Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 532430, 19 pages
doi:10.1155/2012/532430

*Research Article*

# Multiple Solutions for Degenerate Elliptic Systems Near Resonance at Higher Eigenvalues

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Received 24 February 2012; Accepted 29 April 2012

Academic Editor: Shaoyong Lai



We study the degenerate semilinear elliptic systems of the form $-\operatorname{div}(h_1(x)\nabla u) = \lambda(a(x)u + b(x)v) + F_u(x,u,v), x \in \Omega, -\operatorname{div}(h_2(x)\nabla v) = \lambda(d(x)v + b(x)u) + F_v(x,u,v), x \in \Omega, u|_{\partial\Omega} = v|_{\partial\Omega} = 0$, where $\Omega \subset R^N (N \geq 2)$ is an open bounded domain with smooth boundary $\partial\Omega$, the measurable, nonnegative diffusion coefficients $h_1$, $h_2$ are allowed to vanish in $\Omega$ (as well as at the boundary $\partial\Omega$) and/or to blow up in $\overline{\Omega}$. Some multiplicity results of solutions are obtained for the degenerate elliptic systems which are near resonance at higher eigenvalues by the classical saddle point theorem and a local saddle point theorem in critical point theory.

## 1. Introduction

In this paper, we study a class of degenerate elliptic systems:

$$\begin{gathered}
-\operatorname{div}(h_1(x)\nabla u) = \lambda(a(x)u + b(x)v) + F_u(x,u,v), \qquad x \in \Omega, \\
-\operatorname{div}(h_2(x)\nabla v) = \lambda(d(x)v + b(x)u) + F_v(x,u,v), \qquad x \in \Omega, \\
u|_{\partial\Omega} = v|_{\partial\Omega} = 0,
\end{gathered} \tag{1.1}$$

where $\Omega \subset R^N (N \geq 2)$ is an open bounded domain with smooth boundary $\partial\Omega$, $F \in C^1(\overline{\Omega} \times R^2, R)$ satisfies the following sublinear growth condition:

$$\lim_{|s| \to \infty} \frac{|\nabla F(x,s)|}{|s|} = 0 \tag{1.2}$$

uniformly in $x \in \overline{\Omega}$, where $\nabla F = (F_u, F_v)$ denotes the gradient of $F$ with respect to $(u, v) \in R^2$. The degeneracy of this system is considered in the sense that the measurable, nonnegative diffusion coefficients $h_1, h_2$ are allowed to vanish in $\Omega$ (as well as at the boundary $\partial\Omega$) and/or to blow up in $\overline{\Omega}$. The consideration of suitable assumptions on the diffusion coefficients will be based on the work [1], where the degenerate scalar equation was studied. We introduce the function space $(N)_h$, which consists of functions $h : \Omega \subset R^N \to R$, such that $h \in L^1(\Omega)$, $h^{-1} \in L^1(\Omega)$, and $h^{-s} \in L^1(\Omega)$, for some $s > N/2$.

Then for the weight functions $h_1$, $h_2$ we assume the following hypothesis. $(N)$ there exist functions $\mu$ satisfying condition $(N)_\mu$, for some $s_\mu$, and $\nu$ satisfying condition $(N)_\nu$, for some $s_\nu$, such that

$$\frac{\mu(x)}{k_1} \le h_1(x) \le k_1\mu(x), \qquad \frac{\nu(x)}{k_2} \le h_2(x) \le k_2\nu(x), \tag{1.3}$$

a.e. in $\Omega$, for some constants $k_1 > 1$ and $k_2 > 1$.

The mathematical modeling of various physical processes, ranging from physics to biology, where spatial heterogeneity plays a primary role, is reduced to nonlinear evolution equations with variable diffusion or dispersion. Also note that problem (1.1) is closely related (see [1]) to the following system:

$$\begin{gathered} -\operatorname{div}(h_1(x,u,v)\nabla u) = f(\lambda, x, u, v, \nabla u, \nabla v), \qquad x \in \Omega, \\ -\operatorname{div}(h_2(x,u,v)\nabla v) = g(\lambda, x, u, v, \nabla u, \nabla v), \qquad x \in \Omega, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \end{gathered} \tag{1.4}$$

Problems of such a type have been successfully applied to the heat propagation in heterogeneous materials, to the study of transport of electron temperature in a confined plasma, to the propagation of varying amplitude waves in a nonlinear medium, to the study of electromagnetic phenomena in nonhomogeneous superconductors and the dynamics of Josephson junctions, to electrochemistry, to nuclear reaction kinetics, to image segmentation, to the spread of microorganisms, to the growth and control of brain tumors, and to population dynamics (see [2–4] and the references therein).

An example of the physical motivation of the assumptions $(N)$, $(N)_h$ may be found in [3]. These assumptions are related to the modeling of reaction diffusion processes in composite materials occupying a bounded domain $\Omega$, which at some point they behave as perfect insulators. When at some point the medium is perfectly insulated, it is natural to assume that $h_1(x)$ and/or $h_2(x)$ vanish in $\Omega$. For more information we refer the reader to [4] and the references therein.

For the perturbed problem, Mawhin and Schmitt [5] first considered the following two point boundary value problem:

$$-u'' - \lambda u = f(x, u) + h(x), \qquad u(0) = u(\pi) = 0. \tag{1.5}$$

Under the assumption that $f$ is bounded and satisfies a sign condition, if the parameter $\lambda$ is sufficiently close to $\lambda_1$ from left, problem (1.5) has at least three solutions, if $\lambda_1 \le \lambda < \lambda_2$, problem (1.5) has at least one solution, where $\lambda_1, \lambda_2$ are the first and the second eigenvalues of the corresponding linear problem. Ma et al. [6] considered the boundary value problem

$\Delta u + \lambda u + f(x,u) = h(x)$ defined on a bounded open set $\Omega \subset R^N$, no matter whether the boundary conditions are Dirichlet or Neumann condition, as the parameter $\lambda$ approaches $\lambda_1$ from left, there exist three solutions. Moreover, existence of three solutions was obtained for the quasilinear problem in bounded domains as the parameter $\lambda$ approaches $\lambda_1$ from left. In [7, 8], these results were extended to the perturbed $p$-Laplacian equation in $R^N$. In [9], Ou and Tang extended above some results to some elliptic systems with the Dirichlet boundary conditions. Especially, de Paiva and Massa in [10] studied the semilinear elliptic boundary value problem in any spatial dimension and by using variational techniques, they showed that a suitable perturbation will turn the almost resonant situation ($\lambda$ near to $\lambda_k$, i.e., near resonance with a nonprincipal eigenvalue) in a situation where the solutions are at least two. In [11], those results were extended to the cooperative elliptic systems in the bounded domain. Motivated by the idea above, we have the goal in this paper of extending these results in [10, 11] to some degenerate elliptic systems with the Dirichlet boundary conditions.

