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Research Article

On a Third-Order System of Difference Equations with Variable Coefficients

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We show that the system of three difference equations $x_{n+1} = a_n^{(1)} x_{n-2}/(b_n^{(1)} y_n z_{n-1} x_{n-2} + c_n^{(1)})$, $y_{n+1} = a_n^{(2)} y_{n-2}/(b_n^{(2)} z_n x_{n-1} y_{n-2} + c_n^{(2)})$, and $z_{n+1} = a_n^{(3)} z_{n-2}/(b_n^{(3)} x_n y_{n-1} z_{n-2} + c_n^{(3)})$, $n \in \mathbb{N}_0$, where all elements of the sequences $a_n^{(i)}$, $b_n^{(i)}$, $c_n^{(i)}$, $n \in \mathbb{N}_0$, $i \in \{1,2,3\}$, and initial values x_{-j}, y_{-j}, z_{-j}, $j \in \{0,1,2\}$, are real numbers, can be solved. Explicit formulae for solutions of the system are derived, and some consequences on asymptotic behavior of solutions for the case when coefficients are periodic with period three are deduced.

1. Introduction

Studying nonlinear difference equations and systems is an area of a great interest nowadays (see, e.g., [1–39] and the references therein).

This paper studies the system of three difference equations

$$x_{n+1} = \frac{a_n^{(1)} x_{n-2}}{b_n^{(1)} y_n z_{n-1} x_{n-2} + c_n^{(1)}}, \quad y_{n+1} = \frac{a_n^{(2)} y_{n-2}}{b_n^{(2)} z_n x_{n-1} y_{n-2} + c_n^{(2)}},$$
$$z_{n+1} = \frac{a_n^{(3)} z_{n-2}}{b_n^{(3)} x_n y_{n-1} z_{n-2} + c_n^{(3)}}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where all elements of the sequences $a_n^{(i)}, b_n^{(i)}, c_n^{(i)}$, $n \in \mathbb{N}_0$, $i \in \{1,2,3\}$, and initial values x_{-j}, y_{-j}, z_{-j}, $j \in \{0,1,2\}$ are real numbers. The cases when both $b_n^{(i)}$ and $c_n^{(i)}$ are equal to zero for some fixed $i \in \{1,2,3\}$ and an $n \in \mathbb{N}_0$, are not interesting so they are excluded. In [37] we have shown that system (1.1) for the case when the sequences $a_n^{(i)}$, $b_n^{(i)}$, $c_n^{(i)}$, $n \in \mathbb{N}_0$, $i \in \{1,2,3\}$, are constant can be explicitly solved (if solutions are well defined). Some recent results on solving difference equations can be found, for example, in [6, 7, 12, 24, 25, 34, 35, 38, 39]. For some old results see, for example, classic book [14].

Note that the solutions of (1.1) such that all sequences $a_n^{(i)}, b_n^{(i)}, c_n^{(i)}$, $n \in \mathbb{N}_0$, $i \in \{1,2,3\}$, and initial values in system (1.1) are positive, are also positive, that is,

$$x_n > 0, \quad y_n > 0, \quad z_n > 0 \quad \text{for every } n \geq -2, \tag{1.2}$$

so that there are a lot of well defined solutions of the system. In fact, for "majority" initial values of system (1.1), solutions are well defined, but we will not discuss the problem here. Instead of that we assume, throughout the paper, that solutions of (1.1) are well defined. We also adopt the customary notation $\prod_{i=k+1}^{k} g_i = 1$ and $\sum_{i=k+1}^{k} g_i = 0$.

We show that in the main case, system (1.1) is transformed to a third-order system of nonhomogeneous linear first-order difference equations, which can be explicitly solved.

This idea appeared for the first time in [24] for the case of the scalar equation with constant coefficients corresponding to system (1.1) and was also used later in [1, 4]. Some related transformations are used also in [25, 30]. For a different approach in dealing with the scalar difference equation see [2, 3]. For some related scalar difference equations see, for example, [8, 13, 26] and the related references therein. Some related results on systems of difference equations can be found in [11, 15–22] (see also references therein).

Here we give explicit formulae for solutions of system (1.1) and present some consequences on asymptotic behavior of the solutions for the case when coefficients are periodic with period three.

2. Case $a_n^{(i)} = 0$ for Some $i \in \{1,2,3\}$ and All $n \in \mathbb{N}_0$

If $a_n^{(1)} = 0$, $n \in \mathbb{N}_0$, then the first equation in (1.1) becomes

$$x_n = 0, \quad n \in \mathbb{N}, \tag{2.1}$$

so that from the second and the third equations and since the solution is well defined we get

$$y_{n+2} = \frac{a_{n+1}^{(2)}}{c_{n+1}^{(2)}} y_{n-1}, \quad z_{n+1} = \frac{a_n^{(3)}}{c_n^{(3)}} z_{n-2}, \quad n \in \mathbb{N}, \tag{2.2}$$

$c_{n+1}^{(2)} \neq 0 \neq c_n^{(3)}$, $n \in \mathbb{N}$, from which it follows that

$$y_{3n+1} = y_1 \prod_{j=1}^{n} \frac{a_{3j}^{(2)}}{c_{3j}^{(2)}}, \quad y_{3n+2} = y_2 \prod_{j=1}^{n} \frac{a_{3j+1}^{(2)}}{c_{3j+1}^{(2)}}, \quad y_{3n} = y_0 \prod_{j=1}^{n} \frac{a_{3j-1}^{(2)}}{c_{3j-1}^{(2)}}, \quad n \in \mathbb{N},$$
$$z_{3n+1} = z_1 \prod_{j=1}^{n} \frac{a_{3j}^{(3)}}{c_{3j}^{(3)}}, \quad z_{3n-1} = z_{-1} \prod_{j=0}^{n-1} \frac{a_{3j+1}^{(3)}}{c_{3j+1}^{(3)}}, \quad z_{3n} = z_0 \prod_{j=1}^{n} \frac{a_{3j-1}^{(3)}}{c_{3j-1}^{(3)}}, \quad n \in \mathbb{N}. \tag{2.3}$$

If $a_n^{(2)} = 0$, $n \in \mathbb{N}_0$, then the second equation in (1.1) becomes

$$y_n = 0, \quad n \in \mathbb{N}, \tag{2.4}$$

so that from the first and the third equations we get

$$x_{n+1} = \frac{a_n^{(1)}}{c_n^{(1)}} x_{n-2}, \quad z_{n+2} = \frac{a_{n+1}^{(3)}}{c_{n+1}^{(3)}} z_{n-1}, \quad n \in \mathbb{N}, \tag{2.5}$$

$c_n^{(1)} \neq 0 \neq c_{n+1}^{(3)}$, from which it follows that

$$x_{3n+1} = x_1 \prod_{j=1}^{n} \frac{a_{3j}^{(1)}}{c_{3j}^{(1)}}, \quad x_{3n-1} = x_{-1} \prod_{j=0}^{n-1} \frac{a_{3j+1}^{(1)}}{c_{3j+1}^{(1)}}, \quad x_{3n} = x_0 \prod_{j=1}^{n} \frac{a_{3j-1}^{(1)}}{c_{3j-1}^{(1)}}, \quad n \in \mathbb{N},$$
$$z_{3n+1} = z_1 \prod_{j=1}^{n} \frac{a_{3j}^{(3)}}{c_{3j}^{(3)}}, \quad z_{3n+2} = z_2 \prod_{j=1}^{n} \frac{a_{3j+1}^{(3)}}{c_{3j+1}^{(3)}}, \quad z_{3n} = z_0 \prod_{j=1}^{n} \frac{a_{3j-1}^{(3)}}{c_{3j-1}^{(3)}}, \quad n \in \mathbb{N}. \tag{2.6}$$

Finally, if $a_n^{(3)} = 0$, $n \in \mathbb{N}_0$, then the third equation in (1.1) becomes

$$z_n = 0, \quad n \in \mathbb{N}, \tag{2.7}$$

so that from the first and the second equation we get

$$x_{n+2} = \frac{a_{n+1}^{(1)}}{c_{n+1}^{(1)}} x_{n-1}, \quad y_{n+1} = \frac{a_n^{(2)}}{c_n^{(2)}} y_{n-2}, \quad n \in \mathbb{N}, \tag{2.8}$$

$c_{n+1}^{(1)} \neq 0 \neq c_n^{(2)}$, from which it follows that

$$x_{3n+1} = x_1 \prod_{j=1}^{n} \frac{a_{3j}^{(1)}}{c_{3j}^{(1)}}, \quad x_{3n+2} = x_2 \prod_{j=1}^{n} \frac{a_{3j+1}^{(1)}}{c_{3j+1}^{(1)}}, \quad x_{3n} = x_0 \prod_{j=1}^{n} \frac{a_{3j-1}^{(1)}}{c_{3j-1}^{(1)}}, \quad n \in \mathbb{N},$$
$$y_{3n+1} = y_1 \prod_{j=1}^{n} \frac{a_{3j}^{(2)}}{c_{3j}^{(2)}}, \quad y_{3n-1} = y_{-1} \prod_{j=0}^{n-1} \frac{a_{3j+1}^{(2)}}{c_{3j+1}^{(2)}}, \quad y_{3n} = y_0 \prod_{j=1}^{n} \frac{a_{3j-1}^{(2)}}{c_{3j-1}^{(2)}}, \quad n \in \mathbb{N}. \tag{2.9}$$

3. Explicit Formulae for the Case $a_n^{(i)} \neq 0$ for All $i \in \{1,2,3\}$ and $n \in \mathbb{N}_0$

Here we consider system (1.1) in the case when $a_n^{(i)} \neq 0$ for all $i \in \{1,2,3\}$ and $n \in \mathbb{N}_0$. Noticing that in this case, system (1.1) can be written in the form

$$x_{n+1} = \frac{x_{n-2}}{\widehat{b}_n^{(1)} y_n z_{n-1} x_{n-2} + \widehat{c}_n^{(1)}}, \quad y_{n+1} = \frac{y_{n-2}}{\widehat{b}_n^{(2)} z_n x_{n-1} y_{n-2} + \widehat{c}_n^{(2)}},$$
$$z_{n+1} = \frac{z_{n-2}}{\widehat{b}_n^{(3)} x_n y_{n-1} z_{n-2} + \widehat{c}_n^{(3)}}, \quad n \in \mathbb{N}_0, \tag{3.1}$$

where $\widehat{b}_n^{(i)} = b_n^{(i)}/a_n^{(i)}$, $\widehat{c}_n^{(i)} = c_n^{(i)}/a_n^{(i)}$, $i \in \{1,2,3\}$, we see that we may assume that $a_n^{(i)} = 1$, for every $n \in \mathbb{N}_0$ and for each $i \in \{1,2,3\}$.

Hence we consider, without loss of generality, the system

$$x_{n+1} = \frac{x_{n-2}}{b_n^{(1)} y_n z_{n-1} x_{n-2} + c_n^{(1)}}, \quad y_{n+1} = \frac{y_{n-2}}{b_n^{(2)} z_n x_{n-1} y_{n-2} + c_n^{(2)}},$$
$$z_{n+1} = \frac{z_{n-2}}{b_n^{(3)} x_n y_{n-1} z_{n-2} + c_n^{(3)}}, \quad n \in \mathbb{N}_0 \tag{3.2}$$

using the same notation for coefficients as in (1.1) except for the coefficients $a_n^{(i)}$, assuming that $a_n^{(i)} = 1$ for all $i \in \{1,2,3\}$ and $n \in \mathbb{N}_0$.

