

Research Article

Probing Fractionalized Charges

Mohsen Alishahiha¹ and Davood Allahbakhshi²

¹ School of Physics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5531, Tehran, Iran

² School of Particles and Accelerators, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5531, Tehran, Iran

Correspondence should be addressed to Mohsen Alishahiha; alishah@ipm.ir

Received 26 June 2013; Accepted 20 September 2013

Academic Editor: Aninda Sinha

Copyright © 2013 M. Alishahiha and D. Allahbakhshi. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Inspired by the holographic entanglement entropy, for geometries with nonzero abelian charges, we define a quantity which is sensitive to the background charges. One observes that there is a critical charge below the system that is mainly described by the metric, and the effects of the background charges are just via metric's components. For charges above the critical one, the background gauge field plays an essential role. This, in turn, might be used to define an order parameter to probe phases of a system with fractionalized charges.

1. Introduction

In application of AdS/CFT correspondence [1] to condensed matter physics, one typically is interested in a gravity dual which describes a system at finite temperature and density. Following [2] a natural guess for the dual gravity would be a charged black hole. The existence of the charged horizon would result in a dual theory at finite temperature and finite density.

We note, however, that this is not the only way to construct a gravity model whose dual theory is a system at finite density. Indeed, finite density holographic duals may be obtained by two, rather distinctive, ways. Actually, the asymptotic electric flux—to be identified with the chemical potential at the boundary theory—may be supported by either nonzero charges from behind an event horizon or charged matter in the bulk geometry. If we are interested in a phase with unbroken $U(1)$ global group, the matter field in the bulk is charged fermions (see, e.g., [3]).

Of course one can distinguish between these two cases due to the fact that in the first case (fractionalized phase), the charge density is of order N^2 , while in the second case (mesonic phase), it is of order $\mathcal{O}(N^0)$, where N is the number of degrees of freedom (the number of color for $U(N)$ gauge theory). Alternatively, when the $U(1)$ is unbroken, the

fractionalized phase may also be identified by the violation of the Luttinger theorem [4–6].

Since the charge density of a system may be originated from both behind an event horizon and a charged matter, it could be in different phases depending on the origin of the asymptotic flux. To classify possible phases, an order parameter has been introduced in [7]. This order parameter at leading order is essentially the holographic entanglement entropy taking into account the electric fluxes through the hypersurface of holographic entanglement entropy. In the present paper, we would like to introduce an order parameter which may probe a system with the fractionalized charges.

To proceed, let us consider a $d + 2$ dimensional Einstein-Dilaton-Maxwell theory whose action, in minimal form, may be written as follows:

$$I = \frac{1}{16\pi G_{d+2}} \int d^{d+2}x \sqrt{-g} \times \left[\mathbf{R} - \frac{1}{2}(\partial\phi)^2 + V(\phi) - \frac{1}{4} \sum_{i=1}^n e^{\lambda_i \phi} F_i^2 \right], \quad (1)$$

where G_{d+2} is the $d + 2$ dimensional Newton constant and λ_i 's are parameters of model. This is, indeed, a typical action we get from compactification of low energy effective action of

string theory. Of course this is the case for particular values of the parameters λ_i and a specific form of the potential. Nevertheless, in what follows, we will not restrict ourselves to these particular values.

A generic solution of the equations of motion of the above action could be a charged black hole (brane) with nontrivial dilaton profile. We may assume the background solution to be an asymptotically locally AdS $_{d+2}$. Therefore, the solution may provide a gravitational dual for a $d + 1$ dimensional theory at finite charge and temperature with a UV fixed point.

The gravity description may be used to extract certain information about the dual field theory. In particular, one may study certain nonlocal observables. Prototype examples include holographic entanglement entropy [8] and Wilson loop [9, 10]. In both cases, the gravitational dual is found useful for extracting the corresponding information. In both cases, the problem reduces to minimizing an area of a hypersurface in the bulk gravity. Actually, motivated by these quantities, we would like to define a similar object which is also sensitive to the background gauge field.

