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*Research Article*

# An Extension of Young's Inequality

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Young's inequality is extended to the context of absolutely continuous measures. Several applications are included.

## 1. Introduction

Young's inequality [1] asserts that every strictly increasing continuous function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $\lim_{x\to\infty} f(x) = \infty$ verifies an inequality of the following form:

$$ab \leq \int_0^a f(x)dx + \int_0^b f^{-1}(y)dy, \tag{1.1}$$

whenever $a$ and $b$ are nonnegative real numbers. The equality occurs if and only if $f(a) = b$; see [2–5] for details and significant applications.

Several questions arise naturally in connection with this classical result.

(Q1): Is the restriction on strict monotonicity (or on continuity) really necessary?

(Q2): Is there any weighted analogue of Young's inequality?

(Q3): Can Young's inequality be improved?

Cunningham and Grossman [6] noticed that question (Q1) has a positive answer (correcting the prevalent belief that Young's inequality is the business of strictly increasing continuous functions). The aim of the present paper is to extend the entire discussion to the framework of locally absolutely continuous measures and to prove several improvements.

As is well known, Young's inequality is an illustration of the Legendre duality. Precisely, the functions

$$F(a) = \int_0^a f(x)dx, \qquad G(b) = \int_0^b f^{-1}(x)dx \tag{1.2}$$

are both continuous and convex on $[0,\infty)$, and (1.1) can be restated as

$$ab \leq F(a) + G(b) \quad \forall\ a, b \in [0,\infty), \tag{1.3}$$

with equality if and only if $f(a) = b$. Because of the equality case, formula (1.3) leads to the following connection between the functions $F$ and $G$:

$$\begin{aligned} F(a) &= \sup\{ab - G(b) : b \geq 0\}, \\ G(b) &= \sup\{ab - F(a) : a \geq 0\}. \end{aligned} \tag{1.4}$$

It turns out that each of these formulas produces a convex function (possibly on a different interval). Some details are in order.

By definition, the *conjugate* of a convex function $F$ defined on a nondegenerate interval $I$ is the function

$$F^* : I^* \longrightarrow \mathbb{R}, \qquad F^*(y) = \sup\{xy - F(x) : x \in I\}, \tag{1.5}$$

with domain $I^* = \{y \in \mathbb{R} : \ F^*(y) < \infty\}$. Necessarily $I^*$ is a nonempty interval, and $F^*$ is a convex function whose level sets $\{y : F^*(y) \leq \lambda\}$ are closed subsets of $\mathbb{R}$ for each $\lambda \in \mathbb{R}$ (usually such functions are called *closed* convex functions).

A convex function may not be differentiable, but it admits a good substitute for differentiability.

The *subdifferential* of a real function $F$ defined on an interval $I$ is a multivalued function $\partial F : I \to \mathcal{P}(\mathbb{R})$ defined by

$$\partial F(x) = \{\lambda \in \mathbb{R} : F(y) \geq F(x) + \lambda(y - x), \text{ for every } y \in I\}. \tag{1.6}$$

Geometrically, the subdifferential gives us the slopes of the supporting lines for the graph of $F$. The subdifferential at a point is always a convex set, possibly empty, but the convex functions $F : I \to \mathbb{R}$ have the remarkable property that $\partial F(x) \neq \emptyset$ at all interior points. It is worth noticing that $\partial F(x) = \{F'(x)\}$ at each point where $F$ is differentiable (so this formula works for all points of $I$ except for a countable subset), see [4, page 30].

**Lemma 1.1** (Legendre duality, [4, page 41]). *Let $F : I \to \mathbb{R}$ be a closed convex function. Then its conjugate $F^* : I^* \to \mathbb{R}$ is also convex and closed and*

(i) $xy \leq F(x) + F^*(y)$ *for all* $x \in I$, $y \in I^*$;

(ii) $xy = F(x) + F^*(y)$ *if and only if* $y \in \partial F(x)$;

(iii) $\partial F^{*} = (\partial F)^{-1}$ *(as graphs);*

(iv) $F^{**} = F$.

Recall that the inverse of a graph $\Gamma$ is the set $\Gamma^{-1} = \{(y, x) : (x, y) \in \Gamma\}$.

How far is Young's inequality from the Legendre duality? Surprisingly, they are pretty closed in the sense that in most cases the Legendre duality can be converted into a Young-like inequality. Indeed, every continuous convex function admits an integral representation.

**Lemma 1.2** (see [4, page 37]). *Let $F$ be a continuous convex function defined on an interval $I$, and let $\varphi : I \longrightarrow \mathbb{R}$ be a function such that $\varphi(x) \in \partial F(x)$ for every $x \in I$. Then for every $a < b$ in $I$ one has*

$$F(b) - F(a) = \int_a^b \varphi(t)\ dt. \tag{1.7}$$

As a consequence, the heuristic meaning of formula (i) in Lemma 1.1 is the following Young-like inequality:

$$ab \le \int_{a_0}^{a} \varphi(x)dx + \int_{b_0}^{b} \psi(y)dy \quad \forall a \in I,\ b \in I^{*}, \tag{1.8}$$

where $\varphi$ and $\psi$ are selection functions for $\partial F$ and, $(\partial F)^{-1}$ respectively. Now it becomes clear that Young's inequality should work outside strict monotonicity (as well as outside continuity). The details are presented in Section 2. Our approach (based on the geometric meaning of integrals as areas) allows us to extend the framework of integrability to all positive measures $\rho$ which are locally absolutely continuous with respect to the planar Lebesgue measure $dx\,dy$, see Theorem 2.3.