First we consider the case when some of initial values of solutions of system (3.2) is equal to zero.

If $x_{n_0} = 0$ for some $n_0 \in \mathbb{N}$ then from (3.2) it follows that $x_{n_0-3k} = 0$, for each $k \in \mathbb{N}_0$ such that $n_0 - 3k \geq -2$. Hence, we have that $x_{-2} = 0$ or $x_{-1} = 0$ or $x_0 = 0$.

If $x_{-2} = 0$, then $x_{3n-2} = 0$, $n \in \mathbb{N}_0$, which implies

$$y_{3n+3} = \frac{1}{c_{3n+2}^{(2)}} y_{3n}, \quad z_{3n+2} = \frac{1}{c_{3n+1}^{(3)}} z_{3n-1}, \quad n \in \mathbb{N}_0, \tag{3.3}$$

and consequently

$$y_{3n} = \frac{y_0}{\prod_{j=0}^{n-1} c_{3j+2}^{(2)}}, \quad z_{3n+2} = \frac{z_{-1}}{\prod_{j=0}^{n} c_{3j+1}^{(3)}}, \quad n \in \mathbb{N}_0. \tag{3.4}$$

If $x_{-1} = 0$, then $x_{3n-1} = 0$, $n \in \mathbb{N}_0$, which implies

$$y_{3n+1} = \frac{1}{c^{(2)}_{3n}} y_{3n-2}, \qquad z_{3n+3} = \frac{1}{c^{(3)}_{3n+2}} z_{3n}, \qquad n \in \mathbb{N}_0, \tag{3.5}$$

and consequently

$$y_{3n+1} = \frac{y_{-2}}{\prod_{j=0}^{n} c^{(2)}_{3j}}, \qquad z_{3n} = \frac{z_0}{\prod_{j=0}^{n-1} c^{(3)}_{3j+2}}, \qquad n \in \mathbb{N}_0. \tag{3.6}$$

If $x_0 = 0$, then $x_{3n} = 0$, $n \in \mathbb{N}_0$, which implies

$$y_{3n+2} = \frac{1}{c^{(2)}_{3n+1}} y_{3n-1}, \qquad z_{3n+1} = \frac{1}{c^{(3)}_{3n}} z_{3n-2}, \qquad n \in \mathbb{N}_0, \tag{3.7}$$

and consequently

$$y_{3n+2} = \frac{y_{-1}}{\prod_{j=0}^{n} c^{(2)}_{3j+1}}, \qquad z_{3n+1} = \frac{z_{-2}}{\prod_{j=0}^{n} c^{(3)}_{3j}}, \qquad n \in \mathbb{N}_0. \tag{3.8}$$

If $y_{n_1} = 0$ or $z_{n_1} = 0$ for some $n_1 \in \mathbb{N}$ then similar results are obtained analogously.

3.1. Main Case

Here we study well-defined solutions of system (1.1) when neither of the sequences $(a^{(i)}_n)_{n\in\mathbb{N}_0}$, $i \in \{1,2,3\}$, or initial conditions x_{-i}, y_{-i}, z_{-i}, $i \in \{0,1,2\}$, is equal to zero. Recall that we may assume that $a^{(i)}_n = 1$, for every $n \in \mathbb{N}_0$ and for each $i \in \{1,2,3\}$.

Following the idea in [37], we use a transformation which reduces nonlinear systems (1.1) and (3.2) to third-order systems of nonhomogeneous linear difference equations.

If we multiply the first equation in system (3.2) by $y_n z_{n-1}$, the second by $z_n x_{n-1}$ and the third by $x_n y_{n-1}$, and then using in such obtained system the change of variables

$$u_{n+1} = \frac{1}{x_{n+1} y_n z_{n-1}}, \qquad v_{n+1} = \frac{1}{y_{n+1} z_n x_{n-1}}, \qquad w_{n+1} = \frac{1}{z_{n+1} x_n y_{n-1}}, \qquad n \geq -1, \tag{3.9}$$

the system is, for $n \in \mathbb{N}_0$, transformed into

$$\begin{aligned} u_{n+1} &= c^{(1)}_n v_n + b^{(1)}_n, \\ v_{n+1} &= c^{(2)}_n w_n + b^{(2)}_n, \\ w_{n+1} &= c^{(3)}_n u_n + b^{(3)}_n. \end{aligned} \tag{3.10}$$

System (3.10) implies that for $n \geq 2$

$$u_{n+1} = c_n^{(1)} c_{n-1}^{(2)} c_{n-2}^{(3)} u_{n-2} + c_n^{(1)} c_{n-1}^{(2)} b_{n-2}^{(3)} + c_n^{(1)} b_{n-1}^{(2)} + b_n^{(1)}, \tag{3.11}$$

$$v_{n+1} = c_{n-2}^{(1)} c_n^{(2)} c_{n-1}^{(3)} v_{n-2} + c_n^{(2)} c_{n-1}^{(3)} b_{n-2}^{(1)} + c_n^{(2)} b_{n-1}^{(3)} + b_n^{(2)}, \tag{3.12}$$

$$w_{n+1} = c_{n-1}^{(1)} c_{n-2}^{(2)} c_n^{(3)} w_{n-2} + c_{n-1}^{(1)} c_n^{(3)} b_{n-2}^{(2)} + c_n^{(3)} b_{n-1}^{(1)} + b_n^{(3)}, \tag{3.13}$$

where values for u_0, v_0, w_0 are computed by (3.9) with $n = -1$.

Equation (3.11) implies that the sequences $(u_{3n+i})_{n\in\mathbb{N}_0}$, $i \in \{0,1,2\}$, are solutions of the first-order linear difference equation

$$\begin{aligned} u_{3n+i} = {} & c_{3n+i-1}^{(1)} c_{3n+i-2}^{(2)} c_{3n+i-3}^{(3)} u_{3(n-1)+i} + c_{3n+i-1}^{(1)} c_{3n+i-2}^{(2)} b_{3n+i-3}^{(3)} \\ & + c_{3n+i-1}^{(1)} b_{3n+i-2}^{(2)} + b_{3n+i-1}^{(1)}, \quad n \in \mathbb{N}. \end{aligned} \tag{3.14}$$

Applying the well-known formula for solutions of first-order difference equation we have that the general solution of equation (3.14) is

$$\begin{aligned} u_{3n+i} = {} & u_i \prod_{j=1}^{n} \left(c_{3j+i-1}^{(1)} c_{3j+i-2}^{(2)} c_{3j+i-3}^{(3)} \right) \\ & + \sum_{l=1}^{n} \left(c_{3l+i-1}^{(1)} c_{3l+i-2}^{(2)} b_{3l+i-3}^{(3)} + c_{3l+i-1}^{(1)} b_{3l+i-2}^{(2)} + b_{3l+i-1}^{(1)} \right) \prod_{j=l+1}^{n} \left(c_{3j+i-1}^{(1)} c_{3j+i-2}^{(2)} c_{3j+i-3}^{(3)} \right) \end{aligned} \tag{3.15}$$

for every $n \in \mathbb{N}$ and each $i \in \{0,1,2\}$.

From (3.12) we get that the sequences $(v_{3n+i})_{n\in\mathbb{N}_0}$, $i \in \{0,1,2\}$, are solutions of the first-order linear difference equation

$$\begin{aligned} v_{3n+i} = {} & c_{3n+i-3}^{(1)} c_{3n+i-1}^{(2)} c_{3n+i-2}^{(3)} v_{3(n-1)+i} + c_{3n+i-1}^{(2)} c_{3n+i-2}^{(3)} b_{3n+i-3}^{(1)} \\ & + c_{3n+i-1}^{(2)} b_{3n+i-2}^{(3)} + b_{3n+i-1}^{(2)}, \quad n \in \mathbb{N}, \end{aligned} \tag{3.16}$$

from which it follows that

$$\begin{aligned} v_{3n+i} = {} & v_i \prod_{j=1}^{n} \left(c_{3j+i-3}^{(1)} c_{3j+i-1}^{(2)} c_{3j+i-2}^{(3)} \right) \\ & + \sum_{l=1}^{n} \left(c_{3l+i-1}^{(2)} c_{3l+i-2}^{(3)} b_{3l+i-3}^{(1)} + c_{3l+i-1}^{(2)} b_{3l+i-2}^{(3)} + b_{3l+i-1}^{(2)} \right) \prod_{j=l+1}^{n} \left(c_{3j+i-3}^{(1)} c_{3j+i-1}^{(2)} c_{3j+i-2}^{(3)} \right), \end{aligned} \tag{3.17}$$

for every $n \in \mathbb{N}$ and each $i \in \{0,1,2\}$.

From (3.13) we get that the sequences $(w_{3n+i})_{n\in\mathbb{N}_0}$, $i \in \{0,1,2\}$, are solutions of the first-order linear difference equation

$$\begin{aligned} w_{3n+i} &= c^{(1)}_{3n+i-2}c^{(2)}_{3n+i-3}c^{(3)}_{3n+i-1}w_{3(n-1)+i} \\ &\quad + c^{(1)}_{3n+i-2}c^{(3)}_{3n+i-1}b^{(2)}_{3n+i-3} + c^{(3)}_{3n+i-1}b^{(1)}_{3n+i-2} + b^{(3)}_{3n+i-1}, \quad n \in \mathbb{N}. \end{aligned} \tag{3.18}$$

Hence, for every $n \in \mathbb{N}$ and each $i \in \{0,1,2\}$, we have that

$$\begin{aligned} w_{3n+i} &= w_i \prod_{j=1}^{n}\left(c^{(1)}_{3j+i-2}c^{(2)}_{3j+i-3}c^{(3)}_{3j+i-1}\right) \\ &\quad + \sum_{l=1}^{n}\left(c^{(1)}_{3l+i-2}c^{(3)}_{3l+i-1}b^{(2)}_{3l+i-3} + c^{(3)}_{3l+i-1}b^{(1)}_{3l+i-2} + b^{(3)}_{3l+i-1}\right)\prod_{j=l+1}^{n}\left(c^{(1)}_{3j+i-2}c^{(2)}_{3j+i-3}c^{(3)}_{3j+i-1}\right). \end{aligned} \tag{3.19}$$

Now note that from (3.9) we have

$$\begin{aligned} x_{n+1} &= \frac{1}{u_{n+1}y_n z_{n-1}} = \frac{v_n}{u_{n+1}}x_{n-2}, \qquad y_{n+1} = \frac{1}{v_{n+1}z_n x_{n-1}} = \frac{w_n}{v_{n+1}}y_{n-2}, \\ z_{n+1} &= \frac{1}{w_{n+1}x_n y_{n-1}} = \frac{u_n}{w_{n+1}}z_{n-2}, \qquad n \in \mathbb{N}_0. \end{aligned} \tag{3.20}$$

Hence

$$x_{3n+i+1} = x_{i-2}\prod_{j=0}^{n}\frac{v_{3j+i}}{u_{3j+i+1}}, \qquad n \in \mathbb{N}_0,\ i \in \{0,1,2\}, \tag{3.21}$$

$$y_{3n+i+1} = y_{i-2}\prod_{j=0}^{n}\frac{w_{3j+i}}{v_{3j+i+1}}, \qquad n \in \mathbb{N}_0,\ i \in \{0,1,2\}, \tag{3.22}$$

$$z_{3n+i+1} = z_{i-2}\prod_{j=0}^{n}\frac{u_{3j+i}}{w_{3j+i+1}}, \qquad n \in \mathbb{N}_0,\ i \in \{0,1,2\}. \tag{3.23}$$

Applying (3.15), (3.17) and (3.19) in (3.21)–(3.23), we get explicit solutions of system (3.2) in terms of sequences $b_n^{(i)}$, $c_n^{(i)}$, $n \in \mathbb{N}_0$, $i \in \{0,1,2\}$.