We note, however, that since typically we are interested in backgrounds with electric field, it is not appropriate to work with fixed time as one does for the holographic entanglement entropy. In other words, it would be more natural to consider the geometric entropy [11, 12] which is defined as follows. To be specific, consider a finite temperature four-dimensional quantum field theory on $S^1 \times S^3$. The metric of S^3 sphere may be parametrized as follows

$$d\Omega_d^2 = d\theta + \sin^2\theta (d\psi^2 + \sin^2\psi d\phi^2), \quad (2)$$

with $0 \leq \theta, \psi \leq \pi$ and $0 \leq \phi \leq 2\pi$.

Let us change the periodicity of ϕ into $0 \leq \phi \leq 2\pi k$ which results to conical singularities at $\psi = 0, \pi$ for $k \neq 1$ with the deficit angle $2\pi(1 - k)$. Let us denote by $Z[k]$ the partition function of the theory on this singular space. Then one may define a density matrix as follows:

$$\text{Tr } \rho^k = \frac{Z[k]}{(Z[1])^k}, \quad (3)$$

where $Z[1]$ is the partition function of theory on $S^1 \times S^3$. Using the definition of von-Neumann entropy, the geometric entropy is defined by

$$S_G = -\text{Tr}(\rho \log \rho) = -\partial_k \log \left(\frac{Z[k]}{(Z[1])^k} \right) \Big|_{k=1}. \quad (4)$$

Restricting to a subsystem, one can also define a reduced density matrix. Of course it is clear that the corresponding entropy is different from the entanglement entropy, though it may be related to it by a double Wick rotation.

From gravity point of view, it is essentially similar to the entanglement entropy, where one should minimize a codimension two hypersurface in the bulk. However, in the present case—one considers a hypersurface with a spatial direction fixed. Indeed, to compute the geometric entropy, one usually utilizes a double Wick rotation to promote a spatial direction to time direction. Of course as far as the

computations in the gravity side are concerned, it is not necessary to do that.

Now consider a codimension two hypersurface in the bulk parametrized by coordinates ξ_a for $a = 1, \dots, d$ (in what follows, we use a notation in which the bulk coordinates are given by $x^\mu = (t, r, x^i)$ for $i = 1, \dots, d$). Then one may define two natural quantities: the induced metric and the pull back of the gauge field on the world volume of the hypersurface which are given by

$$\tilde{g}_{ab} = \frac{\partial x^\mu}{\partial \xi_a} \frac{\partial x^\nu}{\partial \xi_b} g_{\mu\nu}, \quad F_{ab}^i = \frac{\partial x^\mu}{\partial \xi_a} \frac{\partial x^\nu}{\partial \xi_b} F_{\mu\nu}^i, \quad \text{for } x^d = \text{fixed}. \quad (5)$$

The geometric entropy can be defined in terms of the induced metric as $S_G = \int d^d \xi \sqrt{\det(\tilde{g})}$ when the area of the hypersurface is minimized. On the other hand, motivated by DBI action in the string theory, it is natural to define the following quantity (in general, one could have put free parameters in front of each F_{ab}^i 's in the square root, and therefore, one has an n -parameter family object: we would like to thank D. Tong for suggesting such a possibility);

$$\Gamma = \frac{1}{G_{d+2}} \int d^d \xi \sqrt{\det \left(\tilde{g} + R \sum_i F_{ab}^i \right)}, \quad (6)$$

where R is a typical scale of the theory (e.g., the radius of curvature). An advantage of this definition is that it is directly sensitive to the background charge. This is in contrast to the holographic entanglement entropy or Wilson loop, where the effects of the background charge are due to the metric components.

For sufficiently small charges, one may expand the square root which for $n = 1$ and at leading order one arrives at

$$\Gamma = \frac{1}{G_{d+2}} \int d^d \xi \sqrt{\det(\tilde{g})} \left(1 - \frac{1}{4} R^2 F_{ab}^2 \right), \quad (7)$$

which, in turns, shows that in this limit it essentially contains the same information as the geometric entropy, as we will explicitly demonstrate in the next section.

For arbitrary charges, following the general idea of AdS/CFT correspondence, it is then natural to minimize Γ . The resultant quantity might be used to define an order parameter which could probe different phases of the system as we will demonstrate in the following sections, within a specific model.

The paper is organized as follows. In the next section, we will consider charge black branes with one $U(1)$ charge and then compute the quantity (6), where we will explore its different properties. In Section three, we will redo the same computations for the charge black hole in a global AdS geometry. The last section is devoted to discussions.