## 2. Preliminaries and Main Results

Let $h(x)$ be a nonnegative weight function in $\Omega$ which satisfies condition $(N)_h$. We consider the weighted Sobolev space $D_0^{1,2}(\Omega, h)$ to be defined as the closure of $C_0^\infty(\Omega)$ with respect to the following norm:

$$\|u\|_h = \left(\int_\Omega h(x)|\nabla u|^2 dx\right)^{1/2}, \tag{2.1}$$

and the following scalar product:

$$\langle u, v\rangle_h = \int_\Omega h(x)\nabla u \nabla v\, dx, \tag{2.2}$$

for all $u, v \in D_0^{1,2}(\Omega, h)$. The space $D_0^{1,2}(\Omega, h)$ is a Hilbert space. For a discussion about the space setting we refer to [1] and the references therein. Let

$$2_s^* = \frac{2Ns}{N(s+1) - 2s}. \tag{2.3}$$

**Lemma 2.1.** *Assume that $\Omega$ is a bounded domain in $R^N$ and the weight $h$ satisfies $(N)_h$. Then the following embeddings hold:*

(i) $D_0^{1,2}(\Omega, h) \hookrightarrow L^{2_s^*}(\Omega)$ *continuously,*

(ii) $D_0^{1,2}(\Omega, h) \hookrightarrow L^r(\Omega)$ *compactly for any $r \in [1, 2_s^*)$.*

*In the sequel one denotes by $2_\mu^*$ and $2_\nu^*$ the quantities $2_{s_\mu}^*$ and $2_{s_\nu}^*$, respectively, where $s_\mu$ and $s_\nu$ are induced by condition $(N)$, recall that $h_1$, $h_2$ satisfy $(N)$. The assumptions concerning the coefficient functions of systems* (1.1) *are as follows.*

$(AD)$ *The functions $a, d \in C(\overline{\Omega}, R)$ and there exists $x_0 \in \Omega$, such that $a(x_0) > 0$, $d(x_0) > 0$.*

$(B)$ *The function $b \in C(\overline{\Omega}, (0, +\infty))$.*

The space setting for our problem is the product space $H = D_0^{1,2}(\Omega, h_1) \times D_0^{1,2}(\Omega, h_2)$ equipped with the following norm:

$$\|z\| = \left(\|u\|_{h_1}^2 + \|v\|_{h_2}^2\right)^{1/2}, \quad z = (u, v) \in H, \tag{2.4}$$

and the following scalar product:

$$\langle z, \phi \rangle = \langle u, \xi \rangle_{h_1} + \langle v, \tau \rangle_{h_2}, \tag{2.5}$$

for all $z = (u, v)$, $\phi = (\xi, \tau)$. Observe that inequalities (1.3) in condition $(N)$ imply that the functional spaces $D_0^{1,2}(\Omega, h_1) \times D_0^{1,2}(\Omega, h_2)$ and $D_0^{1,2}(\Omega, \mu) \times D_0^{1,2}(\Omega, \nu)$ are equivalent. Especially, by Lemma 2.1 we know that for any $1 \leq \delta \leq \min\{2_\mu^*, 2_\nu^*\}$, the embedding $H \hookrightarrow L^\delta(\Omega) \times L^\delta(\Omega)$ is continuous and there is a positive constant $S = S(\delta, N, \Omega)$ such that $\int_\Omega |z|^\delta dx \leq S^\delta \|z\|^\delta$ for all $z \in H$. Moreover, if $\delta < \min\{2_\mu^*, 2_\nu^*\}$, the embedding above is also compact. Let

$$H(x) = \operatorname{diag}(h_1(x), h_2(x)), \qquad A(x) = \begin{pmatrix} a(x) & b(x) \\ b(x) & d(x) \end{pmatrix}. \tag{2.6}$$

Assume that hypothesis $(N)$ is satisfied and the coefficient functions $a$, $d$ and $b$ satisfy conditions $(AD)$ and $(B)$, respectively.

We consider the eigenvalue problem with weight $A(x)$,

$$\begin{gathered} -\operatorname{div}(h_1(x)\nabla u) = \lambda(a(x)u + b(x)v), \quad x \in \Omega, \\ -\operatorname{div}(h_2(x)\nabla v) = \lambda(d(x)v + b(x)u), \quad x \in \Omega, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \end{gathered} \tag{2.7}$$

A simple calculation shows that $\lambda$ is an eigenvalue of (2.7) if and only if

$$T_A z = \lambda^{-1} z, \tag{2.8}$$

where $T_A : H \to H$ is the symmetric bounded linear operator defined by

$$\langle T_A z, \phi \rangle = \int_\Omega (A(x)z, \phi)dx, \quad \forall z, \phi \in H. \tag{2.9}$$

Since the coefficient of $A$ are continuous functions and the embedding $H \hookrightarrow L^2(\Omega) \times L^2(\Omega)$ is compact, we can check that the operator $T_A$ is also compact. Thus, we may invoke the spectral theory for compact operators to conclude that $H$ possesses a Hilbertian basis formed by eigenfunctions of (2.7).

Let us denote $z = (u, v)$ and

$$\lambda_1^{-1} = \mu_1 = \sup\{\langle T_A z, z \rangle : \|z\| = 1\}. \tag{2.10}$$

Recalling that $A$ satisfies $(AD)$ and $(B)$, we can use [2, Theorem 1.1] ($p = q = 2$, $\alpha = \beta = 0$) to conclude that the eigenvalue $\mu_1$ is positive, simple, and isolated in the spectrum of $T_A$. Moreover, if we denote by $\phi_1$ the normalized eigenfunction associated to $\lambda_1$, we can suppose that the $\phi_1$ is positive on $\Omega$. By using induction, if we suppose that $\mu_1 > \mu_2 \geq \ldots \geq \mu_{k-1}$ are the $k-1$ first eigenvalues of $T_A$ and $\{\phi_i\}_{i=1}^{k-1}$ are the associated normalized eigenfunctions, we can define

$$\lambda_k^{-1} = \mu_k = \sup\left\{\langle T_A z, z\rangle := \|z\| = 1,\ z \in \left(\operatorname{span}\{\phi_1, \ldots, \phi_{k-1}\}\right)^{\perp}\right\}. \tag{2.11}$$

It is proved in [12, Proposition 1.3], that, if $\mu_k > 0$, then it is an eigenvalue of $T_A$ with associated normalized eigenfunction $\phi_k$. In view of the condition $(AD)$, we can argue as in the proof of [12, Proposition 1.11(c)], and conclude that $\mu_k > 0$. Thus, we obtain a sequence of eigenvalues for (2.7).

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \tag{2.12}$$

such that $\lambda_k \to \infty$ as $k \to \infty$. We denote by $E_k$ the eigenfunction space corresponding to $\lambda_k$. Moreover, if we set $V_k = \operatorname{span}\{\phi_1, \ldots, \phi_k\}$, we can decompose $H = V_k \oplus V_k^{\perp}$. Moreover, the following inequalities hold:

$$\int_\Omega (H(x)\nabla z, \nabla z)dx \leq \lambda_k \int_\Omega (A(x)z, z)dx, \quad \forall z \in V_k, \tag{2.13}$$

$$\int_\Omega (H(x)\nabla z, \nabla z)dx \geq \lambda_{k+1} \int_\Omega (A(x)z, z)dx, \quad \forall z \in V_k^{\perp}. \tag{2.14}$$

**Lemma 2.2** (from Lemma 4.6 of [10]). *Let $X$ be a Hilbert space with orthonormal direct sum splitting $X = V \oplus Z \oplus W$. Moreover, let $\dim(V \oplus Z) < \infty$. For $\rho > R > 0$, set*

$$\begin{gathered} A = \{u \in W : \|u\| \geq R\} \cup \{u \in Z \oplus W : \|u\| = R\}, \\ B = \{u \in V \oplus Z : \|u\| = \rho\}. \end{gathered} \tag{2.15}$$

*Then $A$ links with $B$.*

**Lemma 2.3** (from Theorem 8.1 of [13]). *Let $H = X_1 \oplus X_2$ be a Hilbert space where $X_1$ has finite dimension, $J \in C^1(H, R)$ satisfying the (P.S.) condition and such that, for given $\rho_1, \rho_2 > 0$,*

$$\sup_{z \in \rho_1 S_1} J(z) < a = \inf_{z \in \rho_2 B_2} J(z) \leq b = \sup_{z \in \rho_1 B_1} J(z) < \inf_{z \in \rho_2 S_2} J(z), \tag{2.16}$$

*where $B_i$ and $S_i$ represent the unit ball and the unit sphere in $X_i : i = 1, 2$.*

*Then there exists a critical point $z_0$ such that $J(z_0) \in [a, b]$.*

Next, in order to state our main results, we introduce the following assumptions on the nonlinear term:

$(F1)$ $\lim_{|s|\to\infty}(\nabla F(x,s),s)/|s| = +\infty$ uniformly with respect to $x \in \Omega$.

