\gamma_2-\gamma_3}, \\
e_{33} &= \frac{a^*_{002}c^*_{101}}{2\gamma_3-\gamma_1} + \frac{c^*_{011}b^*_{002}}{2\gamma_3-\gamma_2}, \\
e_{34} &= \frac{2c^*_{020}b^*_{200}}{2\gamma_1-\gamma_2} + \frac{c^*_{011}c^*_{200}}{2\gamma_1-\gamma_3} \\
&\quad + \frac{c^*_{110}\left(a^*_{200}+b^*_{110}-c^*_{101}\right)}{\gamma_1} + \frac{2c^*_{200}a^*_{110}}{\gamma_2}, \\
e_{35} &= \frac{c^*_{110}b^*_{101}}{\gamma_1+\gamma_3-\gamma_2} + \frac{c^*_{101}\left(c^*_{101}+a^*_{200}\right)}{\gamma_1} \\
&\quad + \frac{2c^*_{200}\left(c^*_{002}-a^*_{101}\right)}{2\gamma_1-\gamma_3} + \frac{c^*_{011}b^*_{200}}{2\gamma_1-\gamma_2}, \\
e_{36} &= \frac{2c^*_{020}b^*_{110}}{\gamma_1} + \frac{c^*_{110}\left(a^*_{110}+b^*_{020}-c^*_{011}\right)}{\gamma_2} \\
&\quad + \frac{c^*_{101}c^*_{020}}{2\gamma_2-\gamma_3} + \frac{2c^*_{200}a^*_{020}}{2\gamma_2-\gamma_1}, \\
e_{37} &= \frac{c^*_{011}b^*_{101}}{\gamma_1+\gamma_3-\gamma_2} + \frac{2c^*_{200}a^*_{002}}{2\gamma_3-\gamma_1} \\
&\quad + \frac{c^*_{101}\left(c^*_{002}-a^*_{101}\right)}{\gamma_1} + \frac{c^*_{110}b^*_{002}}{2\gamma_3-\gamma_2}, \\
e_{38} &= \frac{c^*_{110}a^*_{011}}{\gamma_2+\gamma_3-\gamma_1} + \frac{a^*_{020}c^*_{101}}{2\gamma_2-\gamma_1} + \frac{c^*_{011}\left(b^*_{020}+c^*_{011}\right)}{\gamma_2} \\
&\quad + \frac{2c^*_{020}\left(c^*_{002}-b^*_{011}\right)}{2\gamma_2-\gamma_3}, \\
e_{39} &= \frac{a^*_{011}c^*_{101}}{\gamma_2+\gamma_3-\gamma_1} + \frac{2c^*_{020}b^*_{002}}{2\gamma_3-\gamma_2} \\
&\quad + \frac{c^*_{011}\left(c^*_{002}-b^*_{011}\right)}{\gamma_2} + \frac{a^*_{002}c^*_{110}}{2\gamma_3-\gamma_1}, \\
e_{30} &= \frac{2c^*_{200}a^*_{011}}{\gamma_2+\gamma_3-\gamma_1} + \frac{c^*_{011}\left(b^*_{110}+c^*_{101}\right)}{\gamma_2} \\
&\quad + \frac{c^*_{011}\left(b^*_{110}+c^*_{101}\right)}{\gamma_1} + \frac{2c^*_{020}b^*_{101}}{\gamma_1+\gamma_3-\gamma_2}.
\end{aligned}
\tag{52}
$$

We make a further change $z = w + Q(w) \equiv h_2(w)$, where $Q = (Q_1, Q_2, Q_3)$ is homogeneous cubic polynomial, so that (48) becomes

$$
\begin{aligned}
\dot{w} &= \left[Dh_2(w)\right]^{-1} \cdot \dot{z} \\
&= \left(I - DQ + O\left(|DQ|^2\right)\right) \cdot \dot{z}
\end{aligned}
$$