2. Electrically Charged Black Brane Solutions

In this section, in order to explore a possible information encoded in the expression defined by (6), we will consider

a particular model consisting of the Einstein gravity with a negative cosmological constant coupled to a $U(1)$ gauge field. In this case, the action (1) reduces to

$$I = \frac{1}{16\pi G_{d+2}} \int d^{d+2}x \sqrt{-g} \left(\mathbf{R} - 2\Lambda - \frac{1}{4}F^2 \right). \quad (8)$$

This model admits several vacuum solutions which could be either electric or dyonic black branes (holes) charged under the $U(1)$ gauge field. In what follows, we will consider the electric case and will postpone the dyonic one to Section four.

Let us consider a $d+2$ dimensional (Euclidean) Reissner-Nordstrom AdS black brane solution which for $d \geq 2$ may be written as follows [13] (actually for $d = 1$, we still have the same solution but with $f = 1 - r^2 + (Q^2/2)r^2 \ln r$ and $F_{rt} = Q/r$):

$$ds^2 = \frac{R^2}{r^2} \left(-f(r) dt^2 + \frac{dr^2}{f(r)} + \sum_{i=1}^d dx_i^2 \right),$$

$$F_{rt} = -QR\sqrt{2d(d-1)}r^{d-2}, \quad (9)$$

$$f(r) = 1 - \left(1 + Q^2 r_H^{2d}\right) \left(\frac{r}{r_H}\right)^{d+1} + Q^2 r^{2d},$$

where $R = \sqrt{-d(d+1)/2\Lambda}$ and r_H are the radii of curvature and horizon, respectively. The Hawking temperature in terms of the radius of the horizon is

$$T = \frac{d+1}{4\pi r_H} \left(1 - \frac{d-1}{d+1} Q^2 r_H^{2d}\right). \quad (10)$$

This geometry is supposed to provide a gravitational description for a $d+1$ dimensional CFT at finite temperature and density. The corresponding chemical potential is

$$\mu = \sqrt{\frac{2d}{d-1}} QRr_H^{d-1}. \quad (11)$$

Let us consider the following strip as a subsystem in the dual $d+1$ dimensional theory:

$$0 \leq t \leq \tau, \quad -\frac{\ell}{2} \leq x_{d-1} \leq \frac{\ell}{2}, \quad (12)$$

$$0 \leq x_i \leq L, \quad x_d = \text{fixed}$$

for $i = 1, \dots, d-2$. Then there is a hypersurface in the bulk whose intersection with the boundary coincides with the above strip. The profile of the corresponding hypersurface may be given by $x_{d-1} = x(r)$. Thus, the induced (Euclidean) metric on the hypersurface is

$$ds_{\text{ind}}^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu$$

$$= \frac{R^2}{r^2} \left[f dt^2 + \left(\frac{1}{f} + x'^2 \right) dr^2 + \sum_{i=1}^{d-2} dx_i^2 \right], \quad (13)$$

where prime represents derivative with respect to r . In this case, expression (6), taking into account the solution (9) and the boundary subsystem (12), reads

$$\Gamma = \frac{\tau L^{d-2} R^d}{G_{d+2}} \int dr \frac{\sqrt{1 - \phi^2 + f x'^2}}{r^d}, \quad (14)$$

where $\phi = \sqrt{2d(d-1)}Qr^d$.

Now, the aim is to minimize Γ . Actually there is a standard procedure to minimize Γ by which the expression of Γ may be treated as a one-dimensional action for x whose momentum conjugate is a constant of motion. Therefore, one arrives at

$$\frac{f x'}{r^d \sqrt{1 - \phi^2 + f x'^2}} = c, \quad (15)$$

where c is a constant which can be fixed at a particular point. Usually the particular point is chosen to be the turning point where $x' \rightarrow \infty$ in which x' drops from the left-hand side leading to a constant which is given in terms of a function of r evaluated at the turning point. When we are not explicitly considering the effects of gauge field, for example, in the computation of holographic entanglement or geometric entropies where there is no F in the square root, then the position of turning point is located between boundary and horizon; whereas in the present case the situation is different.

