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Research Article

Certain Properties of Multivalent Functions Associated with the Dziok-Srivastava Operator

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By making use of the techniques of the differential subordination, we derive certain properties of p-valent functions associated with the Dziok-Srivastava operator.

1. Introduction

Let $A(p,k)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=k}^{\infty} a_{n+p} z^{n+p} \quad (p,k \in \mathbb{N} = \{1,2,3,\dots\}), \quad (1)$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. We write $A(p,1) = A(p)$.

Suppose that f and g are analytic in U. We say that the function f is subordinate to g in U, or g superordinate to f in U, and we write $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$, if there exists an analytic function ω in U with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $f(z) = g(\omega(z))$ $(z \in U)$. If g is univalent in U, then the following equivalence relationship holds true (see [1–3]):

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0), \qquad f(U) \subset g(U). \quad (2)$$

For functions $f_j \in A(p,k)$ given by

$$f_j(z) = z^p + \sum_{n=k}^{\infty} a_{n+p,j} z^{n+p} \quad (j = 1,2;\ p \in \mathbb{N}), \quad (3)$$

we define the Hadamard product (or convolution) of f_1 and f_2 by

$$(f_1 * f_2)(z) = z^p + \sum_{n=k}^{\infty} a_{n+p,1} a_{n+p,2} z^{n+p} = (f_2 * f_1)(z). \quad (4)$$

For complex parameters $a_1, \dots, a_q$ and $b_1, \dots, b_s$ $(bj \notin \mathbb{Z}_0^- = \{0,-1,-2,\dots\};\ j = 1,\dots,s)$, the generalized hypergeometric function ${}_qF_s$ is defined (see [4]) by the following infinite series:

$${}_qF_s(a_1,\dots,a_i,\dots,a_q;b_1,\dots,b_s;z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_q)_n}{(b_1)_n \cdots (b_s)_n} \frac{z^n}{n!}$$
$$(q \le s+1;\ q,s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ z \in U), \quad (5)$$

where $(\theta)_n$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\theta)_n = \frac{\Gamma(\theta+n)}{\Gamma(\theta)} = \begin{cases} 1, & (\upsilon = 0), \\ \theta(\theta+1)\cdots(\theta+n-1), & (\upsilon \in \mathbb{N}). \end{cases} \quad (6)$$

Corresponding a function $h_p(a_1,\dots,a_i,\dots,a_q;b_1,\dots,b_s;z)$ defined by

$$h_p(a_1,\dots,a_i,\dots,a_q;b_1,\dots,b_s;z) = z^p \cdot {}_qF_s(a_1,\dots,a_i,\dots,a_q;b_1,\dots,b_s;z) \quad (z \in U), \quad (7)$$

Dziok and Srivastava [5] considered a linear operator

$$H_p(a_1,\dots,a_q;b_1,\dots,b_s) : A(p,k) \longrightarrow A(p,k) \quad (8)$$

defined by the following Hadamard product:

$$H_p(a_1,\dots,a_q;b_1,\dots,b_s)f(z),$$
$$=h_p(a_1,\dots,a_i,\dots,a_q;b_1,\dots,b_s;z)*f(z), \quad (9)$$
$$(q\le s+1;\ q,s\in\mathbb{N}_0;\ z\in U).$$

If $f\in A(p,k)$ is given by (1), then we have

$$H_p(a_1,\dots,a_q;b_1,\dots,b_s)f(z) = z^p+\sum_{n=k}^{\infty}\Gamma_n a_{n+p}z^{n+p} \quad (z\in U), \quad (10)$$

where

$$\Gamma_n=\frac{(a_1)_n\cdots(a_q)_n}{(b_1)_n\cdots(b_s)_n}\frac{1}{n!}, \quad (n\in\mathbb{N}). \quad (11)$$

To make the notation simple, we write

$$H_{p,q,s}(a_1)f(z)=H_p(a_1,\dots,a_q;b_1,\dots,b_s)f(z). \quad (12)$$

It easily follows from (9) or (10) that

$$z\left(H_{p,q,s}(a_1)f(z)\right)' = a_1H_{p,q,s}(a_1+1)f(z) - (a_1-p)H_{p,q,s}(a_1)f(z), \quad (z\in U). \quad (13)$$