A special case of Young's inequality is

$$xy \le \frac{x^p}{p} + \frac{y^q}{q}, \tag{1.9}$$

which works for all $x, y \ge 0$, and $p, q > 1$ with $1/p + 1/q = 1$. Theorem 2.3 yields the following companion to this inequality in the case of Gaussian measure $(4/2\pi)e^{-x^2-y^2}dx\,dy$ on $[0,\infty) \times [0,\infty)$:

$$\operatorname{erf}(x)\operatorname{erf}(y) \le \frac{2}{\sqrt{\pi}}\int_0^x \operatorname{erf}\left(s^{p-1}\right)e^{-s^2}ds + \frac{2}{\sqrt{\pi}}\int_0^y \operatorname{erf}\left(t^{q-1}\right)e^{-t^2}dt, \tag{1.10}$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}ds \tag{1.11}$$

is the *Gauss error function* (or the erf function).

The precision of our generalization of Young's inequality makes the objective of Section 3.

In Section 4 we discuss yet another extension of Young's inequality, based on recent work done by Pečarić and Jakšetić [7].

The paper ends by noticing the connection of our result to the theory of $c$-convexity (i.e., of convexity associated to a cost density function).

Last but not least, all results in this paper can be extended verbatim to the framework of nondecreasing functions $f : [a_0, a_1) \to [A_0, A_1)$ such that $a_0 < a_1 \leq \infty$ and $A_0 < A_1 \leq \infty$, $f(a_0) = A_0$ and $\lim_{x \to a_1} f(x) = A_1$. In other words, the interval $[0, \infty)$ plays no special role in Young's inequality.

Besides, there is a straightforward companion of Young's inequality for nonincreasing functions, but this is outside the scope of the present paper.

## 2. Young's Inequality for Weighted Measures

In what follows $f : [0, \infty) \to [0, \infty)$ will denote a nondecreasing function such that $f(0) = 0$ and $\lim_{x \to \infty} f(x) = \infty$. Since $f$ is not necessarily injective we will attach to a *pseudoinverse* $f$ by the following formula:

$$f_{\sup}^{-1} : [0, \infty) \longrightarrow [0, \infty), \qquad f_{\sup}^{-1}(y) = \inf\{x \geq 0 : f(x) > y\}. \tag{2.1}$$

Clearly, $f_{\sup}^{-1}$ is nondecreasing and $f_{\sup}^{-1}(f(x)) \geq x$ for all $x$. Moreover, with the convention $f(0-) = 0$,

$$f_{\sup}^{-1}(y) = \sup\{x : y \in [f(x-), f(x+)]\}. \tag{2.2}$$

Here $f(x-)$ and $f(x+)$ represent the lateral limits at $x$. When $f$ is also continuous,

$$f_{\sup}^{-1}(y) = \max\{x \geq 0 : y = f(x)\}. \tag{2.3}$$

*Remark 2.1* (Cunningham and Grossman [6]). Since pseudoinverses will be used as integrands, it is convenient to enlarge the concept of pseudoinverse by referring to any function g such that

$$f_{\inf}^{-1} \leq g \leq f_{\sup}^{-1}, \tag{2.4}$$

where $f_{\inf}^{-1}(y) = \sup\{x \geq 0 : f(x) < y\}$. Necessarily, g is nondecreasing, and any two pseudoinverses agree except on a countable set (so their integrals will be the same).

Given $0 \leq a < b$, we define the *epigraph* and the *hypograph* of $f|_{[a,b]}$, respectively, by

$$\begin{aligned} \operatorname{epi} f|_{[a,b]} &= \{(x, y) \in [a, b] \times [f(a), f(b)] : y \geq f(x)\}, \\ \operatorname{hyp} f|_{[a,b]} &= \{(x, y) \in [a, b] \times [f(a), f(b)] : y \leq f(x)\}. \end{aligned} \tag{2.5}$$

Their intersection is the *graph* of $f|_{[a,b]}$,

$$\operatorname{graph} f|_{[a,b]} = \{(x,y) \in [a,b] \times [f(a), f(b)] : y = f(x)\}. \tag{2.6}$$

Notice that our definitions of epigraph and hypograph are not the standard ones, but agree with them in the context of monotone functions.

We will next consider a measure $\rho$ on $[0,\infty) \times [0,\infty)$, which is locally absolutely continuous with respect to the Lebesgue measure $dx\,dy$ that is, $\rho$ is of the form

$$\rho(A) = \int_A K(x,y)\,dx\,dy, \tag{2.7}$$

where $K : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a Lebesgue locally integrable function, and $A$ is any compact subset of $[0,\infty) \times [0,\infty)$.

Clearly,

$$\begin{aligned}\rho(\operatorname{hyp} f|_{[a,b]}) + \rho(\operatorname{epi} f|_{[a,b]}) &= \rho([a,b] \times [f(a), f(b)]) \\ &= \int_a^b \int_{f(a)}^{f(b)} K(x,y)\,dy\,dx.\end{aligned} \tag{2.8}$$

Moreover,

$$\begin{aligned}\rho(\operatorname{hyp} f|_{[a,b]}) &= \int_a^b \left( \int_{f(a)}^{f(x)} K(x,y)\,dy \right) dx, \\ \rho(\operatorname{epi} f|_{[a,b]}) &= \int_{f(a)}^{f(b)} \left( \int_a^{f_{\sup}^{-1}(y)} K(x,y)\,dx \right) dy.\end{aligned} \tag{2.9}$$

The discussion above can be summarized as follows.