Actually, as we will see when we increase the background charges the effects of gauge field become important leading to a new scale in the theory which could take over the role of the horizon. More precisely, as it is evident from (15), for a given background charge there is a special point at which $\phi = 1$ that is given by

$$r_\phi = \left(\frac{1}{2d(d-1)Q^2} \right)^{1/2d}. \quad (16)$$

Note that although at this point the x' dependence is dropped from the left-hand side of (15), it is not a turning point. Moreover, we can convince ourselves that the minimization makes sense only for $r \leq r_\phi$. In other words, in the present case, the location of the turning point will be between boundary and r_{\min} , where $r_{\min} = \text{Min}(r_H, r_\phi)$; that is, $0 \leq r_t \leq r_{\min}$, with r_t being the turning point. In what follows, we will consider both $r_{\min} = r_H$ and $r_{\min} = r_\phi$ cases.

2.1. $r_{\min} = r_H$ Case. Let us assume $r_{\min} = r_H$, which happens if

$$Q \leq Q_c = \frac{1}{\sqrt{2d(d-1)}r_H^d}, \quad \text{or} \quad \mu \leq \mu_c = \frac{R}{(d-1)r_H}. \quad (17)$$

In this case, one finds

$$\ell = 2 \int_0^{r_t} dr \left(\frac{f_t}{f^2} \right)^{1/2}$$

$$\times \left(\frac{r}{r_t} \right)^d \frac{\sqrt{1 - \phi^2}}{\sqrt{1 - (r/r_t)^{2d} (f_t/f)}}, \quad (18)$$

where $f_t = f(r_t)$. On the other hand, using (15), one arrives at

$$\Gamma = \frac{\tau L^{d-2} R^d}{G_{d+2}} \int_{\epsilon}^{r_t} dr \frac{\sqrt{1-\phi^2}}{r^d \sqrt{1-(r/r_t)^{2d} (f_t/f)}}, \quad (19)$$

where ϵ is a UV cut-off. From these expressions, it is clear that there is a new scale in the theory that controls the effects of the background field, as we anticipated. Of course since at the moment we are in the range of $Q \leq Q_c$, the new scale is irrelevant in what follows. The case of $Q > Q_c$ will be discussed latter.

If one drops the factor of $\sqrt{1-\phi^2}$, the above expressions reduce to that of the geometric entropy studied in [11, 12]. Moreover, for pure AdS _{$d+2$} , $d \geq 2$, one has [8]

$$\Gamma = \frac{\tau L^{d-2} R^d}{G_{d+2}} \left[\frac{1}{(d-1)\epsilon^{d-1}} - \frac{2^{d-1} \pi^{d/2}}{d-1} \left(\frac{\Gamma((d+1)/2d)}{\Gamma(1/2d)} \right)^d \frac{1}{\ell^{d-1}} \right], \quad (20)$$

which is the expression of holographic entanglement entropy. Note also that for $d = 1$, one gets a logarithmic behavior, $\Gamma \sim \ln(\ell/\epsilon)$.

For the RN background given in (9), we cannot find an analytic expression for Γ as a function of ℓ . Nevertheless, we can utilize a numerical method to find $\Gamma(\ell)$ numerically. This is, indeed, what we will do in this subsection. To proceed, let us first explore the behavior of ℓ as a function of r_t .

From expression (18), one finds that for sufficiently small r_t , where Γ probes the UV region of the theory, the width ℓ vanishes as $\ell \sim r_t \rightarrow 0$. Moreover, in the opposite limit, the width ℓ also goes to zero as the turning point approaches the horizon. It is, indeed, due to the fact that the function f_t goes to zero as the turning point approaches the horizon, $r_t \rightarrow r_H$. Moreover in this limit the integrand does not diverge faster than $1/f_t$. Therefore for $0 \leq r_t \leq r_H$ the width ℓ goes to zero at both bounds and reaches a maximum value in this interval.

This behavior can be demonstrated by solving the integral (18) numerically. To do so, by making use of a scaling, without loss of generality, one may set $r_H = 1$. Then, the only parameter of the model is the charge of the solution. Note that in this case, one has $0 \leq Q^2 \leq 1/2d(d-1)$. The neutral black brane corresponds to $Q = 0$, while $Q^2 = 1/2d(d-1)$ is the case where $r_H = r_\phi$. The behavior of ℓ as a function of r_t for different values of Q for $d = 2$ is shown in Figure 1.