The results in this section can be summed up in Table 1.

Remark 3.1. Formulae for the solutions of system (3.2) when some of the numbers $b_n^{(i)}$, $c_n^{(i)}$, $n \in \mathbb{N}_0$, $i \in \{0,1,2\}$ are zero follow from the formulae given in Table 1.

Table 1

Case	Formulas for well-defined solutions of system (3.2)
$x_{-2}=0$	$x_{3n-2}=0,\ y_{3n}=\dfrac{y_0}{\prod_{j=0}^{n-1}c_{3j+2}^{(2)}},\ z_{3n+2}=\dfrac{z_{-1}}{\prod_{j=0}^{n}c_{3j+1}^{(3)}},\ n\in\mathbb{N}_0$
$x_{-1}=0$	$x_{3n-1}=0,\ y_{3n+1}=\dfrac{y_{-2}}{\prod_{j=0}^{n}c_{3j}^{(2)}},\ z_{3n}=\dfrac{z_0}{\prod_{j=0}^{n-1}c_{3j+2}^{(3)}},\ n\in\mathbb{N}_0$
$x_0=0$	$x_{3n}=0,\ y_{3n+2}=\dfrac{y_{-1}}{\prod_{j=0}^{n}c_{3j+1}^{(2)}},\ z_{3n+1}=\dfrac{z_{-2}}{\prod_{j=0}^{n}c_{3j}^{(3)}},\ n\in\mathbb{N}_0$
$y_{-2}=0$	$y_{3n-2}=0,\ z_{3n}=\dfrac{z_0}{\prod_{j=0}^{n-1}c_{3j+2}^{(3)}},\ x_{3n+2}=\dfrac{x_{-1}}{\prod_{j=0}^{n}c_{3j+1}^{(1)}},\ n\in\mathbb{N}_0$
$y_{-1}=0$	$y_{3n-1}=0,\ z_{3n+1}=\dfrac{z_{-2}}{\prod_{j=0}^{n}c_{3j}^{(3)}},x_{3n}=\dfrac{x_0}{\prod_{j=0}^{n-1}c_{3j+2}^{(1)}},\ n\in\mathbb{N}_0$
$y_0=0$	$y_{3n}=0,\ z_{3n+2}=\dfrac{z_{-1}}{\prod_{j=0}^{n}c_{3j+1}^{(3)}},\ x_{3n+1}=\dfrac{x_{-2}}{\prod_{j=0}^{n}c_{3j}^{(1)}},\ n\in\mathbb{N}_0$
$z_{-2}=0$	$z_{3n-2}=0,x_{3n}=\dfrac{x_0}{\prod_{j=0}^{n-1}c_{3j+2}^{(1)}},\ y_{3n+2}=\dfrac{y_{-1}}{\prod_{j=0}^{n}c_{3j+1}^{(2)}},\ n\in\mathbb{N}_0$
$z_{-1}=0$	$z_{3n-1}=0,\ x_{3n+1}=\dfrac{x_{-2}}{\prod_{j=0}^{n}c_{3j}^{(1)}},\ y_{3n}=\dfrac{y_0}{\prod_{j=0}^{n-1}c_{3j+2}^{(2)}},\ n\in\mathbb{N}_0$
$z_0=0$	$z_{3n}=0,\ x_{3n+2}=\dfrac{x_{-1}}{\prod_{j=0}^{n}c_{3j+1}^{(1)}},\ y_{3n+1}=\dfrac{y_{-2}}{\prod_{j=0}^{n}c_{3j}^{(2)}},\ n\in\mathbb{N}_0$
$x_{-i}\neq 0\neq y_{-i}$ $z_{-i}\neq 0,$ $i\in\{0,1,2\}$	$x_{3n+i+1}=x_{i-2}\prod_{j=0}^{n}\dfrac{v_{3j+i}}{u_{3j+i+1}},\ y_{3n+i+1}=y_{i-2}\prod_{j=0}^{n}\dfrac{w_{3j+i}}{v_{3j+i+1}},\ z_{3n+i+1}=z_{i-2}\prod_{j=0}^{n}\dfrac{u_{3j+i}}{w_{3j+i+1}},$ $n\in\mathbb{N}_0,\ i\in\{0,1,2\}$ $u_{3n+i}=u_i\prod_{j=1}^{n}(c_{3j+i-1}^{(1)}c_{3j+i-2}^{(2)}c_{3j+i-3}^{(3)})$ $+\sum_{l=1}^{n}(c_{3l+i-1}^{(1)}c_{3l+i-2}^{(2)}b_{3l+i-3}^{(3)}+c_{3l+i-1}^{(1)}b_{3l+i-2}^{(2)}+b_{3l+i-1}^{(1)})\prod_{j=l+1}^{n}(c_{3j+i-1}^{(1)}c_{3j+i-2}^{(2)}c_{3j+i-3}^{(3)}),$ $v_{3n+i}=v_i\prod_{j=1}^{n}(c_{3j+i-3}^{(1)}c_{3j+i-1}^{(2)}c_{3j+i-2}^{(3)})$ $+\sum_{l=1}^{n}(c_{3l+i-1}^{(2)}c_{3l+i-2}^{(3)}b_{3l+i-3}^{(1)}+c_{3l+i-1}^{(2)}b_{3l+i-2}^{(3)}+b_{3l+i-1}^{(2)})\prod_{j=l+1}^{n}(c_{3j+i-3}^{(1)}c_{3j+i-1}^{(2)}c_{3j+i-2}^{(3)}),$ $w_{3n+i}=w_i\prod_{j=1}^{n}(c_{3j+i-2}^{(1)}c_{3j+i-3}^{(2)}c_{3j+i-1}^{(3)})$ $+\sum_{l=1}^{n}(c_{3l+i-2}^{(1)}c_{3l+i-1}^{(3)}b_{3l+i-3}^{(2)}+c_{3l+i-1}^{(3)}b_{3l+i-2}^{(1)}+b_{3l+i-1}^{(3)})\prod_{j=l+1}^{n}(c_{3j+i-2}^{(1)}c_{3j+i-3}^{(2)}c_{3j+i-1}^{(3)})$

4. Some Consequences

4.1. Case $a_n^{(i)}\neq 0$ for $i\in\{1,2,3\}$ and $n\in\mathbb{N}_0$

First we use the formulae in Section 3 to get solutions of system (1.1), when $a_n^{(i)}\neq 0$ for $i\in\{1,2,3\}$ and $n\in\mathbb{N}_0$.

Table 2

Case	Formulas for well-defined solutions of system (1.1)
$x_{-2}=0$	$x_{3n-2}=0,\ y_{3n}=y_0\prod_{j=0}^{n-1}\left(\frac{a_{3j+2}^{(2)}}{c_{3j+2}^{(2)}}\right),\ z_{3n+2}=z_{-1}\prod_{j=0}^{n}\left(\frac{a_{3j+1}^{(3)}}{c_{3j+1}^{(3)}}\right),\ n\in\mathbb{N}_0$
$x_{-1}=0$	$x_{3n-1}=0,\ y_{3n+1}=y_{-2}\prod_{j=0}^{n}\left(\frac{a_{3j}^{(2)}}{c_{3j}^{(2)}}\right),\ z_{3n}=z_0\prod_{j=0}^{n-1}\left(\frac{a_{3j+2}^{(3)}}{c_{3j+2}^{(3)}}\right),\ n\in\mathbb{N}_0$
$x_0=0$	$x_{3n}=0,\ y_{3n+2}=y_{-1}\prod_{j=0}^{n}\left(\frac{a_{3j+1}^{(2)}}{c_{3j+1}^{(2)}}\right),\ z_{3n+1}=z_{-2}\prod_{j=0}^{n}\left(\frac{a_{3j}^{(3)}}{c_{3j}^{(3)}}\right),\ n\in\mathbb{N}_0$
$y_{-2}=0$	$y_{3n-2}=0,\ z_{3n}=z_0\prod_{j=0}^{n-1}\left(\frac{a_{3j+2}^{(3)}}{c_{3j+2}^{(3)}}\right),\ x_{3n+2}=x_{-1}\prod_{j=0}^{n}\left(\frac{a_{3j+1}^{(1)}}{c_{3j+1}^{(1)}}\right),\ n\in\mathbb{N}_0$
$y_{-1}=0$	$y_{3n-1}=0, z_{3n+1}=z_{-2}\prod_{j=0}^{n}\left(\frac{a_{3j}^{(3)}}{c_{3j}^{(3)}}\right),\ x_{3n}=x_0\prod_{j=0}^{n-1}\left(\frac{a_{3j+2}^{(1)}}{c_{3j+2}^{(1)}}\right),\ n\in\mathbb{N}_0$
$y_0=0$	$y_{3n}=0,\ z_{3n+2}=z_{-1}\prod_{j=0}^{n}\left(\frac{a_{3j+1}^{(3)}}{c_{3j+1}^{(3)}}\right),\ x_{3n+1}=x_{-2}\prod_{j=0}^{n}\left(\frac{a_{3j}^{(1)}}{c_{3j}^{(1)}}\right),\ n\in\mathbb{N}_0$
$z_{-2}=0$	$z_{3n-2}=0,\ x_{3n}=x_0\prod_{j=0}^{n-1}\left(\frac{a_{3j+2}^{(1)}}{c_{3j+2}^{(1)}}\right),\ y_{3n+2}=y_{-1}\prod_{j=0}^{n}\left(\frac{a_{3j+1}^{(2)}}{c_{3j+1}^{(2)}}\right),\ n\in\mathbb{N}_0$
$z_{-1}=0$	$z_{3n-1}=0,\ x_{3n+1}=x_{-2}\prod_{j=0}^{n}\left(\frac{a_{3j}^{(1)}}{c_{3j}^{(1)}}\right),\ y_{3n}=y_0\prod_{j=0}^{n-1}\left(\frac{a_{3j+2}^{(2)}}{c_{3j+2}^{(2)}}\right),\ n\in\mathbb{N}_0$
$z_0=0$	$z_{3n}=0,\ x_{3n+2}=x_{-1}\prod_{j=0}^{n}\left(\frac{a_{3j+1}^{(1)}}{c_{3j+1}^{(1)}}\right),\ y_{3n+1}=y_{-2}\prod_{j=0}^{n}\left(\frac{a_{3j}^{(2)}}{c_{3j}^{(2)}}\right),\ n\in\mathbb{N}_0$
$x_{-i}\neq 0\neq y_{-i}$ $z_{-i}\neq 0,$ $i\in\{0,1,2\}$	$x_{3n+i+1}=x_{i-2}\prod_{j=0}^{n}\frac{v_{3j+i}}{u_{3j+i+1}},\ y_{3n+i+1}=y_{i-2}\prod_{j=0}^{n}\frac{w_{3j+i}}{v_{3j+i+1}},$ $z_{3n+i+1}=z_{i-2}\prod_{j=0}^{n}\frac{u_{3j+i}}{w_{3j+i+1}},\ n\in\mathbb{N}_0,\ i\in\{0,1,2\}$ $u_{3n+i}=u_i\prod_{j=1}^{n}\frac{c_{3j+i-1}^{(1)}c_{3j+i-2}^{(2)}c_{3j+i-3}^{(3)}}{a_{3j+i-1}^{(1)}a_{3j+i-2}^{(2)}a_{3j+i-3}^{(3)}}$ $+\sum_{l=1}^{n}\left(\frac{c_{3l+i-1}^{(1)}c_{3l+i-2}^{(2)}b_{3l+i-3}^{(3)}}{a_{3l+i-1}^{(1)}a_{3l+i-2}^{(2)}a_{3l+i-3}^{(3)}}+\frac{c_{3l+i-1}^{(1)}b_{3l+i-2}^{(2)}}{a_{3l+i-1}^{(1)}a_{3l+i-2}^{(2)}}+\frac{b_{3l+i-1}^{(1)}}{a_{3l+i-1}^{(1)}}\right)\prod_{j=l+1}^{n}\frac{c_{3j+i-1}^{(1)}c_{3j+i-2}^{(2)}c_{3j+i-3}^{(3)}}{a_{3j+i-1}^{(1)}a_{3j+i-2}^{(2)}a_{3j+i-3}^{(3)}},$ $v_{3n+i}=v_i\prod_{j=1}^{n}\frac{c_{3j+i-3}^{(1)}c_{3j+i-1}^{(2)}c_{3j+i-2}^{(3)}}{a_{3j+i-3}^{(1)}a_{3j+i-1}^{(2)}a_{3j+i-2}^{(3)}}$ $+\sum_{l=1}^{n}\left(\frac{c_{3l+i-1}^{(2)}c_{3l+i-2}^{(3)}b_{3l+i-3}^{(1)}}{a_{3l+i-1}^{(2)}a_{3l+i-2}^{(3)}a_{3l+i-3}^{(1)}}+\frac{c_{3l+i-1}^{(2)}b_{3l+i-2}^{(3)}}{a_{3l+i-1}^{(2)}a_{3l+i-2}^{(3)}}+\frac{b_{3l+i-1}^{(2)}}{a_{3l+i-1}^{(2)}}\right)\prod_{j=l+1}^{n}\frac{c_{3j+i-3}^{(1)}c_{3j+i-1}^{(2)}c_{3j+i-2}^{(3)}}{a_{3j+i-3}^{(1)}a_{3j+i-1}^{(2)}a_{3j+i-2}^{(3)}},$