It should be remarked that the linear operator $H_{p,q,s}(a_1)$ is a generalization of many other linear operators considered earlier. In particular, for $f\in A(p)$ we have the following observations:

(i) $H_{1,2,1}(a,b;c)f(z)=(I_c^{a,b})f(z)$ $(a,b\in\mathbb{C};c\notin\mathbb{Z}_0^-)$, where the linear operator $I_c^{a,b}$ was investigated by Hohlov [6];

(ii) $H_{p,2,1}(n+p,1;1)f(z)=D^{n+p-1}f(z)$ $(n\in\mathbb{N};n>-p)$, where the linear operator D^{n+p-1} was studied by Goel and Sohi [7]. In the case when $p=1$, $D^nf(z)$ is the Ruscheweyh derivative of $f(z)$ (see [8]);

(iii) $H_{p,2,1}(\mu+p,1;\mu+p+1)f(z)=J_{p,\delta}(f)(z)=((p+\delta)/z^{\delta})\int_0^z t^{\delta-1}f(t)dt$ $(\delta>-p)$, where $J_{p,\delta}$ is the generalized Bernardi-Libera-Livingston integral operator (see [9]);

(iv) $H_{p,2,1}(p+1,1;p+1-\lambda)f(z)=\Omega_z^{(\lambda,p)}f(z)=(\Gamma(p+1-\lambda)/\Gamma(p+1))z^{\lambda}D_z^{\lambda}f(z)$ $(-\infty\le\lambda<p+1;z\in U)$, where $D_z^{\lambda}f(z)$ is the fractional integral of f of order $-\lambda$ when $-\infty\le\lambda<0$ and fractional derivative of f of order λ when $0\le\lambda<p+1$. The extended fractional differintegral operator $D_z^{(\lambda,p)}$ was introduced and studied by Patel and Mishra [10]. The fractional differential operator $\Omega_z^{(\lambda,p)}$ with $0\le\lambda<1$ was investigated by Srivastava and Aouf [11]. The operator $\Omega_z^{(\lambda,1)}=\Omega_z^{\lambda}$ was introduced by Owa and Srivastava [12] (see also [13–15]).

(v) $H_{p,2,1}(a,1;c)f(z)=L_p(a,c)f(z)$ $(a\in\mathbb{R};c\in\mathbb{R}\setminus\mathbb{Z}_0)$, where the linear operator $L_p(a,c)$ was studied by Saitoh [16] which yields the operator $L(a,c)$ introduced by Carlson and Shaffer [17] for $p=1$;

(vi) $H_{1,2,1}(\mu,1;\lambda+1)f(z)=I_{\lambda,\mu}f(z)$ $(\lambda>-1;\mu>0)$, where $I_{\lambda,\mu}$ is the Choi-Saigo-Srivastava operator [9] which is closely related to the Carlson-Shaffer [17] operator $L(\mu,\lambda+1)f(z)$;

(vii) $H_{p,2,1}(p+1,1;n+p)f(z)=I_{n,p}f(z)$ $(n\in\mathbb{Z};n>-p)$, where the operator $I_{n,p}$ was considered by Liu and Noor [18];

(viii) $H_{p,2,1}(\lambda+p,c;a)f(z)=I_p^{\lambda}(a,c)f(z)$ $(a,c\in\mathbb{R}\setminus\mathbb{Z}_0^-;\lambda>-p)$, where $I_p^{\lambda}(a,c)$ is the Cho-Kwon-Srivastava operator [19].

In recent years, many interesting subclasses of analytic functions, associated with the Dziok-Srivastava operator $H_{p,q,s}(a_1)$ and its many special cases, were investigated by, for example, Dziok and Srivastava [5, 20], Gangadharan et al. [21], Liu and Noor [18], Liu [22], Liu and Srivastava [23], and others (see also [19, 24–26]). In the present paper, we shall use the method based upon the differential subordination to derive inclusion relationships and other interesting properties and characteristics of the Dziok-Srivastava operator $H_{p,q,s}(a_1)$.

2. Main Results

Unless otherwise mentioned, we assume throughout the sequel that $a_i>0$; $a_i\notin\mathbb{Z}_0^-$ $(i=1,\dots,q)$; $\alpha>0$; $\mu>0$ and $-1\le B<A\le 1$.