From (18), one may, in principle, find the turning point as a function of ℓ . Then plugging the result into (19), we get an expression for Γ as a function ℓ . It is important to note that since ℓ is not a one-to-one function of r_t , one has to make sure that the resultant Γ is minimum. Of course it is clear that the minimum Γ is obtained from the minimum r_t .

It should also be noticed that since the space time has a horizon, one could always imagine the case where the function Γ is minimized by another hypersurface consisting of two disconnected parallel surfaces suspending between

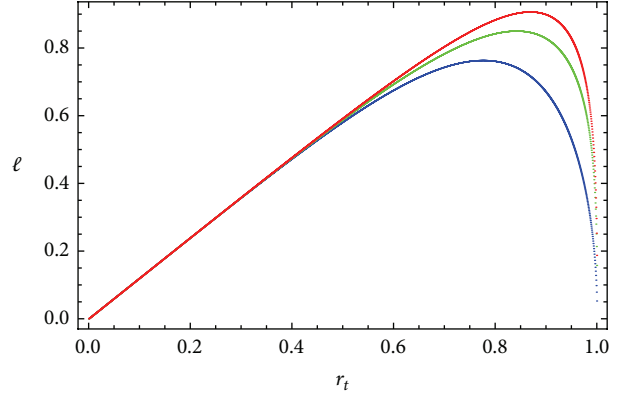


FIGURE 1: ℓ as a function of r_t for the cases of $Q = 0, 0.3, 0.5$ which are shown by red, green, and blue, respectively. The blue curve corresponds to the case of $r_H = r_\phi$, while the red one represents the neutral black brane.

boundary and horizon. Therefore, it is crucial to see which one is smaller.

The disconnected solution is given by setting $r_t = r_H$ in the expression of Γ by which we arrive at

$$\Gamma^{\text{diss}} = \frac{\tau L^{d-2} R^d}{G_{d+2}} \int_{\epsilon}^{r_H} dr \frac{\sqrt{1-\phi^2}}{r^d}. \quad (21)$$

In general, depending on the parameters of the model, either connected or disconnected solutions could be smaller. In order to compare these two solutions, it is useful to define the difference between them as follows:

$$\begin{aligned} \Delta\Gamma &= \Gamma^{\text{con}} - \Gamma^{\text{dis}} \\ &= \frac{\tau L^{d-2} R^d}{G_{d+2}} \\ &\quad \times \left[\int_0^{r_t} dr \left(\frac{\sqrt{1-\phi^2}}{r^d \sqrt{1-(r/r_t)^{2d} (f_t/f)}} - \frac{\sqrt{1-\phi^2}}{r^d} \right) \right. \\ &\quad \left. - \int_{r_t}^{r_H} dr \frac{\sqrt{1-\phi^2}}{r^d} \right]. \end{aligned} \quad (22)$$

Note that although both connected and disconnected solutions are UV divergent, the UV contribution drops out in the difference leading to a finite number. The behaviors of $\Delta\Gamma$ as a function of ℓ for different values of Q for $d = 2$ case are drawn in Figure 2.

One observes that for sufficiently small ℓ , the closed hypersurface minimizes the expression of Γ , though there is a critical width over which the disconnected solution is favored. Moreover, the critical width is always smaller than

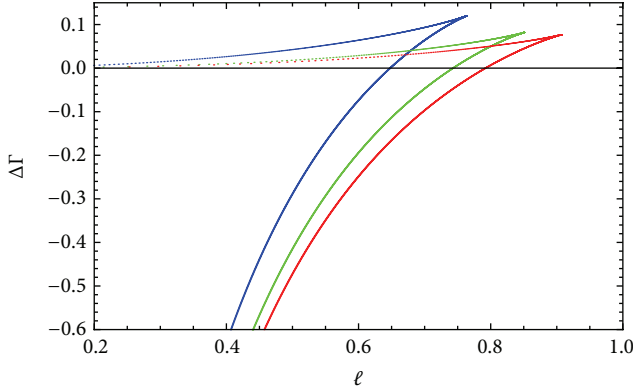


FIGURE 2: $\Delta\Gamma$ as a function of ℓ for the cases of $Q = 0, 0.3, 0.5$ which are shown by red, green, and blue, respectively. The blue curve corresponds to the case where $r_H = r_\phi$, and the red one corresponds to the neutral case. Here, we have set $r_H = 1$ and $\tau L^{d-2} R^d / G_{d+2} = 1$.

the maximum value that the width can reach. Therefore, one may conclude that Γ undergoes a sort of a phase transition before it reaches the maximum ℓ . It is worth to note that as we increase the charge, the maximum width becomes smaller and the phase transition occurs at smaller width; nevertheless as long as $Q \leq Q_c$, the behavior is universal which is that of geometric entropy.