Table 2: Continued.

Case	Formulas for well-defined solutions of system (1.1)
	$w_{3n+i} = w_i \prod_{j=1}^{n} \frac{c^{(1)}_{3j+i-2} c^{(2)}_{3j+i-3} c^{(3)}_{3j+i-1}}{a^{(1)}_{3j+i-2} a^{(2)}_{3j+i-3} a^{(3)}_{3j+i-1}} + \sum_{l=1}^{n} \left(\frac{c^{(1)}_{3l+i-2} c^{(3)}_{3l+i-1} b^{(2)}_{3l+i-3}}{a^{(1)}_{3l+i-2} a^{(3)}_{3l+i-1} a^{(2)}_{3l+i-3}} + \frac{c^{(3)}_{3l+i-1} b^{(1)}_{3l+i-2}}{a^{(3)}_{3l+i-1} a^{(1)}_{3l+i-2}} + \frac{b^{(3)}_{3l+i-1}}{a^{(3)}_{3l+i-1}} \right) \prod_{j=l+1}^{n} \frac{c^{(1)}_{3j+i-2} c^{(2)}_{3j+i-3} c^{(3)}_{3j+i-1}}{a^{(1)}_{3j+i-2} a^{(2)}_{3j+i-3} a^{(3)}_{3j+i-1}}.$

For this we replace sequences

$$b_n^{(i)}, \quad c_n^{(i)}, \quad i \in \{1,2,3\}, \tag{4.1}$$

in formulas of Section 3 with sequences

$$\frac{b_n^{(i)}}{a_n^{(i)}}, \quad \frac{c_n^{(i)}}{a_n^{(i)}}, \quad i \in \{1,2,3\}. \tag{4.2}$$

We arrange these formulae in Table 2.

4.2. Case $b_n^{(i)}$, $c_n^{(i)}$, $n \in \mathbb{N}_0$, $i \in \{1,2,3\}$ Are Period-Three Sequences

Now we get formulae for solutions of system (3.2) when the sequences $b_n^{(i)}$, $c_n^{(i)}$, $n \in \mathbb{N}_0$, $i \in \{1,2,3\}$ are periodic with period three.

If this holds then from (3.15) we have that

$$\begin{aligned} u_{3n+i} &= u_i \prod_{j=1}^{n} \left(c^{(1)}_{i+2} c^{(2)}_{i+1} c^{(3)}_{i} \right) + \sum_{l=1}^{n} \left(c^{(1)}_{i+2} c^{(2)}_{i+1} b^{(3)}_{i} + c^{(1)}_{i+2} b^{(2)}_{i+1} + b^{(1)}_{i+2} \right) \prod_{j=l+1}^{n} \left(c^{(1)}_{i+2} c^{(2)}_{i+1} c^{(3)}_{i} \right) \\ &= u_i \left(c^{(1)}_{i+2} c^{(2)}_{i+1} c^{(3)}_{i} \right)^n + \left(c^{(1)}_{i+2} c^{(2)}_{i+1} b^{(3)}_{i} + c^{(1)}_{i+2} b^{(2)}_{i+1} + b^{(1)}_{i+2} \right) \sum_{l=1}^{n} \left(c^{(1)}_{i+2} c^{(2)}_{i+1} c^{(3)}_{i} \right)^{n-l} \\ &= u_i \left(c^{(1)}_{i+2} c^{(2)}_{i+1} c^{(3)}_{i} \right)^n + \left(c^{(1)}_{i+2} c^{(2)}_{i+1} b^{(3)}_{i} + c^{(1)}_{i+2} b^{(2)}_{i+1} + b^{(1)}_{i+2} \right) \frac{1 - \left(c^{(1)}_{i+2} c^{(2)}_{i+1} c^{(3)}_{i} \right)^n}{1 - c^{(1)}_{i+2} c^{(2)}_{i+1} c^{(3)}_{i}} \end{aligned} \tag{4.3}$$

for every $n \in \mathbb{N}_0$, when $c^{(1)}_{i+2} c^{(2)}_{i+1} c^{(3)}_{i} \neq 1$, for some $i \in \{0,1,2\}$, while

$$u_{3n+i} = u_i + \left(c^{(1)}_{i+2} c^{(2)}_{i+1} b^{(3)}_{i} + c^{(1)}_{i+2} b^{(2)}_{i+1} + b^{(1)}_{i+2} \right) n, \tag{4.4}$$

when $c^{(1)}_{i+2} c^{(2)}_{i+1} c^{(3)}_{i} = 1$, for some $i \in \{0,1,2\}$. Here we regard that $c_k^{(j)} = c_i^{(j)}$, for some $j \in \{1,2,3\}$, $i \in \{0,1,2\}$ and $k \geq 3$, when $k \equiv i \pmod 3$.

Similarly, we get

$$v_{3n+i} = v_i\left(c_i^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}\right)^n + \left(c_{i+2}^{(2)}c_{i+1}^{(3)}b_i^{(1)} + c_{i+2}^{(2)}b_{i+1}^{(3)} + b_{i+2}^{(2)}\right)\frac{1-\left(c_i^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}\right)^n}{1-c_i^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}} \tag{4.5}$$

for every $n \in \mathbb{N}_0$, if $c_i^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)} \neq 1$, for some $i \in \{0,1,2\}$, and

$$v_{3n+i} = v_i + \left(c_{i+2}^{(2)}c_{i+1}^{(3)}b_i^{(1)} + c_{i+2}^{(2)}b_{i+1}^{(3)} + b_{i+2}^{(2)}\right)n \tag{4.6}$$

for every $n \in \mathbb{N}_0$, when $c_i^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)} = 1$, for some $i \in \{0,1,2\}$.

Finally

$$w_{3n+i} = w_i\left(c_{i+1}^{(1)}c_i^{(2)}c_{i+2}^{(3)}\right)^n + \left(c_{i+1}^{(1)}c_{i+2}^{(3)}b_i^{(2)} + c_{i+2}^{(3)}b_{i+1}^{(1)} + b_{i+2}^{(3)}\right)\frac{1-\left(c_{i+1}^{(1)}c_i^{(2)}c_{i+2}^{(3)}\right)^n}{1-c_{i+1}^{(1)}c_i^{(2)}c_{i+2}^{(3)}}, \tag{4.7}$$

for every $n \in \mathbb{N}_0$, when $c_{i+1}^{(1)}c_i^{(2)}c_{i+2}^{(3)} \neq 1$, for some $i \in \{0,1,2\}$, and

$$w_{3n+i} = w_i + \left(c_{i+1}^{(1)}c_{i+2}^{(3)}b_i^{(2)} + c_{i+2}^{(3)}b_{i+1}^{(1)} + b_{i+2}^{(3)}\right)n, \tag{4.8}$$

for every $n \in \mathbb{N}_0$, when $c_{i+1}^{(1)}c_i^{(2)}c_{i+2}^{(3)} = 1$, for some $i \in \{0,1,2\}$.

If we assume that $a_n^{(i)} \neq 0$ for all $i \in \{1,2,3\}$ and $n \in \mathbb{N}_0$, that these three sequences are also periodic with period three, and replace the sequences in (4.1) with the corresponding in (4.2) we get formulae for solutions of system (1.1) when the sequences $a_n^{(i)}$, $b_n^{(i)}$, and $c_n^{(i)}$, $n \in \mathbb{N}_0$, $i \in \{1,2,3\}$ are periodic with period three.

These formulae follows from above obtained ones and are summarized in Table 3.

5. Some Applications

Using above listed formulae the behavior of solutions of system (1.1) or (3.2) can be described. We will present here some results which can be obtained from these formulae, to demonstrate how they can be used. Before we formulate the results note that if $c_n^{(j)}$, $n \in \mathbb{N}_0$, $j \in \{1,2,3\}$ are periodic with period three, then

$$c_i^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)} = c_{i+3}^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}, \qquad c_{i+1}^{(1)}c_{i+3}^{(2)}c_{i+2}^{(3)} = c_{i+1}^{(1)}c_i^{(2)}c_{i+2}^{(3)}, \tag{5.1}$$

$$c_{i+2}^{(1)}c_{i+1}^{(2)}c_i^{(3)} = c_{i+2}^{(1)}c_{i+1}^{(2)}c_{i+3}^{(3)}. \tag{5.2}$$