$$= \begin{pmatrix} \gamma_1 w_1 + a_{101}^{*} w_1 w_3 \\ \gamma_2 w_2 + b_{011}^{*} w_2 w_3 \\ \gamma_3 w_3 + c_{002}^{*} w_3^2 + c_{110}^{*} w_1 w_2 \end{pmatrix} + \begin{pmatrix} \gamma_1 Q_1 - \sum_{j=1}^{3} Q_{1j}\gamma_j w_j \\ \gamma_2 Q_2 - \sum_{j=1}^{3} Q_{2j}\gamma_j w_j \\ \gamma_3 Q_3 - \sum_{j=1}^{3} Q_{3j}\gamma_j w_j \end{pmatrix} + L(w) + O\left(w^4\right), \tag{53}$$

where $Q_{ij} = \partial Q_i/\partial w_j$, $i, j = 1, 2, 3$ and L has the form as before. In order to eliminate some possibly cubic terms, we consider the equations below

$$\gamma_i Q_i - \sum_{j=1}^{3} Q_{ij}\gamma_j w_j + L_i(w) = 0, \quad i = 1, 2, 3. \tag{54}$$

Suppose that for $i = 1, 2, 3$,

$$Q_i = q_{i1} w_1^3 + q_{i2} w_2^3 + q_{i3} w_3^3 + q_{i4} w_1^2 w_2 + q_{i5} w_1^2 w_3 + q_{i6} w_1 w_2^2 + q_{i7} w_1 w_3^2 + q_{i8} w_2^2 w_3 + q_{i9} w_2 w_3^2 + q_{i0} w_1 w_2 w_3. \tag{55}$$

By inserting these representations into (54), we can solve as before

$$\begin{aligned} Q_1 = {} & \frac{e_{11}}{2\gamma_1} w_1^3 + \frac{e_{12}}{3\gamma_2 - \gamma_1} w_2^3 + \frac{e_{13}}{3\gamma_3 - \gamma_1} w_3^3 \\ & + \frac{e_{15}}{\gamma_1 + \gamma_3} w_1^2 w_3 + \frac{e_{16}}{2\gamma_2} w_1 w_2^2 \\ & + \frac{e_{18}}{2\gamma_2 + \gamma_3 - \gamma_1} w_2^2 w_3 \\ & + \frac{e_{19}}{\gamma_2 + 2\gamma_3 - \gamma_1} w_2 w_3^2 \\ & + \frac{e_{10}}{\gamma_2 + \gamma_3} w_1 w_2 w_3, \\ Q_2 = {} & \frac{e_{21}}{3\gamma_1 - \gamma_2} w_1^3 + \frac{e_{22}}{2\gamma_2} w_2^3 + \frac{e_{23}}{3\gamma_3 - \gamma_2} w_3^3 \\ & + \frac{e_{24}}{2\gamma_1} w_1^2 w_2 + \frac{e_{25}}{2\gamma_1 + \gamma_3 - \gamma_2} w_1^2 w_3 \\ & + \frac{e_{27}}{\gamma_1 + 2\gamma_3 - \gamma_2} w_1 w_3^2 + \frac{e_{28}}{\gamma_2 + \gamma_3} w_2^2 w_3 \\ & + \frac{e_{20}}{\gamma_1 + \gamma_3} w_1 w_2 w_3, \\ Q_3 = {} & \frac{e_{31}}{3\gamma_1 - \gamma_3} w_1^3 + \frac{e_{32}}{3\gamma_2 - \gamma_3} w_2^3 \\ & + \frac{e_{34}}{2\gamma_1 + \gamma_2 - \gamma_3} w_1^2 w_2 + \frac{e_{35}}{2\gamma_1} w_1^2 w_3 \\ & + \frac{e_{36}}{\gamma_1 + 2\gamma_2 - \gamma_3} w_1 w_2^2 + \frac{e_{37}}{\gamma_1 + \gamma_3} w_1 w_3^2 \\ & + \frac{e_{38}}{2\gamma_2} w_2^2 w_3 + \frac{e_{39}}{\gamma_2 + \gamma_3} w_2 w_3^2. \end{aligned} \tag{56}$$