$(F2)$ $\lim_{|s|\to\infty}F(x,s) = +\infty$ uniformly with respect to $x \in \Omega$.

$(F3)$ $\lim_{|s|\to\infty}(\nabla F(x,s),s)/|s| = -\infty$ uniformly with respect to $x \in \Omega$.

$(F4)$ $\lim_{|s|\to\infty}F(x,s) = -\infty$ uniformly with respect to $x \in \Omega$.

Our main results are given by the following theorems.

**Theorem 2.4.** *Let $\lambda_k (k \geq 2)$ be an eigenvalue of multiplicity $m$. Suppose that condition $(N)$ and the coefficient functions $a$, $d$ and $b$ satisfy conditions $(AD)$ and $(B)$, respectively. Assume, in addition, that $F$ satisfies* (1.2) *and $\nabla F(x,0) = 0$ for all $x \in \Omega$, and one of the sets of hypotheses $(F1)$ or $(F2)$. Then there exists $\delta_0 > 0$ such that for $\lambda \in (\lambda_k - \delta_0, \lambda_k)$ problem* (1.1) *has at least two solutions.*

**Theorem 2.5.** *Let $\lambda_k (k \geq 2)$ be an eigenvalue of multiplicity $m$. Suppose that conditions $(N)$ and the coefficient functions $a$, $d$ and $b$ satisfy conditions $(AD)$ and $(B)$, respectively. Assume, in addition, that $F$ satisfies* (1.2) *and $\nabla F(x,0) = 0$ for all $x \in \Omega$, and one of the sets of hypotheses $(F3)$ or $(F4)$. Then there exists $\delta_1 > 0$ such that for $\lambda \in (\lambda_k, \lambda_k + \delta_1)$ problem* (1.1) *has at least two solutions.*

## 3. Proof of Theorems

Consider the $C^1$ functional $J : H \to R, \forall z \in H$,

$$J(z) = \frac{1}{2}\int_\Omega (H(x)\nabla z, \nabla z)dx - \frac{\lambda}{2}\int_\Omega (A(x)z,z)dx - \int_\Omega F(x,z)dx. \tag{3.1}$$

Since the problems in Theorems 2.4 and 2.5 are not resonant, $J$ satisfies the Palais-Smale condition of compactness (see, e.g., in [14]). In addition, $z \in H$ is a weak solution of problem (1.1) if and only if $z$ is a critical point of $J$.

We set

$$V = \operatorname{span}\{\phi_1, \ldots, \phi_{k-1}\}, \qquad Z = \operatorname{span}\{\phi_k, \ldots, \phi_{k+m-1}\} = E_k, \qquad W = (V \oplus Z)^\perp, \tag{3.2}$$

and we define

$$B_V = \{z \in V : \|z\| \leq 1\}, \qquad B_{VZ} = \{z \in V \oplus Z : \|z\| \leq 1\}, \qquad B_{ZW} = \{z \in Z \oplus W : \|z\| \leq 1\}, \tag{3.3}$$

and $S_V, S_{VZ}, S_{ZW}$, respectively, their relative boundaries.

Theorems 2.4 and 2.5 will be a consequence of the geometry in Propositions 3.1 and 3.2, whose proofs will be postponed to Sections 4 and 5.

**Proposition 3.1.** *If $\lambda \in (\lambda_{k-1}, \lambda_k)$ and hypothesis* (1.2) *is satisfied, then there exist constants $D_\lambda$ and $\rho_\lambda$ such that*

$$J(z) \geq D_\lambda, \quad \text{for } z \in Z \oplus W, \tag{3.4}$$

$$J(z) < D_\lambda, \quad \text{for } z \in \rho_\lambda S_V. \tag{3.5}$$

*Moreover, if one of the sets of hypotheses $(F1)$ or $(F2)$ is satisfied, then there exists $\delta_0$ such that for $\lambda \in (\lambda_k - \delta_0, \lambda_k)$ there exist $D_W, D_\lambda \in R$, $\rho_\lambda > R_1 > 0$ such that, in addition to* (3.4) *and* (3.5),

$$J(z) \geq D_W, \quad \text{for } z \in W, \tag{3.6}$$

$$J(z) < D_W, \quad \text{for } z \in R_1 S_{VZ}, \tag{3.7}$$

$$J(z) < D_W, \quad \text{for } z \in V, \ \|z\| \geq R_1. \tag{3.8}$$

*(The values with index $\lambda$ depend on $\lambda$, the others may be fixed uniformly.)*

*Based on this geometry one gives the following proof.*

*Proof of Theorem 2.4.* Since the functional $J$ satisfies the $(P.S.)$ condition, we can apply the saddle point theorem (see, e.g., in [15]) for two times, let

$$\begin{aligned} \Gamma_{k-1} &= \left\{\gamma \in C^0(\rho_\lambda B_V; H) \text{s.t. } \gamma\,|_{\rho_\lambda S_V} = \text{id}\right\}, \\ \Gamma_k &= \left\{\gamma \in C^0(R_1 B_{VZ}; H) \text{s.t. } \gamma|_{R_1 S_{VZ}} = \text{id}\right\}. \end{aligned} \tag{3.9}$$

The first solution, that we denote by $z_{k-1}$ and may be obtained for any $\lambda \in (\lambda_{k-1}, \lambda_k)$ with just hypothesis (1.2), corresponds to a critical point at the level

$$c_{k-1} = \inf_{\gamma \in \Gamma_{k-1}} \sup_{w \in \rho_\lambda B_V} J(\gamma(w)), \tag{3.10}$$

the criticality of this level is guaranteed by the estimates (3.4) and (3.5), since $\rho_\lambda S_V$ and $Z \oplus W$ link, that is, the image of any map in $\Gamma_{k-1}$ intersects $Z \oplus W$.

The second solution, that we denote by $z_k$, corresponds to a critical point at the critical level

$$c_k = \inf_{\gamma \in \Gamma_k} \sup_{w \in R_1 B_{VZ}} J(\gamma(w)), \tag{3.11}$$

actually, this is a critical level because of the estimates (3.6) and (3.7), since $R_1 S_{VZ}$ and $W$ link.

To conclude the proof, we need to show that these two solutions are distinct.