**Lemma 2.2.** *Let $f : [0,\infty) \to [0,\infty)$ be a nondecreasing function such that $f(0) = 0$ and $\lim_{x\to\infty} f(x) = \infty$. Then for every Lebesgue locally integrable function $K : [0,\infty)\times[0,\infty) \to [0,\infty)$ and every pair of nonnegative numbers $a < b$,*

$$\begin{aligned}&\int_a^b \left( \int_{f(a)}^{f(x)} K(x,y)\,dy \right) dx + \int_{f(a)}^{f(b)} \left( \int_a^{f_{\sup}^{-1}(y)} K(x,y)\,dx \right) dy \\ &\quad = \int_a^b \int_{f(a)}^{f(b)} K(x,y)\,dy\,dx.\end{aligned} \tag{2.10}$$

We can now state the main result of this section.

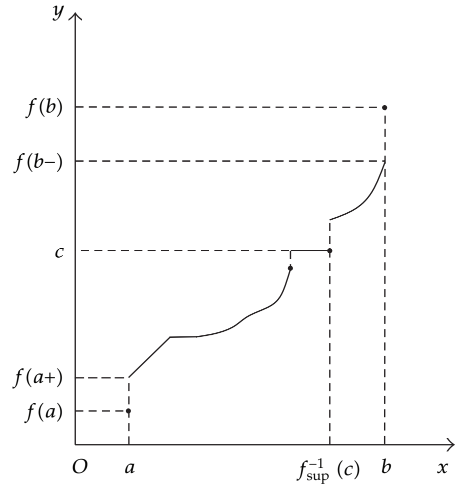


**Figure 1:** The geometry of Young's inequality when $f(a) \leq c \leq f(b-)$.

**Theorem 2.3** (Young's inequality for nondecreasing functions). *Under the assumptions of Lemma 2.2, for every pair of nonnegative numbers $a < b$ and every number $c \geq f(a)$, one has*

$$\int_a^b \int_{f(a)}^c K(x,y)\,dy\,dx \leq \int_a^b \left( \int_{f(a)}^{f(x)} K(x,y)\,dy \right) dx + \int_{f(a)}^c \left( \int_a^{f_{\sup}^{-1}(y)} K(x,y)\,dx \right) dy. \tag{2.11}$$

*If in addition $K$ is strictly positive almost everywhere, then the equality occurs if and only if $c \in [f(b-), f(b+)]$.*

*Proof.* We start with the case where $f(a) \leq c \leq f(b-)$, see Figure 1. In this case,

$$\begin{aligned}
&\int_a^b \left( \int_{f(a)}^{f(x)} K(x,y)\,dy \right) dx + \int_{f(a)}^c \left( \int_a^{f_{\sup}^{-1}(y)} K(x,y)\,dx \right) dy \\
&\quad = \int_a^{f_{\sup}^{-1}(c)} \left( \int_{f(a)}^{f(x)} K(x,y)\,dy \right) dx + \int_{f(a)}^c \left( \int_a^{f_{\sup}^{-1}(y)} K(x,y)\,dx \right) dy \\
&\qquad + \int_{f_{\sup}^{-1}(c)}^b \left( \int_{f(a)}^{f(x)} K(x,y)\,dy \right) dx \\
&\quad = \int_a^{f_{\sup}^{-1}(c)} \int_{f(a)}^c K(x,y)\,dy\,dx + \int_{f_{\sup}^{-1}(c)}^b \left( \int_c^{f(x)} K(x,y)\,dy \right) dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_{f_{\sup}^{-1}(c)}^{b}\int_{f(a)}^{c}K(x,y)dy\,dx \\
&\geq \int_{a}^{b}\int_{f(a)}^{c}K(x,y)dy\,dx,
\end{aligned}
\tag{2.12}
$$

with equality if and only if $\int_{f_{\sup}^{-1}(c)}^{b}(\int_{c}^{f(x)}K(x,y)dy)dx = 0$. When $K$ is strictly positive almost everywhere, this means that $c = f(b-)$.

If $c \geq f(b+)$, then

$$
\begin{aligned}
&\int_{a}^{b}\left(\int_{f(a)}^{f(x)}K(x,y)dy\right)dx+\int_{f(a)}^{c}\left(\int_{a}^{f_{\sup}^{-1}(y)}K(x,y)dx\right)dy \\
&\quad=\int_{a}^{f_{\sup}^{-1}(c)}\left(\int_{f(a)}^{f(x)}K(x,y)dy\right)dx+\int_{f(a)}^{c}\left(\int_{a}^{f_{\sup}^{-1}(y)}K(x,y)dx\right)dy \\
&\qquad-\int_{b}^{f_{\sup}^{-1}(c)}\left(\int_{f(a)}^{f(x)}K(x,y)dy\right)dx \\
&\quad=\int_{a}^{f_{\sup}^{-1}(c)}\int_{f(a)}^{c}K(x,y)dy\,dx \\
&\qquad-\left(\int_{b}^{f_{\sup}^{-1}(c)}\left(\int_{f(a)}^{c}K(x,y)dy\right)dx-\int_{f(b+)}^{c}\left(\int_{a}^{f_{\sup}^{-1}(y)}K(x,y)dx\right)\mathrm{dy}\right) \\
&\quad\geq\int_{a}^{b}\int_{f(a)}^{c}K(x,y)dy\,dx.
\end{aligned}
\tag{2.13}
$$

The equality holds if and only if $\int_{f(b+)}^{c}(\int_{a}^{f_{\sup}^{-1}(y)}K(x,y)dx)dy = 0$, that is, when $c = f(b+)$ (provided that $K$ is strictly positive almost everywhere), see Figure 2.