Table 3

Case	Formulas for well-defined solutions of system (1.1) with period-three coefficients
$x_{-i} \neq 0 \neq y_{-i}$ $z_{-i} \neq 0,\ i \in \{0,1,2\}$	$x_{3n+i+1} = x_{i-2}\prod_{j=0}^{n}\frac{v_{3j+i}}{u_{3j+i+1}},\ y_{3n+i+1} = y_{i-2}\prod_{j=0}^{n}\frac{w_{3j+i}}{v_{3j+i+1}},$ $z_{3n+i+1} = z_{i-2}\prod_{j=0}^{n}\frac{u_{3j+i}}{w_{3j+i+1}},\ n \in \mathbb{N}_0,\ i \in \{0,1,2\}$
$q_1 = \frac{c^{(1)}_{i+2}c^{(2)}_{i+1}c^{(3)}_{i}}{a^{(1)}_{i+2}a^{(2)}_{i+1}a^{(3)}_{i}} \neq 1$	$u_{3n+i} = u_i q_1^n + \left(\frac{c^{(1)}_{i+2}c^{(2)}_{i+1}b^{(3)}_{i}}{a^{(1)}_{i+2}a^{(2)}_{i+1}a^{(3)}_{i}} + \frac{c^{(1)}_{i+2}b^{(2)}_{i+1}}{a^{(1)}_{i+2}a^{(2)}_{i+1}} + \frac{b^{(1)}_{i+2}}{a^{(1)}_{i+2}}\right)\frac{1-q_1^n}{1-q_1},\ n \in \mathbb{N}_0,\ i \in \{0,1,2\}$
$q_1 = \frac{c^{(1)}_{i+2}c^{(2)}_{i+1}c^{(3)}_{i}}{a^{(1)}_{i+2}a^{(2)}_{i+1}a^{(3)}_{i}} = 1$	$u_{3n+i} = u_i + \left(\frac{c^{(1)}_{i+2}c^{(2)}_{i+1}b^{(3)}_{i}}{a^{(1)}_{i+2}a^{(2)}_{i+1}a^{(3)}_{i}} + \frac{c^{(1)}_{i+2}b^{(2)}_{i+1}}{a^{(1)}_{i+2}a^{(2)}_{i+1}} + \frac{b^{(1)}_{i+2}}{a^{(1)}_{i+2}}\right)n,\ n \in \mathbb{N}_0,\ i \in \{0,1,2\}$
$q_2 = \frac{c^{(1)}_{i}c^{(2)}_{i+2}c^{(3)}_{i+1}}{a^{(1)}_{i}a^{(2)}_{i+2}a^{(3)}_{i+1}} \neq 1$	$v_{3n+i} = v_i q_2^n + \left(\frac{c^{(2)}_{i+2}c^{(3)}_{i+1}b^{(1)}_{i}}{a^{(2)}_{i+2}a^{(3)}_{i+1}a^{(1)}_{i}} + \frac{c^{(2)}_{i+2}b^{(3)}_{i+1}}{a^{(2)}_{i+2}a^{(3)}_{i+1}} + \frac{b^{(2)}_{i+2}}{a^{(2)}_{i+2}}\right)\frac{1-q_2^n}{1-q_2},\ n \in \mathbb{N}_0,\ i \in \{0,1,2\}$
$q_2 = \frac{c^{(1)}_{i}c^{(2)}_{i+2}c^{(3)}_{i+1}}{a^{(1)}_{i}a^{(2)}_{i+2}a^{(3)}_{i+1}} = 1$	$v_{3n+i} = v_i + \left(\frac{c^{(2)}_{i+2}c^{(3)}_{i+1}b^{(1)}_{i}}{a^{(2)}_{i+2}a^{(3)}_{i+1}a^{(1)}_{i}} + \frac{c^{(2)}_{i+2}b^{(3)}_{i+1}}{a^{(2)}_{i+2}a^{(3)}_{i+1}} + \frac{b^{(2)}_{i+2}}{a^{(2)}_{i+2}}\right)n,\ n \in \mathbb{N}_0,\ i \in \{0,1,2\}$
$q_3 = \frac{c^{(1)}_{i+1}c^{(2)}_{i}c^{(3)}_{i+2}}{a^{(1)}_{i+1}a^{(2)}_{i}a^{(3)}_{i+2}} \neq 1$	$w_{3n+i} = w_i q_3^n + \left(\frac{c^{(1)}_{i+1}c^{(3)}_{i+2}b^{(2)}_{i}}{a^{(1)}_{i+1}a^{(3)}_{i+2}a^{(2)}_{i}} + \frac{c^{(3)}_{i+2}b^{(1)}_{i+1}}{a^{(3)}_{i+2}a^{(1)}_{i+1}} + \frac{b^{(3)}_{i+2}}{a^{(3)}_{i+2}}\right)\frac{1-q_3^n}{1-q_3},\ n \in \mathbb{N}_0,\ i \in \{0,1,2\}$
$q_3 = \frac{c^{(1)}_{i+1}c^{(2)}_{i}c^{(3)}_{i+2}}{a^{(1)}_{i+1}a^{(2)}_{i}a^{(3)}_{i+2}} = 1$	$w_{3n+i} = w_i + \left(\frac{c^{(1)}_{i+1}c^{(3)}_{i+2}b^{(2)}_{i}}{a^{(1)}_{i+1}a^{(3)}_{i+2}a^{(2)}_{i}} + \frac{c^{(3)}_{i+2}b^{(1)}_{i+1}}{a^{(3)}_{i+2}a^{(1)}_{i+1}} + \frac{b^{(3)}_{i+2}}{a^{(3)}_{i+2}}\right)n,\ n \in \mathbb{N}_0,\ i \in \{0,1,2\}$

Theorem 5.1. *Consider system* (3.2). *Let the sequences* $b_n^{(j)}$, $c_n^{(j)}$, $n \in \mathbb{N}_0$, $j \in \{1,2,3\}$ *be periodic with period three,*

$$p_i := \frac{c^{(2)}_{i+2}c^{(3)}_{i+1}b^{(1)}_{i} + c^{(2)}_{i+2}b^{(3)}_{i+1} + b^{(2)}_{i+2}}{c^{(1)}_{i+3}c^{(2)}_{i+2}b^{(3)}_{i+1} + c^{(1)}_{i+3}b^{(2)}_{i+2} + b^{(1)}_{i+3}},$$
$$q_i := \frac{v_i - u_{i+1}}{c^{(1)}_{i+3}c^{(2)}_{i+2}b^{(3)}_{i+1} + c^{(1)}_{i+3}b^{(2)}_{i+2} + b^{(1)}_{i+3}}, \tag{5.3}$$
$$r_i := \frac{v_i + u_{i+1}}{c^{(1)}_{i+3}c^{(2)}_{i+2}b^{(3)}_{i+1} + c^{(1)}_{i+3}b^{(2)}_{i+2} + b^{(1)}_{i+3}},$$

when $c^{(1)}_{i+3}c^{(2)}_{i+2}b^{(3)}_{i+1} + c^{(1)}_{i+3}b^{(2)}_{i+2} + b^{(1)}_{i+3} \neq 0$, *and for some* $i \in \{0,1,2\}$ *the following condition holds:*

$$c^{(1)}_{i}c^{(2)}_{i+2}c^{(3)}_{i+1} = 1. \tag{5.4}$$

Then if $x_{3n+i+1} \neq 0$ *for every* $n \in \mathbb{N}_0$, *the following statements hold:*

(a) *if* $|p_i| < 1$, *then* $x_{3n+i+1} \to 0$, *as* $n \to \infty$;

(b) *if* $|p_i| > 1$, *then* $|x_{3n+i+1}| \to \infty$, *as* $n \to \infty$;

(c) *if* $p_i = 1$ *and* $q_i < 0$, *then* $x_{3n+i+1} \to 0$, *as* $n \to \infty$;

(d) *if* $p_i = 1$ *and* $q_i > 0$, *then* $|x_{3n+i+1}| \to \infty$, *as* $n \to \infty$;

(e) *if* $p_i = 1$ *and* $q_i = 0$, *then* x_{3n+i+1} *is convergent*;

(f) *if* $p_i = -1$ *and* $r_i > 0$, *then* $x_{3n+i+1} \to 0$, *as* $n \to \infty$;

(g) *if* $p_i = -1$ *and* $r_i < 0$, *then* $|x_{3n+i+1}| \to \infty$, *as* $n \to \infty$, *so that* $x_{6n+i+1} \to +\infty$ *and* $x_{6n+i+4} \to -\infty$, *or* $x_{6n+i+1} \to -\infty$ *and* $x_{6n+i+4} \to +\infty$ *as* $n \to \infty$;

(h) *If* $p_i = -1$ *and* $r_i = 0$, *then* x_{6n+i+1} *and* x_{6n+i+4} *are convergent*, *as* $n \to \infty$.

If $c^{(1)}_{i+3}c^{(2)}_{i+2}b^{(3)}_{i+1} + c^{(1)}_{i+3}b^{(2)}_{i+2} + b^{(1)}_{i+3} = 0$, *then the following statements hold:*

(i) *if* $c^{(2)}_{i+2}c^{(3)}_{i+1}b^{(1)}_{i} + c^{(2)}_{i+2}b^{(3)}_{i+1} + b^{(2)}_{i+2} \neq 0$, *then* $|x_{3n+i+1}| \to \infty$, *as* $n \to \infty$;

(j) *if* $c^{(2)}_{i+2}c^{(3)}_{i+1}b^{(1)}_{i} + c^{(2)}_{i+2}b^{(3)}_{i+1} + b^{(2)}_{i+2} = 0$, *then*

$$x_{3n+i+1} = x_{i-2}\left(\frac{v_i}{u_{i+1}}\right)^{n+1}. \tag{5.5}$$

Proof. (a), (b) From (3.21), (5.4) and (5.1), we have that for each $i \in \{0,1,2\}$

$$\begin{aligned} x_{3n+i+1} &= x_{i-2}\prod_{j=0}^{n}\frac{v_{3j+i}}{u_{3j+i+1}} \\ &= x_{i-2}\prod_{j=0}^{n}\frac{v_i + \left(c^{(2)}_{i+2}c^{(3)}_{i+1}b^{(1)}_{i} + c^{(2)}_{i+2}b^{(3)}_{i+1} + b^{(2)}_{i+2}\right)j}{u_{i+1} + \left(c^{(1)}_{i+3}c^{(2)}_{i+2}b^{(3)}_{i+1} + c^{(1)}_{i+3}b^{(2)}_{i+2} + b^{(1)}_{i+3}\right)j}, \end{aligned} \tag{5.6}$$

for every $n \in \mathbb{N}_0$.

Since

$$\lim_{n\to\infty}\frac{v_i + \left(c^{(2)}_{i+2}c^{(3)}_{i+1}b^{(1)}_{i} + c^{(2)}_{i+2}b^{(3)}_{i+1} + b^{(2)}_{i+2}\right)n}{u_{i+1} + \left(c^{(1)}_{i+3}c^{(2)}_{i+2}b^{(3)}_{i+1} + c^{(1)}_{i+3}b^{(2)}_{i+2} + b^{(1)}_{i+3}\right)n} = p_i, \tag{5.7}$$

the results in (a) and (b) follow from (5.6) easily.

(c), (d), (e) Since $p_i = 1$, we have that

$$\begin{aligned} \frac{v_{3n+i}}{u_{3n+i+1}} &= \left(1 + \frac{v_i}{\left(c^{(1)}_{i+3}c^{(2)}_{i+2}b^{(3)}_{i+1} + c^{(1)}_{i+3}b^{(2)}_{i+2} + b^{(1)}_{i+3}\right)n}\right)\left(1 + \frac{u_{i+1}}{\left(c^{(1)}_{i+3}c^{(2)}_{i+2}b^{(3)}_{i+1} + c^{(1)}_{i+3}b^{(2)}_{i+2} + b^{(1)}_{i+3}\right)n}\right)^{-1} \\ &= 1 + \frac{q_i}{n} + O\left(\frac{1}{n^2}\right) = \exp\left(\frac{q_i}{n} + O\left(\frac{1}{n^2}\right)\right), \quad n \longrightarrow \infty. \end{aligned} \tag{5.8}$$

From (5.8) and by using well-known asymptotic relation

$$\sum_{j=n_0}^{n}\frac{1}{j}\sim \ln n, \tag{5.9}$$

where we assume $n_0 \geq 1$ and $n \to \infty$; the results in (c), (d), and (e) easily follow.