Therefore, as far as the qualitative behavior of Γ is concerned, the effects of gauge field are not important and the main contributions come from the metric. In fact, the background gauge field only affects the position of the horizon. We note, however, that as we increase the background charge, one expects the effects of background charges to become important as we will explore in the following subsection.

2.2. $r_{\min} = r_\phi$ Case. To study the effects of the background gauge field, one may increase the background charge so that $Q > Q_c$, where $r_{\min} = r_\phi$. Since in our notations we have set $r_H = 1$, there is an upper bound on the background charge. More precisely, the allowed values of background charge are $1/(2d(d-1)) \leq Q^2 \leq (d+1)/(d-1)$. Note that $Q^2 = (d+1)/(d-1)$ corresponds to the extremal case where $T = 0$. This indicates that the maximum value that the turning point can get is r_ϕ . More precisely, one has $0 \leq r_t \leq r_\phi < r_H$. In other words, since the turning point cannot reach the horizon, we will not have the disconnected solution.

Indeed looking at (18), one finds that although the width vanishes in the limit of $r_t \rightarrow 0$, it terminates at a nonzero value as one approaches r_ϕ . By making use of the numerical method, the width ℓ can be found as a function of turning point which has been depicted in Figure 3(a).

Moreover, since in the present case we do not have the disconnected solution, it does not make sense to compute the difference $\Delta\Gamma$. Indeed the function Γ is the quantity we may want to compute. We note that due to the UV contribution,

Γ diverges and has to be regulated by a UV cutoff. More precisely, one gets

$$\begin{aligned} \Gamma &= \frac{\tau L^{d-2} R^d}{G_{d+2}} \int_\epsilon^{r_t} dr \frac{\sqrt{1-\phi^2}}{r^d \sqrt{1-(r/r_t)^{2d}} (f_t/f)} \\ &= \frac{1}{G_{d+2}} \frac{\tau L^{d-2} R^d}{(d-1) \epsilon^{d-1}} + \Gamma_{\text{finite}}. \end{aligned} \quad (23)$$

Subtracting the divergence part, it is then straightforward to calculate the finite part, Γ_{finite} , numerically. The results are shown in Figure 3(b).

From our numerical results, one observes that as long as we are in the range of $Q_c < Q \leq \sqrt{(d+1)/(d-1)}$, qualitatively the behavior of Γ is universal, though it decreases as one increases the charge. Indeed, there is a critical width ℓ_c ; above that, both Γ and ℓ are not single valued functions. In other words, for each width $\ell > \ell_c$, there are two turning points. Of course the favored Γ corresponds to the smaller turning point. Moreover, there is a maximum width over which there is no closed hypersurface. It is important to note that the width gets its maximum value before the turning point reaches its maximum value at r_ϕ .

An interesting observation we have made is as follows. Although there is a maximum width (or correspondingly a maximum turning point) over which there is no closed hypersurface which minimizes Γ , there is a single closed hypersurface when $r_t = r_\phi$. Actually, as we have already mentioned, in this case, r_t is not a turning point and indeed the hypersurface can cross the $r = r_\phi$ point and reaches the horizon. In fact, it is easy to see that for this case, the horizon is a turning point. Therefore, we will get a single distinctive closed hypersurface which can probe the charged horizon while the effects of charges are important. In this case, the corresponding expressions for ℓ and Γ are given by

$$\begin{aligned} \frac{\ell}{2} &= \int_0^{r_H} dr \frac{(f_\phi/f^2)^{1/2} (r/r_\phi)^d \sqrt{1-\phi^2}}{\sqrt{1-(r/r_\phi)^{2d}} (f_\phi/f)}, \\ \Gamma &= \frac{\tau L^{d-2} R^d}{G_{d+2}} \int_\epsilon^{r_H} dr \frac{r^{-d} \sqrt{1-\phi^2}}{\sqrt{1-(r/r_\phi)^{2d}} (f_\phi/f)}, \end{aligned} \quad (24)$$

where $f_\phi = f(r_\phi)$. In the Figure 4 we have depicted the behaviors of ℓ and finite part of Γ as functions of Q . Note that as we increase the background charges the width also increases linearly though the finite part of Γ decreases linearly.