If $c \in (f(b-), f(b+))$, then $f_{\sup}^{-1}(c) = b$ and the inequality in the statement of Theorem 2.3 is actually an equality, see Figure 3. □

**Corollary 2.4.** *(Young's inequality for continuous increasing functions). If $f : [0,\infty) \to [0,\infty)$ is also continuous and increasing, then*

$$
\int_{a}^{b}\int_{f(a)}^{c}K(x,y)dy\,dx \leq \int_{a}^{b}\left(\int_{f(a)}^{f(x)}K(x,y)dy\right)dx+\int_{f(a)}^{c}\left(\int_{a}^{f^{-1}(y)}K(x,y)dx\right)dy
\tag{2.14}
$$

*for every real number $c \geq f(a)$. Assuming $K$ is strictly positive almost everywhere, the equality occurs if and only if $c = f(b)$.*

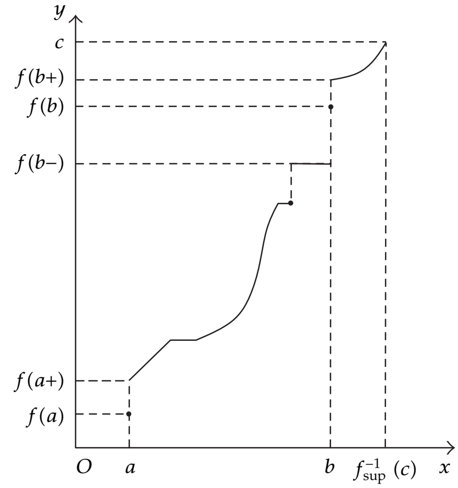


**Figure 2:** The case $c \geq f(b+)$.

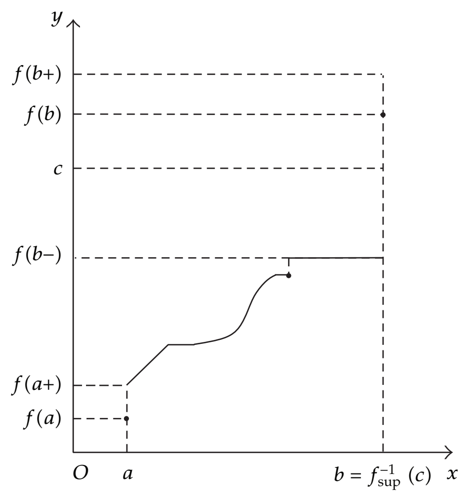


**Figure 3:** The equality case.

If $K(x, y) = 1$ for every $x, y \in [0, \infty)$, then Corollary 2.4 asserts that

$$bc - af(a) < \int_a^b f(x)dx + \int_{f(a)}^c f^{-1}(y)dy \quad \forall\, 0 < a < b,\ c > f(a). \tag{2.15}$$

The equality occurs if and only if $c = f(b)$. In the special case where $a = f(a) = 0$, this reduces to the classical inequality of Young.

*Remark 2.5* (the probabilistic companion of Theorem 2.3). Suppose there is a given non-negative random variable $X : [0, \infty) \to [0, \infty)$ whose cumulative distribution function

$F_X(x) = P(X \le x)$ admits a density, that is, a nonnegative Lebesgue integrable function $\rho_X$, such that

$$P(x \le X \le y) = \int_x^y \rho_X(u)du \quad \forall x \le y. \tag{2.16}$$

The quantile function of the distribution function $F_X$ (also known as the increasing rearrangement of the random variable $X$) is defined by

$$Q_X(x) = \inf\{y : F_X(y) \ge x\}. \tag{2.17}$$

Thus, a quantile function is nothing but a pseudoinverse of $F_X$. Motivated by statistics, a number of fast algorithms were developed for computing the quantile functions with high accuracy; see [8]. Without going through the details, we recall here the remarkable formula (due to G. Steinbrecher) for the quantile function of the normal distribution:

$$\operatorname{erf}^{-1}(z) = \sum_{k=0}^{\infty} \frac{c_k\left(\left(\sqrt{\pi}/2\right)z\right)^{2k+1}}{2k+1}, \tag{2.18}$$

where $c_0 = 1$ and

$$c_k = \sum_{m=0}^{k-1} \frac{c_m c_{k-m-1}}{(m+1)(2m+1)} \quad \forall k \ge 1. \tag{2.19}$$

According to Theorem 2.3, for every pair of continuous random variables $Y, Z : [0,\infty) \to [0,\infty)$ with density $\rho_{Y,Z}$ and all positive numbers $b$ and $c$, the following inequality holds:

$$P(Y \le b; Z \le c) \le \int_0^b \left(\int_0^{F_X(x)} \rho_{Y,Z}(x,y)dy\right)dx + \int_0^c \left(\int_0^{Q_X(y)} \rho_{Y,Z}(x,y)dx\right)dy. \tag{2.20}$$

This can be seen as a principle of uncertainty, since it shows that the functions

$$x \longrightarrow \int_0^{F_X(x)} \rho_{Y,Z}(x,y)dy, \qquad y \longrightarrow \int_0^{Q_X(y)} \rho_{Y,Z}(x,y)dx \tag{2.21}$$

cannot be made simultaneously small.