(f), (g), (h) Since $p_i = -1$, we have that

$$\begin{aligned}\frac{v_{3n+i}}{u_{3n+i+1}} &= -\left(1-\frac{v_i}{\left(c_{i+3}^{(1)}c_{i+2}^{(2)}b_{i+1}^{(3)}+c_{i+3}^{(1)}b_{i+2}^{(2)}+b_{i+3}^{(1)}\right)n}\right)\left(1+\frac{u_{i+1}}{\left(c_{i+3}^{(1)}c_{i+2}^{(2)}b_{i+1}^{(3)}+c_{i+3}^{(1)}b_{i+2}^{(2)}+b_{i+3}^{(1)}\right)n}\right)^{-1}\\ &= -\left(1-\frac{r_i}{n}+O\left(\frac{1}{n^2}\right)\right) = -\exp\left(-\frac{r_i}{n}+O\left(\frac{1}{n^2}\right)\right), \quad n\longrightarrow\infty.\end{aligned} \tag{5.10}$$

From (5.6), (5.10) and asymptotic relation (5.9) the results in (f), (g), and (h) follow.

(i), (j) These two statements follow easily from (5.6). □

Remark 5.2. If $x_{3n_0+i+1} = 0$ for an $n_0 \in \mathbb{N}_0$, then by (3.2) we get $x_{3n+i+1} = 0$, for $n \geq n_0$, which is the reason why we posed the condition $x_{3n+i+1} \neq 0$, for every $n \in \mathbb{N}_0$, in Theorem 5.1. Similar conditions will be posed in the theorems which follow.

Theorem 5.3. *Consider system* (3.2). *Let the sequences* $b_n^{(j)}$, $c_n^{(j)}$, $n \in \mathbb{N}_0$, $j \in \{1,2,3\}$ *be periodic with period three,*

$$\begin{aligned}\widehat{p}_i &:= \frac{c_{i+1}^{(1)}c_{i+2}^{(3)}b_i^{(2)}+c_{i+2}^{(3)}b_{i+1}^{(1)}+b_{i+2}^{(3)}}{c_{i+3}^{(2)}c_{i+2}^{(3)}b_{i+1}^{(1)}+c_{i+3}^{(2)}b_{i+2}^{(3)}+b_{i+3}^{(2)}},\\ \widehat{q}_i &:= \frac{w_i-v_{i+1}}{c_{i+3}^{(2)}c_{i+2}^{(3)}b_{i+1}^{(1)}+c_{i+3}^{(2)}b_{i+2}^{(3)}+b_{i+3}^{(2)}},\\ \widehat{r}_i &:= \frac{w_i+v_{i+1}}{c_{i+3}^{(2)}c_{i+2}^{(3)}b_{i+1}^{(1)}+c_{i+3}^{(2)}b_{i+2}^{(3)}+b_{i+3}^{(2)}},\end{aligned} \tag{5.11}$$

when $c_{i+3}^{(2)}c_{i+2}^{(3)}b_{i+1}^{(1)}+c_{i+3}^{(2)}b_{i+2}^{(3)}+b_{i+3}^{(2)} \neq 0$, *and that for some* $i \in \{0,1,2\}$ *the following condition holds*

$$c_{i+1}^{(1)}c_{i+3}^{(2)}c_{i+2}^{(3)} = 1. \tag{5.12}$$

Then if $y_{3n+i+1} \neq 0$ *for every* $n \in \mathbb{N}_0$, *the following statements hold:*

(a) *if* $|\widehat{p}_i| < 1$, *then* $y_{3n+i+1} \to 0$, *as* $n \to \infty$;

(b) *if* $|\widehat{p}_i| > 1$, *then* $|y_{3n+i+1}| \to \infty$, *as* $n \to \infty$;

(c) *if* $\widehat{p}_i = 1$ *and* $\widehat{q}_i < 0$, *then* $y_{3n+i+1} \to 0$, *as* $n \to \infty$;

(d) *if* $\widehat{p}_i = 1$ *and* $\widehat{q}_i > 0$, *then* $|y_{3n+i+1}| \to \infty$, *as* $n \to \infty$;

(e) *if* $\widehat{p}_i = 1$ *and* $\widehat{q}_i = 0$, *then* y_{3n+i+1} *is convergent;*

(f) *if* $\widehat{p}_i = -1$ *and* $\widehat{r}_i > 0$, *then* $y_{3n+i+1} \to 0$, *as* $n \to \infty$;

(g) *if* $\widehat{p}_i = -1$ *and* $\widehat{r}_i < 0$, *then* $|y_{3n+i+1}| \to \infty$, *as* $n \to \infty$, *so that* $y_{6n+i+1} \to +\infty$ *and* $y_{6n+i+4} \to -\infty$, *or* $y_{6n+i+1} \to -\infty$ *and* $y_{6n+i+4} \to +\infty$ *as* $n \to \infty$;

(h) *if* $\widehat{p}_i = -1$ *and* $\widehat{r}_i = 0$, *then* y_{6n+i+1} *and* y_{6n+i+4} *are convergent, as* $n \to \infty$.

If $c_{i+3}^{(2)}c_{i+2}^{(3)}b_{i+1}^{(1)} + c_{i+3}^{(2)}b_{i+2}^{(3)} + b_{i+3}^{(2)} = 0$, *then the following statements hold:*

(i) *if* $c_{i+1}^{(1)}c_{i+2}^{(3)}b_{i}^{(2)} + c_{i+2}^{(3)}b_{i+1}^{(1)} + b_{i+2}^{(3)} \neq 0$, *then* $|y_{3n+i+1}| \to \infty$, *as* $n \to \infty$;

(j) *if* $c_{i+1}^{(1)}c_{i+2}^{(3)}b_{i}^{(2)} + c_{i+2}^{(3)}b_{i+1}^{(1)} + b_{i+2}^{(3)} = 0$, *then*

$$y_{3n+i+1} = y_{i-2}\left(\frac{w_i}{v_{i+1}}\right)^{n+1}. \tag{5.13}$$

Proof. From (3.22), by using condition (5.12) and the second equality in (5.1), we have that

$$y_{3n+i+1} = y_{i-2}\prod_{j=0}^{n}\frac{w_i + \left(c_{i+1}^{(1)}c_{i+2}^{(3)}b_{i}^{(2)} + c_{i+2}^{(3)}b_{i+1}^{(1)} + b_{i+2}^{(3)}\right)j}{v_{i+1} + \left(c_{i+3}^{(2)}c_{i+2}^{(3)}b_{i+1}^{(1)} + c_{i+3}^{(2)}b_{i+2}^{(3)} + b_{i+3}^{(2)}\right)j}, \tag{5.14}$$

for each $i \in \{0,1,2\}$ and every $n \in \mathbb{N}_0$, from which the results in this theorem follows similar to Theorem 5.1. □

Theorem 5.4. *Consider system* (3.2). *Let the sequences* $b_n^{(j)}$, $c_n^{(j)}$, $n \in \mathbb{N}_0$, $j \in \{1,2,3\}$ *be periodic with period three,*

$$\begin{aligned}
p_i^* &:= \frac{c_{i+2}^{(1)}c_{i+1}^{(2)}b_{i}^{(3)} + c_{i+2}^{(1)}b_{i+1}^{(2)} + b_{i+2}^{(1)}}{c_{i+2}^{(1)}c_{i+3}^{(3)}b_{i+1}^{(2)} + c_{i+3}^{(3)}b_{i+2}^{(1)} + b_{i+3}^{(3)}},\\
q_i^* &:= \frac{u_i - w_{i+1}}{c_{i+2}^{(1)}c_{i+3}^{(3)}b_{i+1}^{(2)} + c_{i+3}^{(3)}b_{i+2}^{(1)} + b_{i+3}^{(3)}},\\
r_i^* &:= \frac{u_i + w_{i+1}}{c_{i+2}^{(1)}c_{i+3}^{(3)}b_{i+1}^{(2)} + c_{i+3}^{(3)}b_{i+2}^{(1)} + b_{i+3}^{(3)}},
\end{aligned} \tag{5.15}$$

when $c_{i+2}^{(1)}c_{i+3}^{(3)}b_{i+1}^{(2)} + c_{i+3}^{(3)}b_{i+2}^{(1)} + b_{i+3}^{(3)} \neq 0$, *and for some* $i \in \{0,1,2\}$ *the following condition holds*

$$c_{i+2}^{(1)}c_{i+1}^{(2)}c_{i}^{(3)} = 1. \tag{5.16}$$

Then if $z_{3n+i+1} \neq 0$ *for every* $n \in \mathbb{N}_0$, *the following statements hold:*

(a) *if* $|p_i^*| < 1$, *then* $z_{3n+i+1} \to 0$, *as* $n \to \infty$;

(b) *if* $|p_i^*| > 1$, *then* $|z_{3n+i+1}| \to \infty$, *as* $n \to \infty$;

(c) *if* $p_i^* = 1$ *and* $q_i^* < 0$, *then* $z_{3n+i+1} \to 0$, *as* $n \to \infty$;

(d) *if* $p_i^* = 1$ *and* $q_i^* > 0$, *then* $|z_{3n+i+1}| \to \infty$, *as* $n \to \infty$;

(e) *if* $p_i^* = 1$ *and* $q_i^* = 0$, *then* z_{3n+i+1} *is convergent;*

(f) *if* $p_i^* = -1$ *and* $r_i^* > 0$, *then* $z_{3n+i+1} \to 0$, *as* $n \to \infty$;

(g) *if* $p_i^* = -1$ *and* $r_i^* < 0$, *then* $|z_{3n+i+1}| \to \infty$, *as* $n \to \infty$, *so that* $z_{6n+i+1} \to +\infty$ *and* $z_{6n+i+4} \to -\infty$, *or* $z_{6n+i+1} \to -\infty$ *and* $z_{6n+i+4} \to +\infty$ *as* $n \to \infty$;

(h) *if* $p_i^* = -1$ *and* $r_i^* = 0$, *then* z_{6n+i+1} *and* z_{6n+i+4} *are convergent, as* $n \to \infty$.

If $c_{i+2}^{(1)}c_{i+3}^{(3)}b_{i+1}^{(2)} + c_{i+3}^{(3)}b_{i+2}^{(1)} + b_{i+3}^{(3)} = 0$, *then the following statements hold:*

(i) *if* $c_{i+2}^{(1)}c_{i+1}^{(2)}b_i^{(3)} + c_{i+2}^{(1)}b_{i+1}^{(2)} + b_{i+2}^{(1)} \neq 0$, *then* $|z_{3n+i+1}| \to \infty$, *as* $n \to \infty$;

(j) *if* $c_{i+2}^{(1)}c_{i+1}^{(2)}b_i^{(3)} + c_{i+2}^{(1)}b_{i+1}^{(2)} + b_{i+2}^{(1)} = 0$, *then*

$$z_{3n+i+1} = z_{i-2}\left(\frac{u_i}{w_{i+1}}\right)^{n+1}. \tag{5.17}$$

Proof. From (3.23), by using conditions (5.16) and (5.2), we have that

$$z_{3n+i+1} = z_{i-2}\prod_{j=0}^{n}\frac{u_i + \left(c_{i+2}^{(1)}c_{i+1}^{(2)}b_i^{(3)} + c_{i+2}^{(1)}b_{i+1}^{(2)} + b_{i+2}^{(1)}\right)j}{w_{i+1} + \left(c_{i+2}^{(1)}c_{i+3}^{(3)}b_{i+1}^{(2)} + c_{i+3}^{(3)}b_{i+2}^{(1)} + b_{i+3}^{(3)}\right)j}, \tag{5.18}$$

for each $i \in \{0,1,2\}$ and every $n \in \mathbb{N}_0$, from which the results in this theorem follows similar to Theorem 5.1. □