We observe first that by estimate (3.6) we have that $c_k \geq D_W$, then we observe that we may build a map $\gamma_0 \in \Gamma_{k-1}$ in such a way that its image is the union between the annulus $\{z \in V : \|z\| \in [R_1, \rho_\lambda]\}$ and the image of a $(k-1)$-dimensional ball in $R_1 S_{VZ}$ whose boundary is $R_1 S_V$. By the estimates (3.7) and (3.8), we deduce that $\sup_{w \in \rho_\lambda B_V} J(\gamma_0(w)) < D_W$, and as a consequence $c_{k-1} < D_W$, proving that the two solutions are distinct, for being at different critical levels. □

**Proposition 3.2.** *If $\lambda \in (\lambda_k, \lambda_{k+m})$ and hypothesis* (1.2) *is satisfied, then there exist constants $K_\lambda$ and $\beta_\lambda$ such that*

$$J(z) \geq K_\lambda, \quad \text{for } z \in W, \tag{3.12}$$

$$J(z) < K_\lambda, \quad \text{for } z \in \beta_\lambda S_{VZ}. \tag{3.13}$$

*Moreover, if one of the sets of hypotheses* $(F3)$ *or* $(F4)$ *is satisfied, then there exists $\delta_1$ such that for $\lambda \in (\lambda_k, \lambda_k + \delta_1)$ there exist $K_\lambda, K_V, E \in R$, $\beta_\lambda > R_2 > 0$, $\xi > 0$ such that, in addition to* (3.12) *and* (3.13),

$$J(z) < K_V, \quad \text{for } z \in V, \tag{3.14}$$

$$J(z) > K_V, \quad \text{for } z \in R_2 S_{ZW}, \tag{3.15}$$

$$J(z) > K_V, \quad \text{for } z \in W,\ \|z\| \geq R_2, \tag{3.16}$$

$$J(z) > E, \quad \text{for } z \in R_2 B_{ZW}, \tag{3.17}$$

$$J(z) < E, \quad \text{for } z \in \xi S_V. \tag{3.18}$$

*The values with index $\lambda$ depend on $\lambda$, the others may be fixed uniformly.*

*This geometry, along with* Lemma 2.2*, allows one to give the following.*

*Proof of Theorem 2.5.* Since the functional $J$ satisfies the $(P.S.)$ condition, we can apply the saddle point theorem and Lemma 2.3.

The first solution that we denote by $w_k$ and may be obtained for any $\lambda \in (\lambda_k, \lambda_{k+m})$ with just hypothesis (1.2) is again obtained through the saddle point theorem and corresponds to a critical point at the critical level

$$d_k = \inf_{\gamma \in \Gamma_k} \sup_{w \in \beta_\lambda B_{VZ}} J(\gamma(w)), \tag{3.19}$$

where now

$$\Gamma_k = \left\{ \gamma \in C^0(\beta_\lambda B_{VZ}; H) \text{s.t. } \gamma|_{\beta_\lambda S_{VZ}} = \mathrm{id} \right\}, \tag{3.20}$$

the criticality is guaranteed by estimates (3.12) and (3.13), since $\beta_\lambda S_{VZ}$ and $W$ link.

The second solution that we denote by $w_{k-1}$ comes from Lemma 2.3, where we set $X_1 = V$ and $X_2 = Z \oplus W$, actually we have the following structure:

$$\sup_{\xi S_V} J(z) < E = \inf_{R_2 B_{ZW}} J(z) \leq \sup_{\xi B_V} J(z) < K_V < \inf_{R_2 S_{ZW}} J(z), \tag{3.21}$$

and then we have a critical point $w_{k-1}$ at the level $d_{k-1} \leq K_V$.

Finally, in order to prove that these two solutions are distinct, we need a sharper estimate for $d_k$ than that given by (3.13). For this we use Lemma 2.2 to guarantee that for any map $\gamma \in \Gamma_k$, since $\beta_\lambda > R_2$, one has that the image of $\gamma$ either intersects $R_2 S_{ZW}$ or has a point $z \in W$ with $\|z\| \geq R_2$. This implies that $\sup_{w \in \beta_\lambda B_{VZ}} J(\gamma(w)) > K_V$, by estimates (3.15) and (3.16), and then $d_k > K_V$ proving that the two solutions are distinct, for being at different critical levels. □

## 4. Proof of Estimates

In this section we will prove all the estimates in Propositions 3.1 and 3.2.

From (1.2) and the continuity of the potential $F$, for any $\varepsilon > 0$, there exists a positive constant $M_\varepsilon = M(\varepsilon)$ such that

$$|\nabla F(x,s)| \leq \varepsilon |s| + M_\varepsilon, \tag{4.1}$$

for all $(x,s) \in \overline{\Omega} \times R^2$. By (4.1), Hölder's inequality, we have

$$\begin{aligned} \left| \int_\Omega F(x,z)dx \right| &\leq \int_\Omega \left| \int_0^1 (\nabla F(x,tz), z) dt \right| dx \leq \int_\Omega \int_0^1 |\nabla F(x,tz)| |z| dt\, dx \\ &\leq \int_\Omega \left( \varepsilon |z|^2 + M_\varepsilon |z| \right) dx \leq \varepsilon \|z\|_{L^2}^2 + M_\varepsilon |\Omega|^{1/2} \|z\|_{L^2} \\ &\leq \varepsilon S^2 \|z\|^2 + M_\varepsilon S |\Omega|^{1/2} \|z\|, \end{aligned} \tag{4.2}$$

where $S$ is the best embedding constant.

### 4.1. Estimates of the Saddle Geometry

**Lemma 4.1.** *Under hypothesis* (1.2)*, one gets the following:*

(i) *for* $\lambda \in (\lambda_{k-1}, \lambda_k)$*, there exists* $D_\lambda$ *satisfying* (3.4) *and* $D_W \in R$ *satisfying* (3.6)*;*

(ii) *for* $\lambda \in (\lambda_k, \lambda_{k+m})$*:*

- (a) *there exists* $K_\lambda \in R$ *satisfying* (3.12)*,*
- (b) *for a given* $R_2 > 0$*, there exists* $E \in R$ *satisfying* (3.17)*.*

*Proof.* Let $z \in W$: using estimates (4.2) and (2.14) we get

$$J(z) \geq \left( \frac{\lambda_{k+m} - \lambda}{2\lambda_{k+m}} - \varepsilon S^2 \right) \|z\|^2 - M_\varepsilon |\Omega|^{1/2} S \|z\|. \tag{4.3}$$

For $\lambda \in (\lambda_{k-1}, \lambda_k)$, letting $\varepsilon < (\lambda_{k+m} - \lambda_k)/2S^2\lambda_{k+m} < (\lambda_{k+m} - \lambda)/2S^2\lambda_{k+m}$, it follows that $J$ is bounded below in $W$, that is, there exists a $D_W$ as in (3.6).

For $\lambda \in (\lambda_k, \lambda_{k+m})$, then the same estimate holds but the constant cannot be made independent of $\lambda$, giving (3.12).

In the same way, let $z \in Z \oplus W$ and set $\delta = \lambda_k - \lambda > 0$, we get

$$\begin{aligned} J(z) &\geq \left( \frac{\lambda_k - \lambda}{2\lambda_k} - \varepsilon S^2 \right) \|z\|^2 - M_\varepsilon |\Omega|^{1/2} S \|z\| \\ &\geq \left( \frac{\delta}{2\lambda_k} - \varepsilon S^2 \right) \|z\|^2 - M_\varepsilon |\Omega|^{1/2} S \|z\|. \end{aligned} \tag{4.4}$$

Letting $\varepsilon < \delta / 2S^2 \lambda_k$, it follows that $J$ is bounded below in $Z \oplus W$, that is, there exists a $D_\lambda$ such that for all $z \in Z \oplus W$ we have (3.4), where again the constant $D_\lambda$ depends on $\delta$, that is, on $\lambda$.