*Remark 2.6* (the higher dimensional analogue of Theorem 2.3). Consider a locally absolutely continuous kernel $K : [0,\infty) \times \cdots \times [0,\infty) \to [0,\infty)$, $K = K(s_1, s_2, \ldots, s_n)$, and a family $\phi_1, \ldots, \phi_n : [a_i, b_i] \to \mathbb{R}$ of nondecreasing functions defined on subintervals of $[0,\infty)$. Then

$$\begin{aligned}&\int_{\phi_1(a_1)}^{\phi_1(b_1)}\int_{\phi_2(a_2)}^{\phi_2(b_2)}\cdots\int_{\phi_n(a_n)}^{\phi_n(b_n)} K(s_1, s_2, \ldots, s_n)ds_n\cdots ds_2 ds_1\\ &\quad\leq \sum_{i=1}^{n}\int_{\phi_i(a_i)}^{\phi_i(b_i)}\left(\int_{\phi_1(a_1)}^{\phi_1(s)}\cdots\int_{\phi_n(a_n)}^{\phi_n(s)} K(s_1, \ldots, s_n)ds_n \ldots ds_{i+1}ds_{i-1}\cdots ds_1\right)ds.\end{aligned} \tag{2.22}$$

The proof is based on mathematical induction (which is left to the reader). The above inequality cover the n-variable generalization of Young's inequality as obtained by Oppenheim [9] (as well as the main result in [10]).

The following stronger version of Corollary 2.4 incorporates the Legendre duality.

**Theorem 2.7.** *Let $f : [0,\infty) \to [0,\infty)$ be a continuous nondecreasing function, and $\Phi : [0,\infty) \to \mathbb{R}$ be a convex function whose conjugate is also defined on $[0,\infty)$. Then for all $b > a \geq 0, c \geq f(a)$, and $\varepsilon > 0$ one has*

$$\begin{aligned}&\int_a^b \Phi\left(\varepsilon\int_{f(a)}^{f(x)} K(x,y)dy\right)dx + \int_{f(a)}^{c}\Phi^*\left(\frac{1}{\varepsilon}\int_a^{f_{\sup}^{-1}(y)} K(x,y)dx\right)dy\\ &\quad\geq \int_a^b\int_{f(a)}^{c} K(x,y)dy\,dx - (c - f(a))\Phi(\varepsilon) - (b-a)\Phi^*\left(\frac{1}{\varepsilon}\right).\end{aligned} \tag{2.23}$$

*Proof.* According to the Legendre duality,

$$\Phi(\varepsilon u) + \Phi^*\left(\frac{v}{\varepsilon}\right) \geq uv \quad \forall u, v, \varepsilon \geq 0. \tag{2.24}$$

For $u = \int_{f(a)}^{f(x)} K(x,y)dy$ and $v = 1$, we get

$$\Phi\left(\varepsilon\int_{f(a)}^{f(x)} K(x,y)dy\right) + \Phi^*\left(\frac{1}{\varepsilon}\right) \geq \int_{f(a)}^{f(x)} K(x,y)dy, \tag{2.25}$$

and by integrating both sides from $a$ to $b$, we obtain the inequality

$$\int_a^b \Phi\left(\varepsilon\int_{f(a)}^{f(x)} K(x,y)dy\right)dx + (b-\mathrm{a})\Phi^*\left(\frac{1}{\varepsilon}\right) \geq \int_a^b\left(\int_{f(a)}^{f(x)} K(x,y)dy\right)dx. \tag{2.26}$$

In a similar manner, starting with $u = 1$ and $v = \int_a^{f_{\sup}^{-1}(y)} K(x,y)dx$, we arrive first at the inequality

$$\Phi(\varepsilon) + \Phi^*\left(\frac{1}{\varepsilon}\int_a^{f_{\sup}^{-1}(y)} K(x,y)dx\right) \geq \int_a^{f_{\sup}^{-1}(y)} K(x,y)dx, \tag{2.27}$$

and then to

$$\begin{aligned}(c - f(a))\Phi(\varepsilon) &+ \int_{f(a)}^{c} \Phi^*\left(\frac{1}{\varepsilon}\int_a^{f_{\sup}^{-1}(y)} K(x,y)dx\right)dy \\ &\geq \int_{f(a)}^{c}\left(\int_a^{f_{\sup}^{-1}(y)} K(x,y)dx\right)dy.\end{aligned} \tag{2.28}$$

Therefore,

$$\begin{aligned}\int_a^b \Phi\left(\varepsilon\int_{f(a)}^{f(x)} K(x,y)dy\right)dx &+ \int_{f(a)}^{c} \Phi^*\left(\frac{1}{\varepsilon}\int_a^{f_{\sup}^{-1}(y)} K(x,y)dx\right)dy \\ &\geq \int_a^b\left(\int_{f(a)}^{f(x)} K(x,y)dy\right)dx + \int_{f(a)}^{c}\left(\int_a^{f_{\sup}^{-1}(y)} K(x,y)dx\right)dy \\ &\quad - (b-a)\Phi^*\left(\frac{1}{\varepsilon}\right) - (c - f(a))\Phi(\varepsilon).\end{aligned} \tag{2.29}$$

According to Theorem 2.3,

$$\begin{aligned}\int_a^b\left(\int_{f(a)}^{f(x)} K(x,y)dy\right)dx &+ \int_{f(a)}^{c}\left(\int_a^{f_{\sup}^{-1}(y)} K(x,y)dx\right)dy \\ &\geq \int_a^b\int_{f(a)}^{c} K(x,y)dy\,dx,\end{aligned} \tag{2.30}$$

and the inequality in the statement of Theorem 2.7 is now clear. □

In the special case where $K(x,y) = 1$, $a = f(a) = 0$, and $\Phi(x) = x^p/p$ (for some $p > 1$), Theorem 2.7 yields the following inequality:

$$\int_0^b f^p(x)dx + \int_0^c \left(f_{\sup}^{-1}(y)\right)^p dy \geq pbc - (p-1)(b+c), \quad \text{for every } b, c \geq 0. \tag{2.31}$$

This remark extends a result due to Sulaiman [11].