Theorem 5.5. *Consider system* (3.2). *Let the sequences* $b_n^{(j)}$, $c_n^{(j)}$, $n \in \mathbb{N}_0$, $j \in \{1,2,3\}$ *be periodic with period three and for some* $i \in \{0,1,2\}$ *the following condition holds:*

$$c_i^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)} \neq 1. \tag{5.19}$$

Then if $x_{3n+i+1} \neq 0$ *for every* $n \in \mathbb{N}_0$, *the following statements hold:*

(a) *if* $|c_i^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}| > 1$,

$$\begin{aligned} s_i &:= v_i - \frac{c_{i+2}^{(2)}c_{i+1}^{(3)}b_i^{(1)} + c_{i+2}^{(2)}b_{i+1}^{(3)} + b_{i+2}^{(2)}}{1 - c_i^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}} = 0, \\ t_{i+1} &:= u_{i+1} - \frac{c_{i+3}^{(1)}c_{i+2}^{(2)}b_{i+1}^{(3)} + c_{i+3}^{(1)}b_{i+2}^{(2)} + b_{i+3}^{(1)}}{1 - c_{i+3}^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}} \neq 0 \end{aligned} \tag{5.20}$$

then $x_{3n+i+1} \to 0$, *as* $n \to \infty$;

(b) *if* $|c_i^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}| > 1$, $s_i \neq 0$, *and* $t_{i+1} = 0$, *then* $|x_{3n+i+1}| \to \infty$, *as* $n \to \infty$;

(c) *if* $c_{i+3}^{(1)}c_{i+2}^{(2)}b_{i+1}^{(3)} + c_{i+3}^{(1)}b_{i+2}^{(2)} + b_{i+3}^{(1)} \neq 0$, $s_i = t_{i+1} = 0$, *then*

$$x_{3n+i+1} = x_{i-2}\left(\frac{c_{i+2}^{(2)}c_{i+1}^{(3)}b_i^{(1)} + c_{i+2}^{(2)}b_{i+1}^{(3)} + b_{i+2}^{(2)}}{c_{i+3}^{(1)}c_{i+2}^{(2)}b_{i+1}^{(3)} + c_{i+3}^{(1)}b_{i+2}^{(2)} + b_{i+3}^{(1)}}\right)^{n+1}; \tag{5.21}$$

(d) *if* $c_{i+3}^{(1)}c_{i+2}^{(2)}b_{i+1}^{(3)} + c_{i+3}^{(1)}b_{i+2}^{(2)} + b_{i+3}^{(1)} = c_{i+2}^{(2)}c_{i+1}^{(3)}b_i^{(1)} + c_{i+2}^{(2)}b_{i+1}^{(3)} + b_{i+2}^{(2)} = 0$ *and* (5.20) *holds, then*

$$x_{3n+i+1} = x_{i-2}\left(\frac{s_i}{t_{i+1}}\right)^{n+1}; \tag{5.22}$$

(e) *if* $|c_i^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}| > 1$ *and* $0 < |s_i| < |t_{i+1}|$, *then* $x_{3n+i+1} \to 0$ *as* $n \to \infty$;

(f) *if* $|c_i^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}| > 1$ *and* $|s_i| > |t_{i+1}| > 0$, *then* $|x_{3n+i+1}| \to +\infty$ *as* $n \to \infty$;

(g) *if* $|c_i^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}| > 1$ *and* $s_i = t_{i+1} \neq 0$, *then* x_{3n+i+1} *converges, as* $n \to \infty$;

(h) *if* $|c_i^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}| > 1$ *and* $s_i = -t_{i+1} \neq 0$, *then* x_{6n+i+1} *and* x_{6n+i+4} *converge, as* $n \to \infty$;

(i) *if* $|c_i^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}| < 1$ *and* $0 < |a_i| < |b_i|$, *where*

$$a_i := \frac{c_{i+2}^{(2)}c_{i+1}^{(3)}b_i^{(1)} + c_{i+2}^{(2)}b_{i+1}^{(3)} + b_{i+2}^{(2)}}{1 - c_i^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}}, \qquad b_i = \frac{c_{i+3}^{(1)}c_{i+2}^{(2)}b_{i+1}^{(3)} + c_{i+3}^{(1)}b_{i+2}^{(2)} + b_{i+3}^{(1)}}{1 - c_{i+3}^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}}, \tag{5.23}$$

then $x_{3n+i+1} \to 0$ *as* $n \to \infty$;

(j) *if* $|c_i^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}| < 1$ *and* $|a_i| > |b_i| > 0$, *then* $|x_{3n+i+1}| \to +\infty$ *as* $n \to \infty$;

(k) *if* $|c_i^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}| < 1$ *and* $a_i = b_i \neq 0$, *then* x_{3n+i+1} *converges, as* $n \to \infty$;

(l) *if* $|c_i^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}| < 1$ *and* $a_i = -b_i \neq 0$, *then* x_{6n+i+1} *and* x_{6n+i+4} *converge, as* $n \to \infty$.

Proof. From (3.21) and by using conditions (5.19) and (5.2), we have that

$$x_{3n+i+1} = x_{i-2}\prod_{j=0}^{n}\frac{s_i\left(c_i^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}\right)^j + \left(\left(c_{i+2}^{(2)}c_{i+1}^{(3)}b_i^{(1)} + c_{i+2}^{(2)}b_{i+1}^{(3)} + b_{i+2}^{(2)}\right)/\left(1 - c_i^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}\right)\right)}{t_{i+1}\left(c_{i+3}^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}\right)^j + \left(\left(c_{i+3}^{(1)}c_{i+2}^{(2)}b_{i+1}^{(3)} + c_{i+3}^{(1)}b_{i+2}^{(2)} + b_{i+3}^{(1)}\right)/\left(1 - c_{i+3}^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}\right)\right)}, \tag{5.24}$$

for every $n \in \mathbb{N}_0$ and for each $i \in \{0, 1, 2\}$. Using (5.24) the statements in (a)–(d) easily follows.

(e)–(h) Let $\lambda := c_i^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}$. Then

$$\frac{s_i\left(c_i^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}\right)^j + \left(\left(c_{i+2}^{(2)}c_{i+1}^{(3)}b_i^{(1)} + c_{i+2}^{(2)}b_{i+1}^{(3)} + b_{i+2}^{(2)}\right)/\left(1 - c_i^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}\right)\right)}{t_{i+1}\left(c_{i+3}^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}\right)^j + \left(\left(c_{i+3}^{(1)}c_{i+2}^{(2)}b_{i+1}^{(3)} + c_{i+3}^{(1)}b_{i+2}^{(2)} + b_{i+3}^{(1)}\right)/\left(1 - c_{i+3}^{(1)}c_{i+2}^{(2)}c_{i+1}^{(3)}\right)\right)} = \frac{s_i\lambda^j + a_i}{t_{i+1}\lambda^j + b_i}, \tag{5.25}$$

where a_i, b_i are defined by (5.23). We have

$$\frac{s_i\lambda^j + a_i}{t_{i+1}\lambda^j + b_i} = \frac{s_i}{t_{i+1}}\left(1 + \frac{a_i}{s_i\lambda^j}\right)\left(1 + \frac{b_i}{t_{i+1}\lambda^j}\right)^{-1} = \frac{s_i}{t_{i+1}}\left(1 + \left(\frac{a_i}{s_i} - \frac{b_i}{t_{i+1}}\right)\frac{1}{\lambda^j} + O\left(\frac{1}{\lambda^{2j}}\right)\right). \quad (5.26)$$

From this and since $|\lambda| > 1$, the results in these four cases easily follow.

(i)–(l) We have

$$\frac{s_i\lambda^j + a_i}{t_{i+1}\lambda^j + b_i} = \frac{a_i}{b_i}\left(1 + \frac{s_i}{a_i}\lambda^j\right)\left(1 + \frac{t_{i+1}}{b_i}\lambda^j\right)^{-1} = \frac{a_i}{b_i}\left(1 + \left(\frac{s_i}{a_i} - \frac{t_{i+1}}{b_i}\right)\lambda^j + O\left(\lambda^{2j}\right)\right). \quad (5.27)$$

From this and since $|\lambda| < 1$, the results in these four cases easily follow. □

Theorem 5.6. *Consider system* (3.2). *Let the sequences* $b_n^{(j)}$, $c_n^{(j)}$, $n \in \mathbb{N}_0$, $j \in \{1,2,3\}$ *be periodic with period three and for some* $i \in \{0,1,2\}$ *the following condition holds:*

$$c_{i+1}^{(1)}c_i^{(2)}c_{i+2}^{(3)} \neq 1. \quad (5.28)$$

Then if $y_{3n+i+1} \neq 0$ *for every* $n \in \mathbb{N}_0$, *the following statements hold:*

(a) *if* $|c_{i+1}^{(1)}c_i^{(2)}c_{i+2}^{(3)}| > 1$,

$$\widehat{s}_i := w_i - \frac{c_{i+1}^{(1)}c_{i+2}^{(3)}b_i^{(2)} + c_{i+2}^{(3)}b_{i+1}^{(1)} + b_{i+2}^{(3)}}{1 - c_{i+1}^{(1)}c_i^{(2)}c_{i+2}^{(3)}} = 0,$$
$$\widehat{t}_{i+1} = v_{i+1} - \frac{c_{i+3}^{(2)}c_{i+2}^{(3)}b_{i+1}^{(1)} + c_{i+3}^{(2)}b_{i+2}^{(3)} + b_{i+3}^{(2)}}{1 - c_{i+1}^{(1)}c_{i+3}^{(2)}c_{i+2}^{(3)}} \neq 0, \quad (5.29)$$

then $y_{3n+i+1} \to 0$, *as* $n \to \infty$;

(b) *if* $|c_{i+1}^{(1)}c_i^{(2)}c_{i+2}^{(3)}| > 1$, $\widehat{s}_i \neq 0$, *and* $\widehat{t}_{i+1} = 0$, *then* $|y_{3n+i+1}| \to \infty$, *as* $n \to \infty$;

(c) *if* $c_{i+3}^{(2)}c_{i+2}^{(3)}b_{i+1}^{(1)} + c_{i+3}^{(2)}b_{i+2}^{(3)} + b_{i+3}^{(2)} \neq 0$, $\widehat{s}_i = \widehat{t}_{i+1} = 0$, *then*

$$y_{3n+i+1} = y_{i-2}\left(\frac{c_{i+1}^{(1)}c_{i+2}^{(3)}b_i^{(2)} + c_{i+2}^{(3)}b_{i+1}^{(1)} + b_{i+2}^{(3)}}{c_{i+3}^{(2)}c_{i+2}^{(3)}b_{i+1}^{(1)} + c_{i+3}^{(2)}b_{i+2}^{(3)} + b_{i+3}^{(2)}}\right)^{n+1}; \quad (5.30)$$

(d) *if* $c_{i+1}^{(1)}c_{i+2}^{(3)}b_i^{(2)} + c_{i+2}^{(3)}b_{i+1}^{(1)} + b_{i+2}^{(3)} = c_{i+3}^{(2)}c_{i+2}^{(3)}b_{i+1}^{(1)} + c_{i+3}^{(2)}b_{i+2}^{(3)} + b_{i+3}^{(2)} = 0$ *and* (5.29) *holds, then*

$$y_{3n+i+1} = y_{i-2}\left(\frac{\widehat{s}_i}{\widehat{t}_{i+1}}\right)^{n+1}; \quad (5.31)$$

(e) *if* $|c_{i+1}^{(1)}c_i^{(2)}c_{i+2}^{(3)}| > 1$ *and* $0 < |\widehat{s}_i| < |\widehat{t}_{i+1}|$, *then* $y_{3n+i+1} \to 0$ *as* $n \to \infty$;