Let $P[k]$ denote the class of functions of the form

$$\varphi(z)=1+c_kz^k+c_{k+1}z^{k+1}+\cdots \quad (14)$$

that are analytic in U, we write $P[1]=P$. In our present investigation, we shall require the following lemmas.

Lemma 1 (see [2]). *Let h be analytic and convex (univalent) in U with $h(0)=1$ and $\varphi\in P[k]$. If*

$$\varphi(z)+\frac{z\varphi'(z)}{\gamma}\prec h(z), \quad (15)$$

then, for $\gamma\neq 0$ and $\mathbb{R}(\gamma)\ge 0$,

$$\varphi(z)\prec q(z)=\frac{\gamma}{k}z^{-\gamma/k}\int_0^z t^{\gamma/k-1}h(t)\,dt\prec h(z), \quad (16)$$

and q is the best dominant.

Lemma 2 (see [1]). *Let D be a set in the complex plane $\mathbb{C}$ and b be a complex number satisfying $\mathbb{R}(b)>0$. Suppose that the function $\Psi:\mathbb{C}^2\times U\to\mathbb{C}$ satisfies the condition $\Psi(ix,y)\notin D$ for all real x, $y\le-|b-ix|/2\mathbb{R}(b)$ and for all $z\in U$. If the functions $\varphi\in P$ and $\mathbb{R}\{\Psi(\varphi(z),z\varphi'(z);z)\}\in D$, then $\mathbb{R}\{\varphi(z)\}>0$ in U.*

Lemma 3 (see [27]). *Let ϕ be analytic in U with $\phi(0)=1$ and $\phi(z)\neq 0$ for all $z\in U$. If there exist two points $z_1, z_2 \in U$ such that*

$$-\frac{\pi}{2}\delta_1 = \arg\{\phi(z_1)\} < \arg\{\phi(z)\} < \arg\{\phi(z_2)\} = \frac{\pi}{2}\delta_2 \quad (17)$$

for some δ_1 and δ_2 ($\delta_1, \delta_2 > 0$) and for all z ($|z| < |z_1| = |z_2|$), then

$$\frac{z_1\phi'(z_1)}{\phi(z_1)} = -i\left(\frac{\delta_1+\delta_2}{2}m\right), \qquad \frac{z_2\phi'(z_2)}{\phi(z_2)} = -i\left(\frac{\delta_1+\delta_2}{2}m\right), \quad (18)$$

where

$$m \geq \frac{1-|b|}{1+|b|}, \quad b = i\tan\left(\frac{\delta_2-\delta_1}{\delta_2+\delta_1}\right). \quad (19)$$

Theorem 4. *Let $m\geq 1$, $\gamma > 0$. Let $f\in A(k,p)$, then*

$$\mathbb{R}\left\{\frac{\left(H_{p,q,s}(a_1+1)f(z)\right)^{(j)}}{\left(H_{p,q,s}(a_1)f(z)\right)^{(j)}}\right\} < \frac{a_1+\gamma}{a_1} \quad (z\in U;\ 0\leq j<p), \quad (20)$$

implies

$$\mathbb{R}\left\{\left(\frac{\left(H_{p,q,s}(a_1+1)f(z)\right)^{(j)}}{z^{p-j}}\right)^{-1/2\gamma m}\right\} > 2^{-1/m} \quad (z\in U;\ 0\leq j<p). \quad (21)$$

The bound $2^{-1/m}$ is the best possible.

Proof. It easily follows from (13) that

$$\begin{aligned} z\left(H_{p,q,s}(a_1)f(z)\right)^{(j+1)} &= a_1\left(H_{p,q,s}(a_1+1)f(z)\right)^{(j)} \\ &\quad - (a_1-p+j)\left(H_{p,q,s}(a_1)f(z)\right)^{(j)} \quad (z\in U;\ 0\leq j<p). \end{aligned} \quad (22)$$

From (20) and (22), we have

$$\mathbb{R}\left\{\frac{z\left(H_{p,q,s}(a_1+1)f(z)\right)^{(j+1)}}{\left(H_{p,q,s}(a_1)f(z)\right)^{(j)}}\right\} < \gamma+p-j \quad (z\in U;\ 0\leq j<p). \quad (23)$$

That is,

$$-\frac{1}{2\gamma}\left(\frac{z\left(H_{p,q,s}(a_1+1)f(z)\right)^{(j+1)}}{\left(H_{p,q,s}(a_1)f(z)\right)^{(j)}} - p + j\right) \prec \frac{z}{1-z} \quad (z\in U). \quad (24)$$