We end this section by noticing the following result that complements Theorem 2.3.

**Proposition 2.8.** *Under the assumptions of Lemma 2.2,*

$$\int_a^b \left( \int_{f(a)}^{f(x)} K(x,y)\,dy \right) dx + \int_{f(a)}^{c} \left( \int_a^{f_{\sup}^{-1}(y)} K(x,y)\,dx \right) dy$$
$$\leq \max\left\{ \int_a^b \int_{f(a)}^{f(b)} K(x,y)\,dy\,dx, \int_a^{f_{\sup}^{-1}(c)} \int_{f(a)}^{c} K(x,y)\,dy\,dx \right\}. \tag{2.32}$$

*Assuming $K$ is strictly positive almost everywhere, the equality occurs if and only if $c = f(b)$.*

*Proof.* If $c < f(b)$, then from Lemma 2.2, we infer that

$$\int_a^b \left( \int_{f(a)}^{f(x)} K(x,y)\,dy \right) dx + \int_{f(a)}^{c} \left( \int_a^{f_{\sup}^{-1}(y)} K(x,y)\,dx \right) dy$$
$$= \int_a^b \left( \int_{f(a)}^{f(x)} K(x,y)\,dy \right) dx + \int_{f(a)}^{f(b)} \left( \int_a^{f_{\sup}^{-1}(y)} K(x,y)\,dx \right) dy$$
$$- \int_c^{f(b)} \left( \int_a^{f_{\sup}^{-1}(y)} K(x,y)\,dx \right) dy \tag{2.33}$$
$$\leq \int_a^b \int_{f(a)}^{f(b)} K(x,y)\,dy\,dx.$$

The other case, $c \geq f(b)$, has a similar approach. □

Proposition 2.8 extends a result due to Merkle [12].

## 3. The Precision in Young's Inequality

The main result of this section is as follows.

**Theorem 3.1.** *Under the assumptions of Lemma 2.2, for all $b \geq a \geq 0$ and $c \geq f(a)$,*

$$\int_a^b \left( \int_{f(a)}^{f(x)} K(x,y)\,dy \right) dx + \int_{f(a)}^{c} \left( \int_a^{f_{\sup}^{-1}(y)} K(x,y)\,dx \right) dy$$
$$- \int_a^b \int_{f(a)}^{c} K(x,y)\,dy\,dx \leq \left| \int_{f_{\sup}^{-1}(c)}^{b} \int_c^{f(b)} K(x,y)\,dy\,dx \right|. \tag{3.1}$$

*Assuming $K$ is strictly positive almost everywhere, the equality occurs if and only if $c = f(b)$.*

*Proof.* The case where $f(a) \leq c \leq f(b-)$ is illustrated in Figure 4. The left-hand side of the inequality in the statement of Theorem 3.1 represents the measure of the cross-hatched curvilinear trapezium, while right-hand side is the measure of the $ABCD$ rectangle.

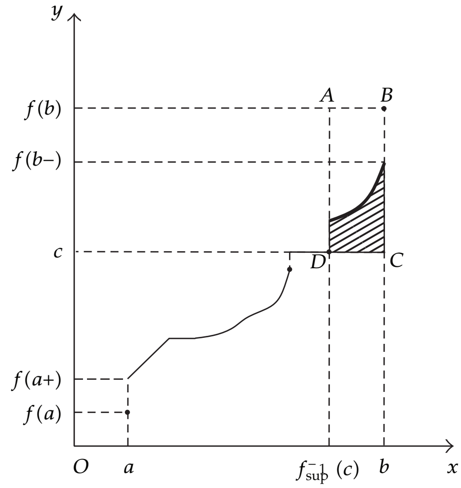


**Figure 4:** The geometry of the case $f(a) \leq c \leq f(b-)$.

Therefore,

$$\begin{aligned}\int_a^b \left( \int_{f(a)}^{f(x)} K(x,y)dy \right) dx &+ \int_{f(a)}^{c} \left( \int_a^{f_{\sup}^{-1}(y)} K(x,y)dx \right) dy \\ &- \int_a^b \int_{f(a)}^{c} K(x,y)dy\,dx = \int_{f_{\sup}^{-1}(c)}^{b} \left( \int_c^{f(x)} K(x,y)dy \right) dx \\ &\leq \int_{f_{\sup}^{-1}(c)}^{b} \int_c^{f(b)} K(x,y)dy\,dx.\end{aligned} \tag{3.2}$$

The equality holds if and only if $\int_{f_{\sup}^{-1}(c)}^{b} (\int_c^{f(x)} K(x,y)dy)dx = 0$, that is, when $f(b-) = c$.

The case where $c \geq f(b+)$ is similar to the precedent one. The first term will be

$$\begin{aligned}\int_a^b \left( \int_{f(a)}^{f(x)} K(x,y)dy \right) dx &+ \int_{f(a)}^{c} \left( \int_a^{f_{\sup}^{-1}(y)} K(x,y)dx \right) dy \\ &- \int_a^b \int_{f(a)}^{c} K(x,y)dy\,dx = \int_b^{f_{\sup}^{-1}(c)} \left( \int_{f(b)}^{f(x)} K(x,y)dy \right) dx \\ &\leq \int_b^{f_{\sup}^{-1}(c)} \int_{f(b)}^{c} K(x,y)dy\,dx.\end{aligned} \tag{3.3}$$

The equality holds if and only if $\int_b^{f_{\sup}^{-1}(c)} \int_{f(b)}^{c} K(x,y)dy\,dx = 0$, so we must have $f(b+) = c$.