(f) *if* $|c_{i+1}^{(1)}c_i^{(2)}c_{i+2}^{(3)}| > 1$ *and* $|\widehat{s}_i| > |\widehat{t}_{i+1}| > 0$, *then* $|y_{3n+i+1}| \to +\infty$ *as* $n \to \infty$;

(g) *if* $|c_{i+1}^{(1)}c_i^{(2)}c_{i+2}^{(3)}| > 1$ *and* $\widehat{s}_i = \widehat{t}_{i+1} \neq 0$, *then* y_{3n+i+1} *converges, as* $n \to \infty$;

(h) *if* $|c_{i+1}^{(1)}c_i^{(2)}c_{i+2}^{(3)}| > 1$ *and* $\widehat{s}_i = -\widehat{t}_{i+1} \neq 0$, *then* y_{6n+i+1} *and* y_{6n+i+4} *converge, as* $n \to \infty$;

(i) *if* $|c_{i+1}^{(1)}c_i^{(2)}c_{i+2}^{(3)}| < 1$ *and* $0 < |\widehat{a}_i| < |\widehat{b}_i|$, *where*

$$\widehat{a}_i := \frac{c_{i+1}^{(1)}c_{i+2}^{(3)}b_i^{(2)} + c_{i+2}^{(3)}b_{i+1}^{(1)} + b_{i+2}^{(3)}}{1 - c_{i+1}^{(1)}c_i^{(2)}c_{i+2}^{(3)}}, \qquad \widehat{b}_i := \frac{c_{i+3}^{(2)}c_{i+2}^{(3)}b_{i+1}^{(1)} + c_{i+3}^{(2)}b_{i+2}^{(3)} + b_{i+3}^{(2)}}{1 - c_{i+1}^{(1)}c_{i+3}^{(2)}c_{i+2}^{(3)}}, \tag{5.32}$$

then $y_{3n+i+1} \to 0$ *as* $n \to \infty$;

(j) *if* $|c_{i+1}^{(1)}c_i^{(2)}c_{i+2}^{(3)}| < 1$ *and* $|\widehat{a}_i| > |\widehat{b}_i| > 0$, *then* $|y_{3n+i+1}| \to +\infty$ *as* $n \to \infty$;

(k) *if* $|c_{i+1}^{(1)}c_i^{(2)}c_{i+2}^{(3)}| < 1$ *and* $\widehat{a}_i = \widehat{b}_i \neq 0$, *then* y_{3n+i+1} *converges, as* $n \to \infty$;

(l) *if* $|c_{i+1}^{(1)}c_i^{(2)}c_{i+2}^{(3)}| < 1$ *and* $\widehat{a}_i = -\widehat{b}_i \neq 0$, *then* y_{6n+i+1} *and* y_{6n+i+4} *converge, as* $n \to \infty$.

Proof. From (3.22) and by using condition (5.28), we have that

$$y_{3n+i+1} = y_{i-2}\prod_{j=0}^{n}\frac{\widehat{s}_i\left(c_{i+1}^{(1)}c_i^{(2)}c_{i+2}^{(3)}\right)^j + \left(\left(c_{i+1}^{(1)}c_{i+2}^{(3)}b_i^{(2)} + c_{i+2}^{(3)}b_{i+1}^{(1)} + b_{i+2}^{(3)}\right)/\left(1 - c_{i+1}^{(1)}c_i^{(2)}c_{i+2}^{(3)}\right)\right)}{\widehat{t}_{i+1}\left(c_{i+1}^{(1)}c_{i+3}^{(2)}c_{i+2}^{(3)}\right)^j + \left(\left(c_{i+3}^{(2)}c_{i+2}^{(3)}b_{i+1}^{(1)} + c_{i+3}^{(2)}b_{i+2}^{(3)} + b_{i+3}^{(2)}\right)/\left(1 - c_{i+1}^{(1)}c_{i+3}^{(2)}c_{i+2}^{(3)}\right)\right)}, \tag{5.33}$$

for every $n \in \mathbb{N}_0$ and for each $i \in \{0,1,2\}$, from which the statements in (a)–(l) follows similar as the corresponding ones in Theorem 5.5. □

Theorem 5.7. *Consider system* (3.2). *Let the sequences* $b_n^{(j)}$, $c_n^{(j)}$, $n \in \mathbb{N}_0$, $j \in \{1,2,3\}$ *be periodic with period three and for some* $i \in \{0,1,2\}$ *the following condition holds*

$$c_{i+2}^{(1)}c_{i+1}^{(2)}c_i^{(3)} \neq 1. \tag{5.34}$$

Then if $z_{3n+i+1} \neq 0$ *for every* $n \in \mathbb{N}_0$, *the following statements hold:*

(a) *if* $|c_{i+2}^{(1)}c_{i+1}^{(2)}c_i^{(3)}| > 1$,

$$\begin{aligned} s_i^* &:= u_i - \frac{c_{i+2}^{(1)}c_{i+1}^{(2)}b_i^{(3)} + c_{i+2}^{(1)}b_{i+1}^{(2)} + b_{i+2}^{(1)}}{1 - c_{i+2}^{(1)}c_{i+1}^{(2)}c_i^{(3)}} = 0, \\ t_{i+1}^* &:= w_{i+1} - \frac{c_{i+2}^{(1)}c_{i+3}^{(3)}b_{i+1}^{(2)} + c_{i+3}^{(3)}b_{i+2}^{(1)} + b_{i+3}^{(3)}}{1 - c_{i+2}^{(1)}c_{i+1}^{(2)}c_{i+3}^{(3)}} \neq 0 \end{aligned} \tag{5.35}$$

then $z_{3n+i+1} \to 0$, *as* $n \to \infty$;

(b) *if* $|c_{i+2}^{(1)}c_{i+1}^{(2)}c_i^{(3)}| > 1$, $s_i^* \neq 0, t_{i+1}^* = 0$, *then* $|z_{3n+i+1}| \to \infty$, *as* $n \to \infty$;

(c) *if* $c^{(1)}_{i+2}c^{(3)}_{i+3}b^{(2)}_{i+1} + c^{(3)}_{i+3}b^{(1)}_{i+2} + b^{(3)}_{i+3} \neq 0$, $s^*_i = t^*_{i+1} = 0$, *then*

$$z_{3n+i+1} = z_{i-2}\left(\frac{c^{(1)}_{i+2}c^{(2)}_{i+1}b^{(3)}_{i} + c^{(1)}_{i+2}b^{(2)}_{i+1} + b^{(1)}_{i+2}}{c^{(1)}_{i+2}c^{(3)}_{i+3}b^{(2)}_{i+1} + c^{(3)}_{i+3}b^{(1)}_{i+2} + b^{(3)}_{i+3}}\right)^{n+1}; \tag{5.36}$$

(d) *if* $c^{(1)}_{i+2}c^{(2)}_{i+1}b^{(3)}_{i} + c^{(1)}_{i+2}b^{(2)}_{i+1} + b^{(1)}_{i+2} = c^{(1)}_{i+2}c^{(3)}_{i+3}b^{(2)}_{i+1} + c^{(3)}_{i+3}b^{(1)}_{i+2} + b^{(3)}_{i+3} = 0$ *and* (5.35) *holds, then*

$$z_{3n+i+1} = z_{i-2}\left(\frac{s^*_i}{t^*_{i+1}}\right)^{n+1}; \tag{5.37}$$

(e) *if* $|c^{(1)}_{i+2}c^{(2)}_{i+1}c^{(3)}_{i}| > 1$ *and* $0 < |s^*_i| < |t^*_{i+1}|$, *then* $z_{3n+i+1} \to 0$ *as* $n \to \infty$;

(f) *if* $|c^{(1)}_{i+2}c^{(2)}_{i+1}c^{(3)}_{i}| > 1$ *and* $|s^*_i| > |t^*_{i+1}| > 0$, *then* $|z_{3n+i+1}| \to +\infty$ *as* $n \to \infty$;

(g) *if* $|c^{(1)}_{i+2}c^{(2)}_{i+1}c^{(3)}_{i}| > 1$ *and* $s^*_i = t^*_{i+1} \neq 0$, *then* z_{3n+i+1} *converges, as* $n \to \infty$;

(h) *if* $|c^{(1)}_{i+2}c^{(2)}_{i+1}c^{(3)}_{i}| > 1$ *and* $s^*_i = -t^*_{i+1} \neq 0$, *then* z_{6n+i+1} *and* z_{6n+i+4} *converge, as* $n \to \infty$;

(i) *if* $|c^{(1)}_{i+2}c^{(2)}_{i+1}c^{(3)}_{i}| < 1$ *and* $0 < |a^*_i| < |b^*_i|$, *where*

$$a^*_i := \frac{c^{(1)}_{i+2}c^{(2)}_{i+1}b^{(3)}_{i} + c^{(1)}_{i+2}b^{(2)}_{i+1} + b^{(1)}_{i+2}}{1 - c^{(1)}_{i+2}c^{(2)}_{i+1}c^{(3)}_{i}}, \qquad b^*_i := \frac{c^{(1)}_{i+2}c^{(3)}_{i+3}b^{(2)}_{i+1} + c^{(3)}_{i+3}b^{(1)}_{i+2} + b^{(3)}_{i+3}}{1 - c^{(1)}_{i+2}c^{(2)}_{i+1}c^{(3)}_{i+3}}, \tag{5.38}$$

then $z_{3n+i+1} \to 0$ *as* $n \to \infty$;

(j) *if* $|c^{(1)}_{i+2}c^{(2)}_{i+1}c^{(3)}_{i}| < 1$ *and* $|a^*_i| > |b^*_i| > 0$, *then* $|z_{3n+i+1}| \to +\infty$ *as* $n \to \infty$;

(k) *if* $|c^{(1)}_{i+2}c^{(2)}_{i+1}c^{(3)}_{i}| < 1$ *and* $a^*_i = b^*_i \neq 0$, *then* z_{3n+i+1} *converges, as* $n \to \infty$;

(l) *if* $|c^{(1)}_{i+2}c^{(2)}_{i+1}c^{(3)}_{i}| < 1$ *and* $a^*_i = -b^*_i \neq 0$, *then* z_{6n+i+1} *and* z_{6n+i+4} *converge, as* $n \to \infty$.

Proof. From (3.23) and by using condition (5.34), we have that

$$z_{3n+i+1} = z_{i-2}\prod_{j=0}^{n}\frac{s^*_i\left(c^{(1)}_{i+2}c^{(2)}_{i+1}c^{(3)}_{i}\right)^j + \left(\left(c^{(1)}_{i+2}c^{(2)}_{i+1}b^{(3)}_{i} + c^{(1)}_{i+2}b^{(2)}_{i+1} + b^{(1)}_{i+2}\right)/\left(1 - c^{(1)}_{i+2}c^{(2)}_{i+1}c^{(3)}_{i}\right)\right)}{t^*_{i+1}\left(c^{(1)}_{i+2}c^{(2)}_{i+1}c^{(3)}_{i+3}\right)^j + \left(\left(c^{(1)}_{i+2}c^{(3)}_{i+3}b^{(2)}_{i+1} + c^{(3)}_{i+3}b^{(1)}_{i+2} + b^{(3)}_{i+3}\right)/\left(1 - c^{(1)}_{i+2}c^{(2)}_{i+1}c^{(3)}_{i+3}\right)\right)}, \tag{5.39}$$

for every $n \in \mathbb{N}_0$ and for each $i \in \{0, 1, 2\}$, from which the statements in (a)–(l) follow similar as the corresponding ones in Theorem 5.5. □

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