Finally, (4.4) with $\lambda \in (\lambda_k, \lambda_{k+m})$ implies

$$J(z) \geq \left( \frac{\lambda_k - \lambda_{k+m}}{2\lambda_k} - \varepsilon S^2 \right) \|z\|^2 - M_\varepsilon |\Omega|^{1/2} S \|z\|, \tag{4.5}$$

then, no matter the value of $\lambda$, $J$ is bounded from below in any bounded subset of $Z \oplus W$, giving (3.17) for a suitable value of $E$. □

**Lemma 4.2.** *Under hypothesis* (1.2), *one gets the following:*

(i) *for* $\lambda \in (\lambda_{k-1}, \lambda_k)$, *given the constant* $D_\lambda \in R$, *there exists* $\rho_\lambda > 0$ *satisfying* (3.5);

(ii) *for* $\lambda \in (\lambda_k, \lambda_{k+m})$:

(a) *there exists* $K_V \in R$ *satisfying* (3.14),

(b) *for a given* $K_\lambda \in R$, *there exists* $\beta_\lambda > 0$ *satisfying* (3.13),

(c) *for a given* $E \in R$, *there exists* $\xi > 0$ *satisfying* (3.18).

*Moreover, given the values* $R_1$, $R_2$, *one may always choose* $\rho_\lambda > R_1$, $\beta_\lambda > R_2$ *as claimed in Propositions 3.1 and 3.2.*

*Proof.* Let $z \in V$, by estimates (4.2) and (2.13) we get

$$J(z) \leq \left( \frac{\lambda_{k-1} - \lambda}{2\lambda_{k-1}} + \varepsilon S^2 \right) \|z\|^2 + M_\varepsilon |\Omega|^{1/2} S \|z\|. \tag{4.6}$$

For $\lambda \in (\lambda_{k-1}, \lambda_k)$, letting $\varepsilon < (\lambda - \lambda_{k-1})/2S^2\lambda_{k-1}$, then one obtains (3.5) for suitably large $\rho_\lambda > R_1$.

For $\lambda \in (\lambda_k, \lambda_{k+m})$, letting $\varepsilon < (\lambda_k - \lambda_{k-1})/2S^2\lambda_{k-1}$, one obtains, for suitable $K_V$ and $\xi > 0$, (3.14) and (3.18).

Finally, let $z \in V \oplus Z$ and set $\delta = \lambda - \lambda_k > 0$, we get

$$\begin{aligned} J(z) &\leq \left( \frac{\lambda_k - \lambda}{2\lambda_k} + \varepsilon S^2 \right) \|z\|^2 + M_\varepsilon |\Omega|^{1/2} S \|z\| \\ &\leq \left( -\frac{\delta}{2\lambda_k} + \varepsilon S^2 \right) \|z\|^2 + M_\varepsilon |\Omega|^{1/2} S \|z\|. \end{aligned} \tag{4.7}$$

Letting $\varepsilon < \delta/2S^2\lambda_k$, it is clear that (once that $\delta$ is fixed) this goes to $-\infty$ and then we may find the claimed $\beta_\lambda > R_2$ such that (3.13) holds.

Observe that $K_V$ and $E$ can be chosen uniformly for $\lambda \in (\lambda_k, \lambda_{k+m})$, while $\rho_\lambda$, $\beta_\lambda$ in fact depend on $\lambda$. □

### *4.2. Estimating the Effect of the Nontrivial Perturbation*

In this section we will prove the remaining inequalities in Propositions 3.1 and 3.2, which rely on the hypotheses $(F1)$, or $(F2)$, or $(F3)$, or $(F4)$, which, roughly speaking, say that the perturbation $F$ is nontrivial in such a way that a new solution arises when it is sufficiently near to the eigenvalue $\lambda_k$. The proof is simpler for Theorem 2.4, since we need to estimate the functional in the compact set $S_{VZ}$, while for Theorem 2.5 the same kind of estimate is required in the noncompact set $S_{ZW}$.

#### *4.2.1. Estimating $J$ in $S_{VZ}$*

For the next estimates, we will need the following lemma.

**Lemma 4.3.** *Hypotheses $(F2)$ implies that there exists a nondecreasing function $D : (0, +\infty) \to R$ such that*

$$\lim_{R \to +\infty} D(R) = +\infty, \qquad \inf_{z \in RS_{VZ}} \int_\Omega F(x, z)dx > D(R). \tag{4.8}$$

*Proof.* First we claim that there exists a constant $\eta > 0$ such that the sets $\Omega_z = \{x \in \Omega : |z(x)| > \eta\}$ have measure $|\Omega_z| > \eta$, for all $z \in S_{VZ}$.

Actually, $V \oplus Z$ is a finite-dimensional subspace and the functions $z \in S_{VZ}$ are smooth, they are uniformly bounded, that is, there exists $M > 0$ such that $|z(x)| \le M$ for all $x \in \Omega$. Suppose that for $\eta_n \to 0 (\eta_n < 1)$ there exists $\{z_n\} \subset S_{VZ}$ such that $|\Omega_{z_n}| \le \eta_n$.

On one hand, by (2.13), one obtains

$$\frac{1}{\lambda_k} \le \int_\Omega (A(x)z_n, z_n)dx. \tag{4.9}$$

On the other hand,

$$\begin{aligned}
\int_\Omega (A(x)z_n, z_n)dx &= \int_\Omega \left(a(x)u_n^2 + 2b(x)u_n v_n + d(x)v_n^2\right)dx \\
&\le \int_\Omega (a(x) + b(x))u_n^2 + \int_\Omega (b(x) + d(x))v_n^2 dx \\
&\le \overline{M} \int_\Omega |z_n|^2 dx \\
&= \overline{M}\left(\int_{\Omega_{z_n}} |z_n|^2 dx + \int_{\Omega - \Omega_{z_n}} |z_n|^2 dx\right) \\
&\le \overline{M}\left(M^2 |\Omega_{z_n}| + \eta_n^2 |\Omega - \Omega_{z_n}|\right) \\
&\le \eta_n C \\
&\longrightarrow 0,
\end{aligned} \tag{4.10}$$

where $\overline{M} = \max_{x\in\Omega}\{|a(x) + b(x)|, |b(x) + d(x)|\}$, $z_n = (u_n, v_n)$, $C = \overline{M}(M^2 + |\Omega|)$. That is a contradiction.

Now for any $H > 0$, we will show that we can find an $\widetilde{R}$ large enough so that $\int_\Omega F(x, Rz)dx \geq H$ for any $z \in S_{VZ}$ and $R \geq \widetilde{R}$, which means that

$$\lim_{R\to\infty} \inf_{z\in RS_{VZ}} \int_\Omega F(x, z)dx = +\infty. \tag{4.11}$$

Actually, letting $M = (H + |\Omega|C_F)\eta^{-1}$, by $(F2)$ we have that there exists $s_0$ such that $F(x, s) > M$ for $|s| > s_0$.

For $R > s_0/\eta$, one has $\Omega_z \subseteq \{x \in \Omega : |Rz(x)| > s_0\}$, and then one gets

$$\int_{|Rz|\geq s_0} F(x, Rz)dx \geq M\eta. \tag{4.12}$$

For $R \leq s_0/\eta$, by $(F2)$ and $F \in C^1(\overline{\Omega}\times R^2, R)$, there exists $C_F > 0$ such that $F(x, s) \geq -C_F$, for all $(x, s) \in (\Omega, R^2)$.

One finally obtains

$$\begin{aligned}\int_\Omega F(x, Rz)dx &= \int_{|Rz|\geq s_0} F(x, Rz)dx + \int_{|Rz|\leq s_0} F(x, Rz)dx \\ &\geq M\eta - |\Omega|C_F \\ &= H,\end{aligned} \tag{4.13}$$

it is elementary that

$$D(R) = \inf_{\rho\geq R} \inf_{z\in RS_{VZ}} \int_\Omega F(x, z)dx \tag{4.14}$$

is well defined and satisfies the claim. □

Now we may prove the following.