Hence, system (53) becomes now

$$\begin{aligned} \dot{w}_1 &= \gamma_1 w_1 + a_{101}^{*} w_1 w_3 + e_{14} w_1^2 w_2 + e_{17} w_1 w_3^2 \\ &\quad + O\left(\left|w_1, w_2, w_3\right|^4\right), \\ \dot{w}_2 &= \gamma_2 w_2 + b_{011}^{*} w_2 w_3 + e_{26} w_1 w_2^2 + e_{29} w_2 w_3^2 \\ &\quad + O\left(\left|w_1, w_2, w_3\right|^4\right), \\ \dot{w}_3 &= \gamma_3 w_3 + c_{002}^{*} w_3^2 + c_{110}^{*} w_1 w_2 + e_{33} w_3^3 + e_{30} w_1 w_2 w_3 \\ &\quad + O\left(\left|w_1, w_2, w_3\right|^4\right), \end{aligned} \tag{57}$$

where w and all of the coefficients are complex. Finally making the change $w = Tx$ and then taking the real parts of x and the coefficients of all terms of the resulting system, we can get a cubic real normal form of the form (7) with

$$\begin{aligned} a_1 &= \frac{1}{4}\left(2\tilde{a}_{101} + \tilde{b}_{011} - \tilde{a}_{011}\right), \\ b_1 &= \frac{1}{4}\left(\tilde{a}_{011} + \tilde{b}_{011} - 2\tilde{b}_{101}\right), \\ c_1 &= \frac{1}{2}\left(\tilde{c}_{200} + \tilde{c}_{020}\right), \\ d_1 &= \tilde{c}_{002}, \\ a_2 &= \frac{1}{8\left(\lambda_1^2 + 1\right)}\left[\lambda_1\left(5\tilde{a}_{200}^2 + 5\tilde{b}_{020}^2 - \tilde{a}_{020}^2 - \tilde{b}_{200}^2 \right.\right. \\ &\qquad\qquad + 4\tilde{a}_{200}\tilde{a}_{020} + 4\tilde{b}_{200}\tilde{b}_{020} + 3\tilde{b}_{110} \\ &\qquad\qquad \left. \times\left(\tilde{a}_{200} + \tilde{a}_{020}\right) + 3\tilde{a}_{110}\left(\tilde{b}_{200} + \tilde{b}_{020}\right)\right) \\ &\qquad - 2\tilde{a}_{200}\tilde{b}_{200} + 2\tilde{a}_{020}\tilde{b}_{020} \\ &\qquad \left. + \tilde{a}_{110}\left(\tilde{a}_{200} + \tilde{a}_{020}\right) - \tilde{b}_{110}\left(\tilde{b}_{200} + \tilde{b}_{020}\right)\right] \\ &\quad - \frac{\lambda_1}{8\left(\lambda_1^2 + 9\right)} \\ &\quad \times\left[\left(\tilde{a}_{020} - \tilde{a}_{200} + \tilde{b}_{110}\right)^2 + \left(-\tilde{a}_{110} + \tilde{b}_{020} - \tilde{b}_{200}\right)^2\right] \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{8\left[\left(2\lambda_1-\lambda_2\right)^2+4\right]}\\
&\times\Big(-2\big(\big(\tilde{a}_{011}+\tilde{b}_{101}\big)\left(\tilde{c}_{200}-\tilde{c}_{020}\right)\\
&\qquad+\big(-\tilde{a}_{101}+\tilde{b}_{011}\big)\tilde{c}_{110}\big)\\
&\quad+\left(2\lambda_1-\lambda_2\right)\big(\big(-\tilde{a}_{101}+\tilde{b}_{011}\big)\left(\tilde{c}_{200}-\tilde{c}_{020}\right)\\
&\qquad-\big(\tilde{a}_{011}+\tilde{b}_{101}\big)\tilde{c}_{110}\big)\Big),\\
b_2={}&\frac{1}{8\left(\lambda_1^2+1\right)}\\
&\times\Big(-\lambda_1\big(6\tilde{a}_{200}\tilde{b}_{200}-6\tilde{a}_{020}\tilde{b}_{020}+3\tilde{b}_{110}\big(\tilde{b}_{200}+\tilde{b}_{020}\big)\\
&\qquad-3\tilde{a}_{110}\left(\tilde{a}_{200}+\tilde{a}_{020}\right)\big)+\tilde{a}_{200}^2+3\tilde{a}_{020}^2+3\tilde{b}_{200}^2\\
&\quad+\tilde{b}_{020}^2+4\tilde{a}_{200}\tilde{a}_{020}+4\tilde{b}_{200}\tilde{b}_{020}-\tilde{b}_{110}\left(\tilde{a}_{200}+\tilde{a}_{020}\right)\\
&\quad-\tilde{a}_{110}\big(\tilde{b}_{200}+\tilde{b}_{020}\big)\Big)\frac{3}{8\left(\lambda_1^2+9\right)}\\
&\times\Big(\big(\tilde{a}_{020}-\tilde{a}_{200}+\tilde{b}_{110}\big)^2+\big(-\tilde{a}_{110}+\tilde{b}_{020}-\tilde{b}_{200}\big)^2\Big)\\
&-\frac{1}{8\left[\left(2\lambda_1-\lambda_2\right)^2+4\right]}\\
&\times\Big(\left(2\lambda_1-\lambda_2\right)\big(\big(\tilde{a}_{011}+\tilde{b}_{101}\big)\left(\tilde{c}_{200}-\tilde{c}_{020}\right)\\