Let

$$\varphi(z) = \left(\frac{(p-j)!}{p!}\frac{\left(H_{p,q,s}(a_1)f(z)\right)^{(j)}}{z^{p-j}}\right)^{-1/2\gamma} \quad (z\in U), \quad (25)$$

then (24) may be written as

$$z\left(\log\varphi(z)\right)' \prec z\left(\log\frac{1}{1-z}\right)'. \quad (26)$$

By using a well-known result (see [28]) to (26) we obtain that

$$\varphi(z) \prec \frac{1}{1-z}, \quad (27)$$

or, equivalently,

$$\left(\frac{(p-j)!}{p!}\frac{\left(H_{p,q,s}(a_1)f(z)\right)^{(j)}}{z^{p-j}}\right)^{-1/2\gamma m} = \left(\frac{1}{1-\omega(z)}\right)^{1/m}, \quad (28)$$

where ω is analytic in U, $\omega(0)=0$ and $|\omega(z)|<1$ for $z\in U$. Since $\mathbb{R}(t^{1/m}) \geq (\mathbb{R}(t))^{1/m}$ for $\mathbb{R}(t)>0$ and $m\geq 1$, (28) yields

$$\mathbb{R}\left(\frac{(p-j)!}{p!}\frac{\left(H_{p,q,s}(a_1)f(z)\right)^{(j)}}{z^{p-j}}\right)^{-1/2\gamma m} \geq \left(\mathbb{R}\left(\frac{1}{1-\omega(z)}\right)\right)^{1/m} \geq 2^{-1/m} \quad (z\in U). \quad (29)$$

To see that the bound $2^{-1/m}$ cannot be increased, we consider the function

$$g(z) = z^p + \frac{p!}{(p-j)!}\sum_{n=1}^{\infty}\frac{(-2\gamma)_n (n+p-j)!}{n!(n+p)!\Gamma_n}z^{n+p}, \quad (z\in U). \quad (30)$$

Since

$$\frac{(p-j)!}{p!}\frac{\left(H_{p,q,s}(a_1)g(z)\right)^{(j)}}{z^{p-j}} = (1-z)^{-2\gamma}, \quad (31)$$

we easily have that g satisfies (20) and

$$\mathbb{R}\left(\frac{(p-j)!}{p!}\frac{\left(H_{p,q,s}(a_1)g(z)\right)^{(j)}}{z^{p-j}}\right)^{-1/2\gamma m} \longrightarrow 2^{-1/m} \quad (32)$$

as $\mathbb{R}(z) = z \to 1^-$. This completes the proof of Theorem 4. ◻

Theorem 5. *Let* $\alpha \geq 0$, $\gamma > 1$. *If* $f \in A(p)$ *satisfies the following inequality*

$$\Re\left\{(1-\alpha)\frac{\left(H_{p,q,s}(a_1+1)f(z)\right)^{(j)}}{\left(H_{p,q,s}(a_1)f(z)\right)^{(j)}} + \alpha\frac{\left(H_{p,q,s}(a_1+2)f(z)\right)^{(j)}}{\left(H_{p,q,s}(a_1+1)f(z)\right)^{(j)}}\right\} < \gamma \quad (0 \leq j < p;\ z \in U), \tag{33}$$

then

$$\Re\left\{\frac{\left(H_{p,q,s}(a_1+1)f(z)\right)^{(j)}}{\left(H_{p,q,s}(a_1)f(z)\right)^{(j)}}\right\} < \beta \quad (0 \leq j < p;\ z \in U), \tag{34}$$

where $\beta \in (1,\infty)$ *is the positive root of the equation*

$$2(a_1-\alpha+1)x^2 + (3\alpha-2\gamma\alpha-2\gamma)x - \alpha = 0. \tag{35}$$

Proof. Let

$$\varphi(z) = \frac{1}{\beta-1}\left[\beta - \frac{\left(H_{p,q,s}(a_1+1)f(z)\right)^{(j)}}{\left(H_{p,q,s}(a_1)f(z)\right)^{(j)}}\right] \quad (z \in U), \tag{36}$$

then $\varphi(z)$ is analytic in U and $\varphi(0) = 1$. Differentiating (36) and using (22), we obtain that