**Lemma 4.4.** *Consider Theorem 2.4 with one of the sets of hypotheses $(F1)$ or $(F2)$. Given the constant $D_W \in R$, there exist $R_1$, $\delta_0 > 0$ such that, for any $\lambda \in (\lambda_k - \delta_0, \lambda_k)$,* (3.7) *and* (3.8) *hold.*

*Proof.* We consider the two sets of hypotheses separately.

(i) In case $(F1)$, assuming (1.2) and $(F1)$ hold, we claim that there exists $C_M$ such that

$$F(x, s) \geq M|s| - C_M, \qquad \forall M \in R, \tag{4.15}$$

uniformly in $x \in \Omega$, in particular we set $M = 1$.

In fact, by $(F1)$, there exits $R_0 > 0$ such that for $|s| \geq R_0$, $(\nabla F(x,s), s) \geq |s|$ uniformly in $x \in \Omega$. For any $s \in R^2 (s \neq 0)$, from (4.1) (letting $\varepsilon = 2$) we have

$$\begin{aligned} F(x,s) &= \int_0^1 (\nabla F(x,ts), s) dt \\ &= \int_{R_0/|s|}^1 (\nabla F(x,ts), s) dt + \int_0^{R_0/|s|} (\nabla F(x,ts), s) dt \\ &\geq \int_{R_0/|s|}^1 |s| dt + \int_0^{R_0/|s|} (\nabla F(x,ts), s) dt \\ &\geq |s| - R_0 - \int_0^{R_0/|s|} \left( 2t|s|^2 + |M_2||s| \right) dt \\ &= |s| - R_0 - R_0^2 - M_2 R_0 \\ &= |s| - C_1, \end{aligned} \tag{4.16}$$

where $C_1 = R_0 + R_0^2 + M_2 R_0$.

Let $z \in RS_{VZ}$, for being in a finite-dimensional subspace, all the norms are equivalent, so that (set $\delta = \lambda_k - \lambda > 0$ and uses estimates (4.15) and (2.13))

$$\begin{aligned} J(z) &\leq \frac{\lambda_k - \lambda}{2\lambda_k} \|z\|^2 - \|z\| + C_1 |\Omega| \\ &\leq \frac{\delta}{2\lambda_k} \|z\|^2 - \|z\| + C_1 |\Omega| \\ &\leq \frac{\delta}{2\lambda_k} R^2 - R + C_1 |\Omega|. \end{aligned} \tag{4.17}$$

(ii) In case $(F2)$, let $D(R)$ be as in Lemma 4.3, for $\|z\| = R$, let $z = w + \phi$ with $w \in V$ and $\phi \in Z = E_k$,

$$\begin{aligned} J(z) &= \frac{1}{2} \int_\Omega (H(x) \nabla z, \nabla z) dx - \frac{\lambda}{2} \int_\Omega (A(x) z, z) - \int_\Omega F(x,z) dx \\ &\leq \frac{\lambda_{k-1} - \lambda}{2\lambda_{k-1}} \|w\|^2 + \frac{\lambda_k - \lambda}{2\lambda_k} \|\phi\|^2 - \int_\Omega F(x,z) dx \\ &\leq \frac{\lambda_{k-1} - \lambda_k + \delta}{2\lambda_{k-1}} \|w\|^2 + \frac{\delta}{2\lambda_k} \|\phi\|^2 - \int_\Omega F(x,z) dx \\ &\leq \frac{\delta}{2\lambda_k} \|\phi\|^2 - \frac{\lambda_k - \lambda_{k-1}}{4\lambda_{k-1}} \|w\|^2 - \int_\Omega F(x,z) dx. \end{aligned} \tag{4.18}$$

Assume that $\delta \leq (\lambda_k - \lambda_{k-1})/2$, it is easy to see that $(-(\lambda_k - \lambda_{k-1})/4\lambda_{k-1}) \|w\|^2 \leq C$ for some constant $C$, we estimate

$$\begin{aligned} J(z) &\leq \frac{\delta}{2\lambda_k} \|\phi\|^2 + C - D(R) \\ &\leq \frac{\delta}{2\lambda_k} R^2 - D(R) + C. \end{aligned} \tag{4.19}$$

Considering (4.17) and (4.19), we see that since $\lim_{R\to\infty} D(R) = +\infty$ by Lemma 4.3, we may fix $R_1$ so that $C - D(R_1) < D_W - 1$ (or $D_M - R_1 < D_W - 1$ for the case $(F1)$) and then for $0 < \delta < \min\{2\lambda_k/R_1^2, (\lambda_k - \lambda_{k-1})/2\}$ one gets (3.7).

To obtain (3.8), we observe that (since $\lambda > \lambda_{k-1}$) if $\phi = 0$, that is, if $z \in V$, then in estimates (4.17) and (4.19) we may avoid the term $(\delta/2\lambda_k)R^2$ so that (remember that $D(R)$ is nondecreasing) $J(z) < D_W - 1$ for $\|z\| > R_1$. □

### 4.2.2. *Estimating $J$ in $S_{ZW}$*

We consider the corresponding of the previous lemma, for Theorem 2.5.

**Lemma 4.5.** *Considering Theorem 2.5 with one of the sets of hypotheses $(F3)$ or $(F4)$. Given the constant $K_V \in R$, there exists $R_2$, $\delta_1 > 0$ such that, for any $\lambda \in (\lambda_k, \lambda_k + \delta_1)$,* (3.15) *and* (3.16) *hold.*

*Proof.* Letting $\lambda = \lambda_k + \delta$, we see from (4.3), that property (3.16) will be satisfied provided that $R_2$ is large enough (say $R_2 > \widetilde{R}$) and observing that this value can be made independent from $\lambda$ once that $\delta$ is small enough.

Now we consider the two sets of hypotheses separately.

(i) In case $(F3)$, suppose $z \in E_k \oplus W$, we can assume that $z = w + \phi$, with $w \in W$ and $\phi \in E_k$. Since $E_k$ is a finite dimension subspace, all the norms are equivalent, so that there exists $C > 0$ such that for all $\phi \in E_k$ we have $\|\phi\| \le C\|\phi\|_{L^1}$. By $(F3)$, from the proof of (4.15), we also have the similar inequality: there exists $C_2$ such that

$$-F(x,s) \ge C|s| - C_2, \tag{4.20}$$

uniformly in $x \in \Omega$. So by (2.14) and (4.20),

$$\begin{aligned}
J(w+\phi) &= \frac{1}{2}\int_\Omega (H(x)\nabla(w+\phi),\ \nabla(w+\phi))dx \\
&\quad - \frac{\lambda}{2}\int_\Omega (A(x)(w+\phi), w+\phi)dx - \int_\Omega F(x, w+\phi)dx \\
&\ge \frac{\lambda_{k+m} - (\lambda_k+\delta)}{2\lambda_{k+m}}\|w\|^2 - \frac{\delta}{2\lambda_k}\|\phi\|^2 + C\|w+\phi\|_{L^1} - C_2|\Omega| \\
&\ge \frac{\lambda_{k+m} - (\lambda_k+\delta)}{2\lambda_{k+m}}\|w\|^2 - \frac{\delta}{2\lambda_k}\|\phi\|^2 + C\|\phi\|_{L^1} - C\|-w\|_{L^1} - C_2|\Omega| \\
&\ge \frac{\lambda_{k+m} - (\lambda_k+\delta)}{2\lambda_{k+m}}\|w\|^2 - \frac{\delta}{2\lambda_k}\|\phi\|^2 + \|\phi\| - C_3\|w\| - C_4,
\end{aligned} \tag{4.21}$$