&\qquad+\big(-\tilde{a}_{101}+\tilde{b}_{011}\big)\tilde{c}_{110}\big)\\
&\quad+2\big(\big(-\tilde{a}_{101}+\tilde{b}_{011}\big)\left(\tilde{c}_{200}-\tilde{c}_{020}\right)\\
&\qquad-\big(\tilde{a}_{011}+\tilde{b}_{101}\big)\tilde{c}_{110}\big)\Big),\\
a_3={}&\frac{1}{2\left[\left(2\lambda_2-\lambda_1\right)^2+1\right]}\\
&\times\Big(\left(2\lambda_2-\lambda_1\right)\\
&\qquad\times\big(\tilde{a}_{002}\big(2\tilde{a}_{200}+\tilde{b}_{110}\big)+\tilde{b}_{002}\big(2\tilde{b}_{020}+\tilde{a}_{110}\big)\big)\\
&\quad+\tilde{a}_{002}\big(2\tilde{b}_{020}+\tilde{a}_{110}\big)-\tilde{b}_{002}\big(2\tilde{a}_{200}+\tilde{b}_{110}\big)\Big)\\
&+\tilde{c}_{011}\Big(\lambda_2\big(\big(\tilde{a}_{011}+\tilde{b}_{101}\big)\big(\tilde{b}_{101}-\tilde{b}_{011}\big)-\big(-\tilde{a}_{101}+\tilde{b}_{011}\big)\\
&\qquad\times\left(\tilde{a}_{101}-\tilde{a}_{011}\right)\big)\\
&\quad+2\big(\big(\tilde{a}_{011}+\tilde{b}_{101}\big)\left(\tilde{a}_{101}-\tilde{a}_{011}\right)\\
&\qquad+\big(-\tilde{a}_{101}+\tilde{b}_{011}\big)\big(\tilde{b}_{101}-\tilde{b}_{011}\big)\big)\Big)\\
&+\frac{1}{\lambda_1^2+1}\big(\lambda_1\tilde{a}_{002}+\tilde{b}_{002}\big)\left(\tilde{a}_{101}-\tilde{a}_{011}\right),\\
b_3={}&\frac{1}{2\left[\left(2\lambda_2-\lambda_1\right)^2+1\right]}\\
&\times\Big(\big(\tilde{a}_{002}\big(2\tilde{a}_{020}-\tilde{b}_{110}\big)+\tilde{b}_{002}\big(2\tilde{b}_{200}-\tilde{a}_{110}\big)\big)\\
&\quad-\left(2\lambda_2-\lambda_1\right)\\
&\quad\times\big(\tilde{a}_{002}\big(2\tilde{b}_{200}-\tilde{a}_{110}\big)-\tilde{b}_{002}\big(2\tilde{a}_{020}-\tilde{b}_{110}\big)\big)\Big)\\
&+\frac{1}{4\left(\lambda_2^2+4\right)}\Big(2\big(\big(\tilde{a}_{011}+\tilde{b}_{101}\big)\big(\tilde{b}_{101}-\tilde{b}_{011}\big)\\
&\qquad-\big(-\tilde{a}_{101}+\tilde{b}_{011}\big)\left(\tilde{a}_{101}-\tilde{a}_{011}\right)\big)\\
&\quad-\lambda_2\big(\big(\tilde{a}_{011}+\tilde{b}_{101}\big)\left(\tilde{a}_{101}-\tilde{a}_{011}\right)\\
&\qquad+\big(-\tilde{a}_{101}+\tilde{b}_{011}\big)\\
&\qquad\times\big(\tilde{b}_{101}-\tilde{b}_{011}\big)\big)\Big)+\frac{1}{\lambda_1^2+1}\\
&\times\big(\tilde{a}_{002}-\lambda_1\tilde{b}_{002}\big)\left(\tilde{a}_{101}-\tilde{a}_{011}\right),\\
c_2={}&\frac{\lambda_1}{2\left(\lambda_1^2+1\right)}\Big[\tilde{c}_{011}\big(\tilde{b}_{200}+\tilde{b}_{020}\big)\\
&\qquad+\tilde{c}_{101}\left(\tilde{a}_{200}+\tilde{a}_{020}+\tilde{c}_{101}-\tilde{c}_{011}\right)\Big]\\
&+\frac{\lambda_2}{4\left(\lambda_2^2+4\right)}\Big(\tilde{c}_{110}\big(\tilde{a}_{011}+2\tilde{b}_{101}-\tilde{b}_{011}\big)\\
&\qquad+\big(2\tilde{a}_{101}-\tilde{b}_{011}-\tilde{a}_{011}\big)\\
&\qquad\times\left(\tilde{c}_{200}-\tilde{c}_{020}\right)\Big)\\
&+\frac{1}{2\left(\lambda_2^2+4\right)}\Big(\tilde{c}_{110}\big(2\tilde{a}_{101}-\tilde{b}_{011}-\tilde{a}_{011}\big)\\
&\qquad-\big(\tilde{a}_{011}+2\tilde{b}_{101}-\tilde{b}_{011}\big)\\
&\qquad\times\left(\tilde{c}_{200}-\tilde{c}_{020}\right)\Big),\\
d_2={}&\frac{2\lambda_2-\lambda_1}{2\left[\left(2\lambda_2-\lambda_1\right)^2+1\right]}\\
&\times\big(2\tilde{a}_{002}\tilde{c}_{101}-\tilde{a}_{002}\tilde{c}_{011}+\tilde{b}_{002}\tilde{c}_{011}\big)\\
&+\frac{1}{2\left[\left(2\lambda_2-\lambda_1\right)^2+1\right]}\\
&\times\big(\tilde{a}_{002}\tilde{c}_{011}-2\tilde{b}_{002}\tilde{c}_{101}+\tilde{b}_{002}\tilde{c}_{011}\big).
\end{aligned}
\tag{58}
$$