$$\begin{aligned}(1-\alpha)&\frac{\left(H_{p,q,s}(a_1+1)f(z)\right)^{(j)}}{\left(H_{p,q,s}(a_1)f(z)\right)^{(j)}} + \alpha\frac{\left(H_{p,q,s}(a_1+2)f(z)\right)^{(j)}}{\left(H_{p,q,s}(a_1+1)f(z)\right)^{(j)}}\\ &= \beta - \frac{\alpha(\beta-1)}{a_1+1} - \frac{(a_1-\alpha+1)(\beta-1)}{a_1+1}\varphi(z) - \frac{\alpha(\beta-1)}{a_1+1}\frac{z\varphi'(z)}{\beta-(\beta-1)\varphi(z)}\\ &= \psi\left(\varphi(z), z\varphi'(z)\right),\end{aligned} \tag{37}$$

where

$$\psi(r,s) = \beta - \frac{\alpha(\beta-1)}{a_1+1} - \frac{(a_1-\alpha+1)(\beta-1)}{a_1+1}r - \frac{\alpha(\beta-1)}{a_1+1}\frac{s}{\beta-(\beta-1)r}. \tag{38}$$

Using (33) and (38), we have

$$\left\{\psi\left(\varphi(z), z\varphi'(z)\right) : z \in U\right\} \subset D = \{z \in \mathbb{C} : \Re(z) < \gamma\}. \tag{39}$$

Now for all real x, $y \leq -(1+x^2)/2$, we have

$$\begin{aligned}\Re\{\psi(ix,y)\} &= \beta - \frac{\alpha(\beta-1)}{a_1+1} - \frac{\alpha(\beta-1)}{a_1+1}\frac{\beta y}{\beta^2+(\beta-1)^2x^2}\\ &\geq \beta - \frac{\alpha(\beta-1)}{a_1+1} + \frac{\alpha\beta(\beta-1)}{2(a_1+1)}\frac{1+x^2}{\beta^2+(\beta-1)^2x^2}\\ &\geq \beta - \frac{\alpha(\beta-1)}{a_1+1} + \frac{\alpha(\beta-1)}{2\beta(a_1+1)}\\ &= \beta - \frac{\alpha(\beta-1)(2\beta-1)}{2\beta(a_1+1)} = \gamma,\end{aligned} \tag{40}$$

where β is the positive root of (35).

Note that for $\alpha \geq 0$, $\gamma > 1$, $a_1 > 0$ and

$$h(x) = 2(a_1-\alpha+1)x^2 + (3\alpha-2\gamma\alpha-2\gamma)x - \alpha, \tag{41}$$

we have $h(0) = -\alpha \leq 0$ and $h(1) = 2a_1(1-\gamma) - 2\gamma < 0$. This shows $\beta \in (0,+\infty)$. Hence for each $z \in U$, $\psi(ix,y) \notin \Omega$. By Lemma 2, we get $\Re\{\varphi(z)\} > 0$ $(z \in U)$, and this proves (34). □

Theorem 6. *Suppose that* $0 \leq j < p$; $\alpha > 0$ *and* $0 < \delta_1, \delta_2 \leq 1$. *If* F_α *given by*

$$\begin{aligned}F_\alpha(z) = &\left(1-\alpha-\alpha a_1+\alpha p\right)H_{p,q,s}(a_1)f(z)\\ &+ \alpha a_1 H_{p,q,s}(a_1+1)f(z)\end{aligned} \tag{42}$$

satisfies

$$-\frac{\pi}{2}\delta_1 < \arg\left\{\frac{F_\alpha^{(j)}(z)}{z^{p-j}}\right\} < \frac{\pi}{2}\delta_2 \quad (z \in U), \tag{43}$$

then

$$-\frac{\pi}{2}\eta_1 < \arg\left\{\frac{\left(H_{p,q,s}(a_1)f(z)\right)^{(j)}}{z^{p-j}}\right\} < \frac{\pi}{2}\eta_2 \quad (z \in U), \tag{44}$$

where η_1 *and* η_2 *are the solution of the equations:*

$$\begin{aligned}\delta_1 &= \eta_1 + \frac{2}{\pi}\arctan\left[\frac{\alpha(\eta_1+\eta_2)}{2(1-\alpha+\alpha p)}\left(\frac{1-|b|}{1+|b|}\right)\right],\\ \delta_2 &= \eta_2 + \frac{2}{\pi}\arctan\left[\frac{\alpha(\eta_1+\eta_2)}{2(1-\alpha+\alpha p)}\left(\frac{1-|b|}{1+|b|}\right)\right],\end{aligned} \tag{45}$$

where b *is given by* (19).