where $C_3 = |\Omega|^{1/2}SC$, $C_4 = C_2|\Omega|$. Since

$$\begin{aligned}
\left(1 - \frac{\delta}{2\lambda_k}\|z\|\right)\|z\| &\le \|w\| + \|\phi\| - \frac{\delta}{2\lambda_k}\left(\|\phi\|^2 + \|w\|^2\right) \\
&\le \left(1 - \frac{\delta}{2\lambda_k}\|\phi\|\right)\|\phi\| + \|w\|,
\end{aligned} \tag{4.22}$$

suppose $\delta \le (\lambda_{k+m} - \lambda_k)/2$, (4.21) becomes

$$J(w+\phi) \ge \frac{\lambda_{k+m} - (\lambda_k + \delta)}{2\lambda_{k+m}} \|w\|^2 - C_3\|w\| - C_4 - \|w\| + \left(1 - \frac{\delta}{2\lambda_k}\|z\|\right)\|z\| \\ \ge \frac{\lambda_{k+m} - \lambda_k}{4\lambda_{k+m}} \|w\|^2 - (C_3+1)\|w\| - C_4 + \left(1 - \frac{\delta}{2\lambda_k}\|z\|\right)\|z\|, \tag{4.23}$$

since $(\lambda_{k+1} - \lambda_k)/4\lambda_{k+1} > 0$, so $((\lambda_{k+1} - \lambda_k)/4\lambda_{k+1})\|w\|^2 - (C_3+1)\|w\| - C_4$ is bounded below for all $w \in W$, that is, there exists $C_5 \in R$ such that

$$\frac{\lambda_{k+m} - \lambda_k}{4\lambda_{k+m}} \|w\|^2 - C_3\|w\| - C_4 \ge C_5, \tag{4.24}$$

by (4.23) one gets

$$J(z) \ge \left(1 - \frac{\delta}{2\lambda_k}\|z\|\right)\|z\| + C_5 \\ = -\frac{\delta}{2\lambda_k}\|z\|^2 + \|z\| + C_5. \tag{4.25}$$

(ii) In case $(F4)$, first we give some conclusions which are similar to Lemma 3 of [16]. Under the property of $F$, there exists a constant $C$, and $G \in C(R^2, R)$ which is subadditive, that is,

$$G(s+t) \le G(s) + G(t), \tag{4.26}$$

for all $s, t \in R^2$, and coercive, that is,

$$G(s) \longrightarrow +\infty, \tag{4.27}$$

as $|s| \to \infty$, and satisfies that

$$G(s) \le |s| + 4, \tag{4.28}$$

for all $s \in R^2$, such that

$$-F(x,s) \ge G(s) - C, \tag{4.29}$$

for all $s \in R^2$ and $x \in \overline{\Omega}$.

In fact, since $-F(x,s) \to +\infty$ as $|s| \to \infty$ uniformly for all $x \in \overline{\Omega}$, there exists a sequence of positive integers $n_k$ with $n_{k+1} > 2n_k$ for all positive integers $k$ such that

$$-F(x,s) \ge k, \tag{4.30}$$

for all $|s| \geq n_k$ and all $x \in \overline{\Omega}$. Let $n_0 = 0$ and define

$$G(s) = k + 2 + \frac{|s| - n_{k-1}}{n_k - n_{k-1}}, \tag{4.31}$$

for $n_{k-1} \leq |s| < n_k$, where $k \in N$.

By the definition of $G$ we have

$$k + 2 \leq G(s) \leq k + 3, \tag{4.32}$$

for all $n_{k-1} \leq |s| < n_k$. By $(F4)$ and $F \in C^1(\overline{\Omega} \times R^2, R)$, there exists $C_F > 0$ such that

$$-F(x, s) \geq -C_F, \quad \forall (x, s) \in \left(\Omega, R^2\right). \tag{4.33}$$

It follows that

$$-F(x, s) \geq G(s) - C, \tag{4.34}$$

where $C = C_F + 4$. In fact, when $n_{k-1} \leq |s| < n_k$ for some $k \geq 2$, one has, by (4.30) and (4.32),

$$-F(x, s) \geq k - 1 \geq G(s) - 4 \geq G(s) - C, \tag{4.35}$$

for all $x \in \overline{\Omega}$. When $|s| < n_1$, we have, by (4.32) and (4.33),

$$-F(x, s) \geq -C_F = 4 - C \geq G(s) - C, \tag{4.36}$$

for all $x \in \overline{\Omega}$.

It is obvious that $G$ is continuous and coercive. Moreover one has

$$G(s) \leq |s| + 4, \tag{4.37}$$

for all $s \in R^2$. In fact, for every $s \in R^2$ there exists $k \in N$ such that

$$n_{k-1} \leq |s| < n_k, \tag{4.38}$$

which implies that

$$G(s) \leq (k - 1) + 4 \leq n_{k-1} + 4 \leq |s| + 4, \tag{4.39}$$

for all $s \in R^2$ by (4.32) and the fact that $n_k \geq k$ for all integers $k \geq 0$.

Now we only need to prove the subadditivity of $G$. Let

$$n_{k-1} \leq |s| < n_k, \qquad n_{j-1} \leq |t| < n_j, \tag{4.40}$$

and $m = \max\{k, j\}$. Then we have

$$|s+t| \le |s| + |t| < n_k + n_j \le 2n_m < n_{m+1}. \tag{4.41}$$

Hence we obtain, by (4.32),

$$G(s+t) \le m+4 \le k+2+j+2 \le G(s)+G(t), \tag{4.42}$$

which shows that $G$ is subadditive.

For $z \in = E_k \oplus W$, assuming that $z = w + \phi$, with $w \in W$ and $\phi \in E_k$, and letting $0 < \delta < (\lambda_{k+m} - \lambda_k)/2$, by (2.13), (4.29), (4.26), and (4.28), one gets

$$\begin{aligned}
J(w+\phi) &= \frac{1}{2}\int_\Omega (H(x)\nabla(w+\phi), \nabla(w+\phi))dx \\
&\quad - \frac{\lambda}{2}\int_\Omega (A(x)(w+\phi), w+\phi)dx - \int_\Omega F(x, w+\phi)dx \\
&\ge \frac{\lambda_{k+m} - (\lambda_k + \delta)}{2\lambda_{k+m}}\|w\|^2 - \frac{\delta}{2\lambda_k}\|\phi\|^2 \\
&\quad + \int_\Omega G(\phi + w)dx - C|\Omega| \\
&\ge \frac{\lambda_{k+m} - (\lambda_k + \delta)}{2\lambda_{k+m}}\|w\|^2 - \frac{\delta}{2\lambda_k}\|\phi\|^2 \\
&\quad + \int_\Omega G(\phi)dx - \int_\Omega G(-w)dx - C|\Omega| \\
&\ge \frac{\lambda_{k+m} - \lambda_k}{4\lambda_{k+m}}\|w\|^2 - \frac{\delta}{2\lambda_k}\|\phi\|^2 \\
&\quad + \int_\Omega G(\phi)dx - \int_\Omega (|w| + 4)dx - C|\Omega| \\
&\ge \frac{\lambda_{k+m} - \lambda_k}{4\lambda_{k+m}}\|w\|^2 - \frac{\delta}{2\lambda_k}\|z\|^2 \\
&\quad + \int_\Omega G(\phi)dx - S|\Omega|\|w\| - C_1 \\
&= g(w) + \int_\Omega G(\phi)dx - \frac{\delta}{2\lambda_k}\|z\|^2,
\end{aligned} \tag{4.43}$$