3. Black Hole in Global AdS

In this section, we extend our study to a charged black hole in a global AdS geometry. The action is still given by (8). The

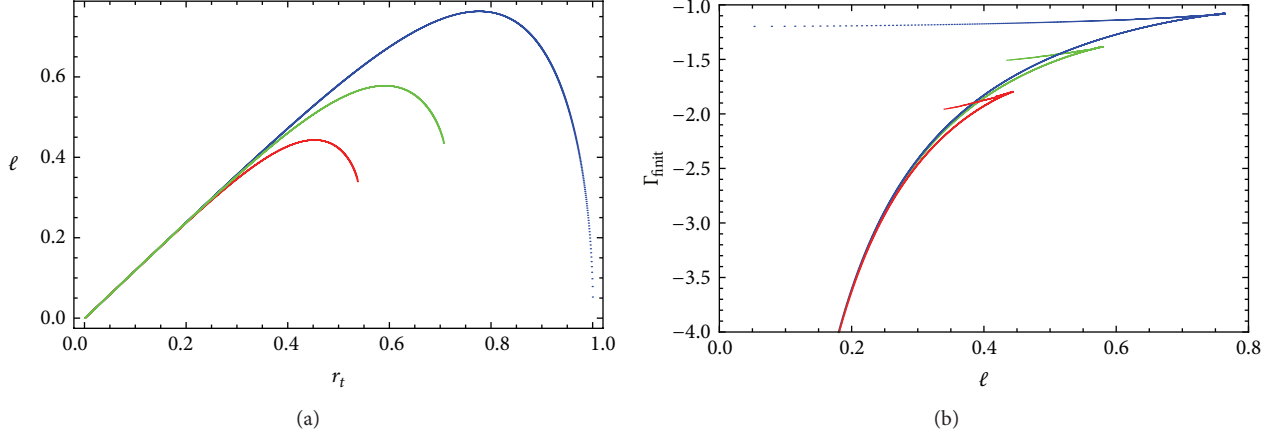


FIGURE 3: ℓ and Γ as functions of r_t and ℓ for $Q = 0.5, 1, \sqrt{3}$ which are shown by blue, green, and red, respectively. Note that $Q = 0.5$ corresponds to the case of $r_H = r_\phi$ and we have plotted it just for a comparison. Here, we have set $r_H = 1$ and $\tau L^{d-2} R^d / G_{d+2} = 1$.

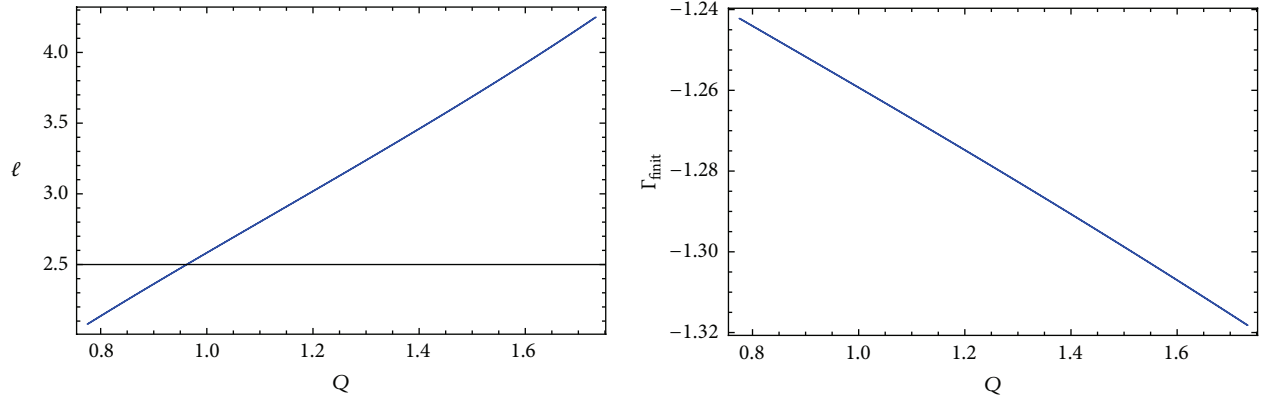


FIGURE 4: ℓ and Γ_{finite} as functions of Q for the case of $r_t = r_\phi$. The numerical values are for $r_H = 1$ and $R = L = 1$. Note that for all values of Q in the above plots, we have $r_\phi < r_H$.