Then, by the equations in (36), we finally get the relationship between the coefficients of the system (2) and of the normal form (7).

4. Examples

4.1. An Example about the Existence of a Limit Cycle in Three-Dimensional Lotka-Volterra Systems. In this section, we construct a concrete example of three-dimensional Lotka-Volterra systems according to Theorem 1. It is shown that this system undergoes nonisolated zero-Hopf bifurcation.

We consider the following three-parameter Lotka-Volterra system in the first octant $\mathbb{R}^3_+$. Consider

$$\begin{aligned}
\frac{dx}{dt} &= x(-vx + vy + vz - v),\\
\frac{dy}{dt} &= y(-2vx - 2vy - vz + 5v),\\
\frac{dz}{dt} &= \frac{z}{2v^2}\Big(-x\big(6v^3 + 6v^2\varepsilon + 10v^2u\varepsilon + 6vu\varepsilon^2 + 3vu^2\varepsilon^2 + u^2\varepsilon^3\big)\\
&\quad - y\big(-6v^3 - 2v^2u\varepsilon + 2vu\varepsilon^2 + vu^2\varepsilon^2 + u^2\varepsilon^3\big)\\
&\quad + z\big(6v^3 + 2v^2\varepsilon + 4v^2u\varepsilon\big)\\
&\quad + \big(-6v^3 + 4v^2\varepsilon + 4v^2u\varepsilon + 8vu\varepsilon^2 + 4vu^2\varepsilon^2 + 2u^2\varepsilon^3\big)\Big),
\end{aligned} \tag{59}$$

where $0 < \varepsilon \ll 1$, $v > 0$ and u are bounded parameters.