where $g(w) = ((\lambda_{k+m} - \lambda_k)/4\lambda_{k+m})\|w\|^2 - S|\Omega|\|w\| - C_1$, $C_1 = (4+C)|\Omega|$. Since $\phi \in E_k$, $E_k$ is a finite-dimensional subspace, and $G$ is coercive, from the proof of (4.8), one can get

$$\lim_{\|\phi\| \to \infty} \int_\Omega G(\phi)dx = +\infty, \tag{4.44}$$

that is, $\int_\Omega G(\phi)dx$ is coercive on $E_k$. Since $(\lambda_{k+m}-\lambda_k)/4\lambda_{k+m} > 0$, so $g(w)$ is coercive on $W$, and $\int_\Omega G(\phi)dx$ and $g(w)$ is bounded below, it is obvious that

$$\lim_{\|z\|\to\infty}\left(g(w)+\int_\Omega G(\phi)dx\right)=+\infty, \tag{4.45}$$

for all $z \in Z \oplus W$.

Considering (4.25), (4.43), and (4.45), we can choose $R_2$ large enough such that for all $\|z\| \geq R_2$ one gets

$$g(w)+\int_\Omega G(\phi)dx > K_V + 1, \tag{4.46}$$

(or $R_2 + C_5 > K_V + 1$ for the case $(F3)$) and property (3.16) holds, then for $0 < \delta < \min\{2\lambda_k/R_2^2, (\lambda_{k+m}-\lambda_k)/2\} = \delta_1$ and $z \in R_2 S_{ZW}$ one gets $J(z) > K_V$, that is, the property (3.15) holds. □

## 5. Proof of the Geometry in Propositions 3.1 and 3.2

We finally give the proof of Propositions 3.1 and 3.2, which is nothing but a resume of the lemmata above, verifying that all the constants can be chosen sequentially without contradictions.

*Proof of Proposition 3.1.* Under hypothesis (1.2), if we fix a value $\lambda$, then we obtain the constant $D_\lambda$ from Lemma 4.1 and with this we get $\rho_\lambda$ from Lemma 4.2. If we also consider one of the two sets of hypotheses $(F1)$ or $(F2)$, then we proceed as follows: first of all, we determine (once for ever) the constant $D_W$ from Lemma 4.1, with this we obtain from Lemma 4.4 the values $R_1$ and $\delta_0$. Then, for any (now fixed) $\lambda \in (\lambda_k - \delta_0, \lambda_k)$, we obtain from Lemma 4.1 the value $D_\lambda$. Finally, we can get from Lemma 4.2 the corresponding value of $\rho_\lambda > R_1$. □

*Proof of Proposition 3.2.* Under hypothesis (1.2), if we fix a value $\lambda \in (\lambda_k, \lambda_{k+m})$, then we obtain the constant $K_\lambda$ from Lemma 4.1 and with this we get $\beta_\lambda$ from Lemma 4.2. If we also consider one of the two sets of hypotheses $(F3)$ or $(F4)$, then we proceed as follows: first of all, we determine (once for ever) the constant $K_V$ from Lemma 4.2, with this we obtain from Lemma 4.5 the values $R_2$ and $\delta_1$. Since we have $R_2$, we can get from Lemma 4.1 the constant $E$ and with this obtain $\xi$ from Lemma 4.2.

Finally, for any (now fixed) $\lambda \in (\lambda_k, \lambda_k + \delta_1)$, we obtain from Lemma 4.1 the constant $K_\lambda$ and with this we get from Lemma 4.2 the corresponding value of $\beta_\lambda > R_2$. □

## Acknowledgments

This paper was supported by National Commission of Ethnic Affairs, Science Foundation of China (no. [2011]02) and by Natural Science Foundation of the Education Department of Guizhou Province (no. [2010]305).

## References


[1] P. Drábek, A. Kufner, and F. Nicolosi, *Quasilinear Elliptic Equations with Degenerations and Singularities*, Walter de Gruyter & Co., Berlin, Germany, 1997.

[2] N. B. Zographopoulos, "On the principal eigenvalue of degenerate quasilinear elliptic systems," *Mathematische Nachrichten*, vol. 281, no. 9, pp. 1351–1365, 2008.

[3] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, vol. 1 of *Physical Origins and Classical Methods*, Springer, Berlin, Germany, 1990.

[4] N. I. Karachalios and N. B. Zographopoulos, "On the dynamics of a degenerate parabolic equation: global bifurcation of stationary states and convergence," *Calculus of Variations and Partial Differential Equations*, vol. 25, no. 3, pp. 361–393, 2006.

[5] J. Mawhin and K. Schmitt, "Nonlinear eigenvalue problems with the parameter near resonance," *Annales Polonici Mathematici*, vol. 51, pp. 241–248, 1990.

[6] T. F. Ma, M. Ramos, and L. Sanchez, "Multiple solutions for a class of nonlinear boundary value problem near resonance: a variational approach," *Nonlinear Analysis*, vol. 30, no. 6, pp. 3301–3311, 1997.

[7] T. F. Ma and M. L. Pelicer, "Perturbations near resonance for the $p$-Laplacian in $R^N$," *Abstract and Applied Analysis*, vol. 7, no. 6, pp. 323–334, 2002.

[8] T. F. Ma and L. Sanchez, "Three solutions of a quasilinear elliptic problem near resonance," *Mathematica Slovaca*, vol. 47, no. 4, pp. 451–457, 1997.

[9] Z.-Q. Ou and C.-L. Tang, "Existence and multiplicity results for some elliptic systems at resonance," *Nonlinear Analysis*, vol. 71, no. 7-8, pp. 2660–2666, 2009.

[10] F. O. de Paiva and E. Massa, "Semilinear elliptic problems near resonance with a nonprincipal eigenvalue," *Journal of Mathematical Analysis and Applications*, vol. 342, no. 1, pp. 638–650, 2008.

[11] H.-M. Suo and C.-L. Tang, "Multiplicity results for some elliptic systems near resonance with a nonprincipal eigenvalue," *Nonlinear Analysis*, vol. 73, no. 7, pp. 1909–1920, 2010.

[12] D. G. de Figueiredo, "Positive solutions of semilinear elliptic problems," in *Differential Equations, São Paulo, 1981*, vol. 957 of *Lecture Notes in Mathematics*, pp. 34–87, Springer, Berlin, Germany, 1982.

[13] A. Marino, A. M. Micheletti, and A. Pistoia, "A nonsymmetric asymptotically linear elliptic problem," *Topological Methods in Nonlinear Analysis*, vol. 4, no. 2, pp. 289–339, 1994.

[14] D. G. de Figueiredo, *Lectures on the Ekeland Variational Principle with Applications and Detours*, vol. 81 of *Tata Institute of Fundamental Research Lectures on Mathematics and Physics*, Tata Institute of Fundamental Research, Bombay, India, 1989.

[15] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory With Applications to Differential Equations*, vol. 65 of *CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Providence, RI, USA, 1986.

[16] C.-L. Tang and X.-P. Wu, "Periodic solutions for second order systems with not uniformly coercive potential," *Journal of Mathematical Analysis and Applications*, vol. 259, no. 2, pp. 386–397, 2001.