The case where $c \in [f(b-), f(b+)]$ is trivial, both sides of our inequality being equal to zero. □

**Corollary 3.2** (Minguzzi [13]). *If, moreover, $K(x,y) = 1$ on $[0,\infty) \times [0,\infty)$, and $f$ is continuous and increasing, then*

$$\int_a^b f(x)dx + \int_{f(a)}^c f^{-1}(y)dy - bc + af(a) \le \left(f^{-1}(c) - b\right) \cdot (c - f(b)). \tag{3.4}$$

*The equality occurs if and only if $c = f(b)$.*

More accurate bounds can be indicated under the presence of convexity.

**Corollary 3.3.** *Let $f$ be a nondecreasing continuous function, which is convex on the interval $[\min\{f_{\sup}^{-1}(c), b\}, \max\{f_{\sup}^{-1}(c), b\}]$. Then*

(i)

$$\begin{aligned}
&\int_a^b \left(\int_{f(a)}^{f(x)} K(x,y)dy\right)dx + \int_{f(a)}^c \left(\int_a^{f_{\sup}^{-1}(y)} K(x,y)dx\right)dy \\
&\quad - \int_a^b \int_{f(a)}^c K(x,y)dy\,dx \\
&\quad \le \int_{f_{\sup}^{-1}(c)}^b \int_c^{c+[(f(b)-c)/(b-f_{\sup}^{-1}(c))](x-f_{\sup}^{-1}(c))} K(x,y)dy\,dx, \text{ for every } c \le f(b),
\end{aligned} \tag{3.5}$$

(ii)

$$\begin{aligned}
&\int_a^b \left(\int_{f(a)}^{f(x)} K(x,y)dy\right)dx + \int_{f(a)}^c \left(\int_a^{f_{\sup}^{-1}(y)} K(x,y)dx\right)dy \\
&\quad - \int_a^b \int_{f(a)}^c K(x,y)dy\,dx \\
&\quad \ge \int_b^{f_{\sup}^{-1}(c)} \int_{f(b)}^{f(b)+[(c-f(b))/(f_{\sup}^{-1}(c)-b)](x-b)} K(x,y)dy\,dx, \text{ for every } c \ge f(b).
\end{aligned} \tag{3.6}$$

*If $f$ is concave on the aforementioned interval, then the inequalities above work in the reverse way.*

*Assuming $K$ is strictly positive almost everywhere, the equality occurs if and only if $f$ is an affine function or $f(b) = c$.*

*Proof.* We will restrict here to the case of convex functions, the argument for the concave functions being similar.

The left-hand side term of each of the inequalities in our statement represents the measure of the cross-hatched surface, see Figures 5 and 6.

As the points of the graph of the convex function $f$ (restricted to the interval of endpoints $b$ and $f_{\sup}^{-1}(c)$) are under the chord joining $(b, f(b))$ and $(f_{\sup}^{-1}(c), c)$, it follows that this measure is less than the measure of the enveloping triangle $MNQ$ when $c \le f(b)$. This yields (i). The assertion (ii) follows in a similar way. □

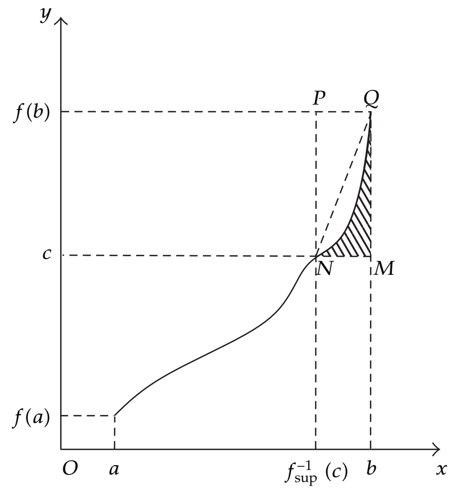


**Figure 5:** The geometry of the case $c \le f(b)$.

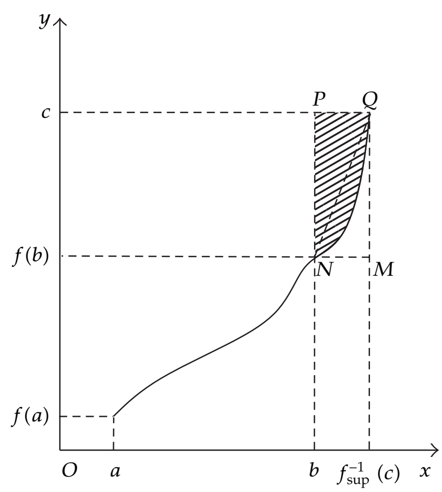


**Figure 6:** The geometry of the case $c \ge f(b)$.

Corollary 3.3 extends a result due to Pečarić and Jakšetić [7]. They considered the special case where $K(x,y) = 1$ on $[0,\infty) \times [0,\infty)$ and $f : [0,\infty) \to [0,\infty)$ is increasing and differentiable, with an increasing derivative on the interval $[\min\{f^{-1}(c), b\}, \max\{f^{-1}(c), b\}]$ and $f(0) = 0$. In this case the conclusion of Corollary 3.3 reads as follows:

(i)

$$\int_0^b f(x)dx + \int_0^c f^{-1}(y)dy - bc \le \frac{1}{2}\left(f^{-1}(c) - b\right)\left(c - f(b)\right) \quad \text{for } c < f(b), \tag{3.7}$$

(ii)

$$\int_0^b f(x)dx + \int_0^c f^{-1}(y)dy - bc \geq \frac{1}{2}\left(f^{-1}(c) - b\right)\left(c - f(b)\right) \quad \text{for } c > f(b). \tag{3.8}$$

The equality holds if $f(b) = c$ or $f$ is an affine function. The inequality sign should be reversed if $f$ has a decreasing derivative on the interval

$$\left[\min\left\{f^{-1}(c), b\right\}, \ \max\left\{f^{-1}(c), b\right\}\right]. \tag{3.9}$$

## 4. The Connection with $c$-Convexity

Motivated by the mass transportation theory, several people [14, 15] drew a parallel to the classical theory of convex functions by extending the Legendre duality. Technically, given two compact metric spaces $X$ and $Y$ and a *cost density* function $c : X \times Y \to \mathbb{R}$ (which is supposed to be continuous), we may consider the following generalization of the notion of convex function.