First of all, we need to change the system (59) to the form of system (2) as in [14]. It can be checked that the point $(1, 1, 1)$ is zero-Hopf equilibrium of system (59). We do the change of variables $X = x - 1$, $Y = y - 1$, and $Z = z - 1$ to obtain

$$\begin{aligned}
\frac{dX}{dt} &= v(1+X)(-X+Y+Z),\\
\frac{dY}{dt} &= v(1+Y)(-2X-2Y-Z),\\
\frac{dZ}{dt} &= \frac{1+Z}{2v^2}\Big(-X\big(6v^3 + 6v^2\varepsilon + 10v^2u\varepsilon + 6vu\varepsilon^2 + 3vu^2\varepsilon^2 + u^2\varepsilon^3\big)\\
&\quad - Y\big(-6v^3 - 2v^2u\varepsilon + 2vu\varepsilon^2 + vu^2\varepsilon^2 + u^2\varepsilon^3\big)\\
&\quad + Z\big(6v^3 + 2v^2\varepsilon + 4v^2u\varepsilon\big)\Big).
\end{aligned} \tag{60}$$

The Jacobian matrix of system (60) at $(0, 0, 0)$ has eigenvalues ε, $\varepsilon u + vi$ and $\varepsilon u - vi$ with $v > 0$. According to [14], in order to obtain the real Jordan normal form of system (60) at the origin, we do the linear transformation

$$\begin{pmatrix} U_1\\ V_1\\ W_1\end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & p_{13}\\ p_{21} & 1 & 0\\ p_{31} & p_{32} & 1\end{pmatrix}\begin{pmatrix} X\\ Y\\ Z\end{pmatrix}, \tag{61}$$

where $p_{11} = -(u\varepsilon^2 + 3v\varepsilon + 3vu\varepsilon + 5v^2)/(v(v+\varepsilon))$, $p_{12} = -(-v^2 + vu\varepsilon + v\varepsilon + u\varepsilon^2)/(v(v+\varepsilon))$, $p_{13} = 2v/(v+\varepsilon)$, $p_{21} = (3v + \varepsilon)/(v+\varepsilon)$, $p_{31} = -(6v^2 + 6vu\varepsilon + u^2\varepsilon^2)/2v^2$, and $p_{32} = -u\varepsilon(2v + u\varepsilon)/2v^2$. Then, in the new variables (U_1, V_1, and W_1) system (60) becomes

$$\begin{aligned}
\frac{dU_1}{dt} &= u\varepsilon U_1 + vV_1 + \sum_{i+j+k=2} a_{ijk}U_1^iV_1^jW_1^k,\\
\frac{dV_1}{dt} &= -vU_1 + u\varepsilon V_1 + \sum_{i+j+k=2} b_{ijk}U_1^iV_1^jW_1^k,\\
\frac{dW_1}{dt} &= \varepsilon W_1 + \sum_{i+j+k=2} c_{ijk}U_1^iV_1^jW_1^k,
\end{aligned} \tag{62}$$

where a_{ijk}, b_{ijk}, and c_{ijk} have the following expressions:

$$\begin{aligned}
a_{011} &= 21v + (-11 + 90u)\varepsilon + O(\varepsilon^2),\\
a_{020} &= 3v + (24 - 17u)\varepsilon + O(\varepsilon^2)\\
a_{002} &= -18v - (36u + 30)\varepsilon + O(\varepsilon^2),\\
a_{110} &= 29v + \left(\frac{16}{3} + 93u\right)\varepsilon + O(\varepsilon^2),\\
a_{101} &= -30v - (60u + 72)\varepsilon + O(\varepsilon^2),\\
a_{200} &= -9v - (22u + 40)\varepsilon + O(\varepsilon^2),\\
b_{101} &= -6v + 22\varepsilon + O(\varepsilon^2),\\
b_{011} &= -12v - (6u + 15)\varepsilon + O(\varepsilon^2),\\
b_{110} &= -9v - 24\varepsilon + O(\varepsilon^2),\\
b_{200} &= -6v + 8\varepsilon + O(\varepsilon^2),\\
b_{020} &= 4v + \left(-\frac{22}{3} + 15u\right)\varepsilon + O(\varepsilon^2),\\
b_{002} &= 12\varepsilon + O(\varepsilon^2),\\
c_{110} &= 36v + (42 + 102u)\varepsilon + O(\varepsilon^2),\\
c_{101} &= -27v - (123 + 54u)\varepsilon + O(\varepsilon^2),\\
c_{011} &= 30v + (10 + 105u)\varepsilon + O(\varepsilon^2),\\
c_{200} &= -6v - (60 + 16u)\varepsilon + O(\varepsilon^2),\\
c_{002} &= -18v - (57 + 36u)\varepsilon + O(\varepsilon^2),\\
c_{020} &= (36 - 32u)\varepsilon + O(\varepsilon^2).
\end{aligned} \tag{63}$$

Next, we need to calculate the partial coefficients of the normal form of system (63). We can get $\tilde{a}_{ijk}$, $\tilde{b}_{ijk}$, and $\tilde{c}_{ijk}$ by (36), and then by the formulas of (58) we have

$$\begin{aligned} a_1 &= \frac{21}{4} + \frac{1}{v}(12+24u)\varepsilon + O(\varepsilon^2), \\ a_2 &= 90 - \frac{1}{16v}(5875+6244u)\varepsilon + O(\varepsilon^2), \\ a_3 &= 45 - 97740v \\ &\quad - \frac{3}{v}(121 - 264u + 31860v + 299790uv)\varepsilon + O(\varepsilon^2), \\ c_1 &= -3 - \frac{12}{v}(1+2u)\varepsilon + O(\varepsilon^2), \\ c_2 &= -\frac{657}{4} + \frac{3}{8v}(-3263+226u)\varepsilon + O(\varepsilon^2), \end{aligned} \tag{64}$$

$$\begin{aligned} d_1 &= -18 - \frac{3}{v}(19+12u)\varepsilon + O(\varepsilon^2), \\ d_2 &= -756 - \frac{9}{v}(436+351u)\varepsilon + O(\varepsilon^2). \end{aligned} \tag{65}$$

By Theorem 1, we have the following conclusion.

Theorem 3. *For any given $\varepsilon_0 > 0$, suppose that $-7/(24+7\varepsilon_0) < u < 0$, and then for $0 < \varepsilon \ll \varepsilon_0$, (59) has a periodic orbit near the origin, which is unstable.*

Proof. In this example, it is easy to see that $\delta_1 = 1$, $\delta_2 = 1/u$. From (64) and (10) we can get $\tilde{c}_1 = \tilde{d}_1 < 0$, thus, in order to satisfy (18), we need

$$\tilde{d}_1 - \delta_1\delta_2 = -\frac{24}{7} - \frac{1}{u} + \frac{4}{49}\frac{-37+108u}{v}\varepsilon + O(\varepsilon^2) > 0. \tag{66}$$

For any given $\varepsilon_0 > 0$, suppose that $-7/(24+7\varepsilon_0) < u < 0$. It can be checked that $\tilde{d}_1 - \delta_1\delta_2 > 0$ for $0 < \varepsilon \ll \varepsilon_0$. Then, by Theorem 1, (59) has a periodic orbit near the origin. Next, we consider the stability of the periodic orbit.

From (64), we can also get

$$\begin{aligned} \Delta &= \frac{49+280u+960u^2}{49u^2} - \frac{16}{343} \\ &\quad \times \frac{-259-502u+3672u^2}{uv} + O(\varepsilon^2) > 0, \end{aligned} \tag{67}$$

when $-7/(24+7\varepsilon_0) < u < 0$ holds, where Δ is given by (20). So none of the conditions (a), (b), or (c) in Theorem 1 holds; further, we know that the periodic orbit is unstable. □

Remark 4. From (63), we can find out that system (59) does not satisfy the conditions mentioned in [14]. Thus, we cannot use the results in [14] to study the existence of a limit cycle in (59).