Proof. Using (42) and the identity (22), it follows that

$$\begin{aligned}F_\alpha^{(j)}(z) = &\left(1-\alpha+\alpha j\right)\left(H_{p,q,s}(a_1)f(z)\right)^{(j)}\\ &+ \alpha z\left(H_{p,q,s}(a_1)f(z)\right)^{(j+1)},\end{aligned} \tag{46}$$

for $0 \le j < p$. Putting

$$\varphi(z) = \frac{(p-j)!}{p!} \frac{\left(H_{p,q,s}(a_1) f(z)\right)^{(j)}}{z^{p-j}} \quad (z \in U). \tag{47}$$

On differentiating (47) followed by a simple calculation, we get

$$\begin{aligned} \frac{F_\alpha^{(j)}(z)}{z^{p-j}} &= \frac{p!(1-\alpha+\alpha p)}{(p-j)!} \\ &\quad \times \left\{\varphi(z) + \frac{\alpha}{1-\alpha+\alpha p} z\varphi'(z)\right\} \quad (z \in U). \end{aligned} \tag{48}$$

Let h be the function which maps U onto the angular domain $\{w \in \mathbb{C} : -(\pi/2)\delta_1 < \arg\{w\} < (\pi/2)\delta_2\}$ with $h(0) = 1$. By using (43) in (48), we get

$$\varphi(z) + \frac{\alpha}{1-\alpha+\alpha p} z\varphi'(z) \prec h(z). \tag{49}$$

Further, an application of Lemma 1 yields $\Re\{\varphi(z)\} > 0$ in U and hence $\varphi(z) \neq 0$ for $z \in U$.

Suppose there exist two points $z_1, z_2 \in U$ such that the condition (28) is satisfied. Then by Lemma 3, we obtain (18) under the constraint (19). Therefore, we have

$$\begin{aligned} &\arg\left\{(1-\alpha+\alpha p)\varphi(z_1) + \alpha z\varphi'(z_1)\right\} \\ &\quad = \arg\{\varphi(z_1)\} + \arg\left\{(1-\alpha+\alpha p) + \alpha\frac{z_1\varphi'(z_1)}{\varphi(z_1)}\right\} \\ &\quad = -\frac{\pi}{2}\eta_1 + \arg\left\{(1-\alpha+\alpha p) - i\frac{\alpha(\eta_1+\eta_2)}{2}m\right\} \\ &\quad = -\frac{\pi}{2}\eta_1 - \arctan\left\{\frac{\alpha(\eta_1+\eta_2)}{2(1-\alpha+\alpha p)}m\right\} \\ &\quad \le -\frac{\pi}{2}\eta_1 - \arctan\left\{\frac{\alpha(\eta_1+\eta_2)}{2(1-\alpha+\alpha p)}\left(\frac{1-|b|}{1+|b|}\right)\right\}, \\ &\arg\left\{(1-\alpha+\alpha p)\varphi(z_2) + \alpha z\varphi'(z_2)\right\} \\ &\quad \ge -\frac{\pi}{2}\eta_2 - \arctan\left\{\frac{\alpha(\eta_1+\eta_2)}{2(1-\alpha+\alpha p)}\left(\frac{1-|b|}{1+|b|}\right)\right\}, \end{aligned} \tag{50}$$

which contradicts the assumption (43). This proves the assertion (44) of the Theorem 6.