*Definition 4.1.* A function $F : X \to \mathbb{R}$ is $c$-convex if there exists a function $G : Y \to \mathbb{R}$ such that

$$F(x) = \sup_{y \in Y}\left\{c(x, y) - G(y)\right\}, \quad \forall x \in X. \tag{4.1}$$

We abbreviate (4.1) by writing $F = G^c$. A useful remark is the equality

$$F^{cc} = F, \tag{4.2}$$

that is,

$$F(x) = \sup_{y \in Y}\left\{c(x, y) - F^c(y)\right\}, \quad \forall x \in X. \tag{4.3}$$

The classical notion of convex function corresponds to the case where $X$ is a compact interval and $c(x, y) = xy$. The details can be found in [4, pages 40–42].

Theorem 2.3 illustrates the theory of $c$-convex functions for the spaces $X = [a, \infty]$, $Y = [f(a), \infty]$ (the Alexandrov one point compactification of $[a, \infty)$ and, respectively, $[f(a), \infty)$), and the cost function

$$c(x, y) = \int_a^x \int_{f(a)}^y K(s, t)dtds. \tag{4.4}$$

In fact, under the hypotheses of this theorem, the functions

$$F(x) = \int_a^x \left( \int_{f(a)}^{f(s)} K(s,t)dt \right) ds, \quad x \geq a,$$
$$G(y) = \int_{f(a)}^{y} \left( \int_{a}^{f^{-1}_{\sup(t)}} K(s,t)ds \right) dt, \quad y \geq f(a), \tag{4.5}$$

verify the relations $F^c = G$ and $G^c = F$ (due to the equality case as specified in the statement of Theorem 2.3), so they are both $c$-convex.

On the other hand, a simple argument shows that $F$ and $G$ are also convex in the usual sense.

Let us call the function $c$ that admits a representation of form (4.4) with $K \in L^1(\mathbb{R} \times \mathbb{R})$, *absolutely continuous in the hyperbolic sense*. With this terminology, Theorem 2.3 can be rephrased as follows.

**Theorem 4.2.** *Suppose that $c : [a,b] \times [A,B] \to \mathbb{R}$ is an absolutely continuous function in the hyperbolic sense with mixed derivative $\partial^2 c/\partial x \partial y \geq 0$, and $f : [a,b] \to [A,B]$ is a nondecreasing function such that $f(a) = A$. Then*

$$c(x,y) - c(a, f(a)) \leq \int_a^x \frac{\partial c}{\partial t}(t, f(t))dt + \int_{f(a)}^{y} \frac{\partial c}{\partial s}\left(f^{-1}_{\sup}(s), s\right) ds, \tag{4.6}$$

*for all $(x,y) \in [a,A] \times [b,B]$.*

*If $\partial^2 c/\partial x \partial y > 0$ almost everywhere, then* (4.6) *becomes an equality if and only if $y \in [f(x-), f(x+)]$; here we made the convention $f(a-) = f(a)$ and $f(b+) = f(b)$.*

Necessarily, an absolutely continuous function $c$ in the hyperbolic sense is continuous. It admits partial derivatives of the first order and a mixed derivative $\partial^2 c/\partial x \partial y$ almost everywhere. Besides, the functions $y \to (\partial c/\partial x)(x,y)$ and $x \to (\partial c/\partial y)(x,y)$ are defined everywhere in their interval of definition and represent absolutely continuous functions; they are also nondecreasing provided that $\partial^2 c/\partial x \partial y \geq 0$ almost everywhere.

A special case of Theorem 4.2 was proved by Páles [10, 16] (assuming $c : [a,A] \times [b,B] \to \mathbb{R}$ is a continuously differentiable function with nondecreasing derivatives $y \to (\partial c/\partial x)(x,y)$ and $x \to (\partial c/\partial y)(x,y)$, and $f : [a,b] \to [A,B]$ is an increasing homeomorphism). An example which escapes his result but is covered by Theorem 4.2 is offered by the function

$$c(x,y) = \int_0^x \left\{ \frac{1}{s} \right\} ds \int_0^y \left\{ \frac{1}{t} \right\} dt, \quad x, y \geq 0, \tag{4.7}$$

where $\{1/s\}$ denotes the fractional part of $1/s$ if $s > 0$, and $\{1/s\} = 0$ if $s = 0$. According to Theorem 4.2,

$$\begin{aligned}
&\int_0^x \left\{\frac{1}{s}\right\} ds \int_0^y \left\{\frac{1}{t}\right\} dt \\
&\quad \le \int_0^x \left(\left\{\frac{1}{s}\right\} \int_0^{f(s)} \left\{\frac{1}{t}\right\} dt\right) ds + \int_0^y \left(\left\{\frac{1}{t}\right\} \int_0^{f_{\sup}^{-1}(t)} \left\{\frac{1}{s}\right\} ds\right) dt,
\end{aligned} \tag{4.8}$$

for every nondecreasing function $f : [0, \infty) \to [0, \infty)$ such that $f(0) = 0$.

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