4.2. An Example about the Existence of an Invariant Torus. For convenience, we give an example about the existence of an invariant torus in a system, which has the form of (2). We consider the following system in the first octant $\mathbb{R}^3_+$:

$$\begin{aligned} \frac{dU_2}{dt} &= -\frac{5}{24}\varepsilon U_2 - \frac{5}{24}V_2 + 3U_2^2 - 2V_2^2 \\ &\quad + 3W_2^2 - 15U_2V_2 + 2U_2W_2 + 5V_2W_2, \\ \frac{dV_2}{dt} &= \frac{5}{24}U_2 - \frac{5}{24}\varepsilon V_2 + 9U_2^2 - 5V_2^2 + 6W_2^2 \\ &\quad + U_2V_2 - 4V_2W_2, \\ \frac{dW_2}{dt} &= \varepsilon W_2 - 4U_2^2 + 8V_2^2 + 3W_2^2 + 510U_2V_2 \\ &\quad - 7U_2W_2 - 5V_2W_2, \end{aligned} \tag{68}$$

where $0 < \varepsilon \ll 1$.

According to Section 3, we have

$$\begin{aligned} a_1 &= 6, \\ a_2 &= -\frac{116136}{25} + \frac{1409296}{125}\varepsilon + O(\varepsilon^2), \\ a_3 &= \frac{180576}{25} + \frac{2081376}{125}\varepsilon + O(\varepsilon^2), \\ c_1 &= -\frac{48}{5}, \\ c_2 &= \frac{117936}{25} - \frac{3953376}{125}\varepsilon + O(\varepsilon^2), \\ d_1 &= -\frac{72}{5}, \\ d_2 &= \frac{11232}{25} + \frac{870048}{125}\varepsilon + O(\varepsilon^2). \end{aligned} \tag{69}$$

Let $A_1 = 1666324684/40625$ and $A_2 = 1731664/3125$. Then, we have the following theorem.

Theorem 5. *For any given $0 < \varepsilon_1 < A_1 - A_2$, there exists an $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, (68) has a unique invariant torus near the origin, which is unstable.*

Proof. By (69) and (29), we can obtain

$$\begin{aligned} \delta_0 &= \frac{14892}{125} - \frac{1753792}{375}\varepsilon - \frac{92710806}{3125}\varepsilon^2 + O(\varepsilon^3), \\ \Delta_0 &= \frac{13251}{50} + \frac{216458}{1875}\varepsilon + \frac{3772653}{12500}\varepsilon^2 + O(\varepsilon^3), \\ \delta_0' &= -\frac{248104}{1625} - \frac{70129552}{14625}\varepsilon - \frac{1217815988}{40625}\varepsilon^2 + O(\varepsilon^2). \end{aligned} \tag{70}$$

Thus, for $0 < \varepsilon \ll 1$, $\Delta_0 > 0$. Further, we can get

$$\Delta_0\phi_1(\lambda_1) - \varepsilon_1\lambda_1^2 = -\frac{159012}{125}\varepsilon - (A_1 - \varepsilon_1)\varepsilon^2 + O(\varepsilon^3),$$

$$\Delta_0\lambda_2 = -\frac{159012}{125}\varepsilon - A_2\varepsilon^2 + O\left(\varepsilon^3\right),$$
$$\Delta_0\phi_0\left(\lambda_1\right) = -\frac{159012}{125}\varepsilon + \frac{96935282}{3125}\varepsilon^2 + O\left(\varepsilon^3\right), \tag{71}$$

where $\phi_1(\lambda_1)$ and $\phi_0(\lambda_1)$ are defined in Theorem 2, and here $\lambda_1 = \varepsilon$ and $\lambda_2 = -(24/5)\varepsilon$. By some easy calculations, we can obtain that for $0 < \varepsilon_1 < A_1 - A_2$ inequality $\Delta_0\phi_1(\lambda_1) - \varepsilon_1\lambda_1^2 < \Delta_0\lambda_2 < \Delta_0\phi_0(\lambda_1)$ holds. Thus, by Theorem 2 we can get the result in this theorem. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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