For $\delta_1 = \delta_2 = \delta$, Theorem 6 reduces to the following corollary. □

Corollary 7. *Suppose that $0 \le j < p$ and $\alpha > 0$. If F_α defined by (42) satisfies*

$$\left|\arg\left\{\frac{F_\alpha^{(j)}(z)}{z^{p-j}}\right\}\right| < \frac{\pi}{2}\delta \quad (0 < \delta \le 1;\ z \in U), \tag{51}$$

then

$$\left|\arg\left\{\frac{\left(H_{p,q,s}(a_1) f(z)\right)^{(j)}}{z^{p-j}}\right\}\right| < \frac{\pi}{2}\eta \quad (z \in U), \tag{52}$$

where η $(0 < \eta \le 1)$ is the solution of the equation:

$$\delta = \eta + \frac{2}{\pi}\arctan\left(\frac{\alpha\eta}{1-\alpha+\alpha p}\right). \tag{53}$$

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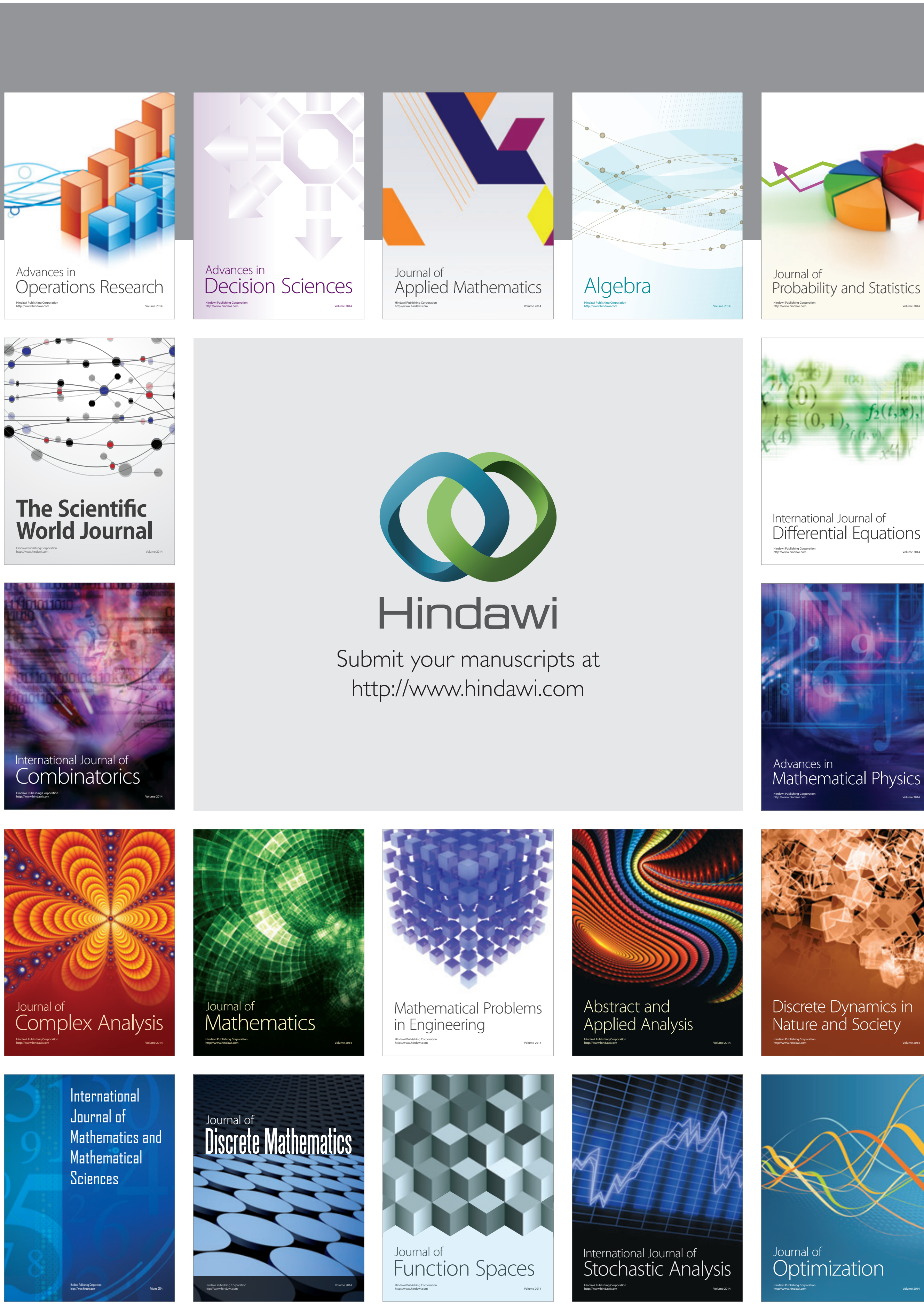
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