corresponding $d + 2$ dimensional charged black hole may be written as [13]

$$\begin{aligned}
 ds^2 &= \frac{R^2}{r^2} \left(-f(r) dt^2 + \frac{dr^2}{f(r)} + R^2 d\Omega_d^2 \right), \\
 F_{rt} &= -QR\sqrt{2d(d-1)}r^{d-2}, \\
 f(r) &= 1 + \frac{r^2}{R^2} - \left(1 + \frac{r_+^2}{R^2} + Q^2 r_+^{2d} \right) \left(\frac{r}{r_+} \right)^{d+1} + Q^2 r^{2d},
 \end{aligned} \tag{25}$$

where in our notation $d\Omega_d^2 = d\theta^2 + \cos^2\theta d\Omega_{d-1}^2$ with $d\Omega_{d-1}^2$ being the metric of a $(d-1)$ -sphere, and r_+ is the location of the horizon which is a solution of $f(r) = 0$. We note that in general $f = 0$ has two real positive solutions and the horizon

is given by the smallest root. The Hawking temperature and chemical potential are

$$\begin{aligned}
 T &= \frac{d+1}{4\pi r_+} \left[1 - \frac{d-1}{d+1} \left(Q^2 r_+^{2d} - \frac{r_+^2}{R^2} \right) \right], \\
 \mu &= \sqrt{\frac{2d}{d-1}} QR r_+^{d-1}.
 \end{aligned} \tag{26}$$

Using the corresponding Euclidean action, the phase space of this system has been studied in [13], where it was shown that the theory has a rich phase space. Indeed the system could be thought of as either a grand canonical ensemble or a canonical ensemble depending on whether one wants to keep chemical potential or electric charge fixed, respectively. In either cases, there are critical values for the parameters over which the model exhibits different behaviors.

Holographic geometric entropy in this background has also been studied [12], where it was shown that it may provide

a useful order parameter to probe different phases of the system. Note that since in this case one, usually, performs a double Wick rotation, there are two different ways to embed the hypersurface in the bulk. One could either consider $r(t)$ or $r(\theta)$. Actually, by making use of this embedding, it was observed in [12] that the resultant phase structures are very similar to those obtained from the Euclidean action [13]. We note, however, that since in what follows, we are interested in the effects of the gauge field; as defined in (6), the $r(t)$ embedding should automatically be excluded.

Therefore, we will consider a subsystem in the form of $S^{d-2} \times \mathbb{R} \times I$, with I being an interval along θ direction given by $0 \leq \theta \leq 2\pi(\ell/R)$ with $\ell < R$. The extension of this subsystem to the bulk leads to a hypersurface whose profile is given by $\theta = \theta(r)$. Thus, the induced (Euclidean) metric on the hypersurface is

$$ds^2 = \frac{R^2}{r^2} \left[f dt^2 + \left(\frac{1}{f} + R^2 \theta'^2 \right) dr^2 + R^2 \cos^2 \theta d\Omega_{d-2}^2 \right]. \quad (27)$$

Therefore, we arrive at

$$\Gamma = \frac{\tau V_{d-2} R^d}{G_{d+2}} \int_{\epsilon}^{r_t} dr \frac{\cos^{d-2} \theta}{r^d} \sqrt{1 - \phi^2 + f R^2 \theta'^2}, \quad (28)$$

where V_{d-2} is the volume of $(d-2)$ -sphere with radius R and r_t is the turning point where $\theta'(r)$ diverges.

Alternatively, for $d \geq 3$, one may use a notation in which

$$d\Omega_d^2 = d\psi^2 + \cos^2 \psi d\theta^2 + \sin^2 \psi (d\phi^2 + \cos^2 \phi d\Omega_{d-3}^2), \quad (29)$$

and thus the corresponding subsystem may be chosen so that $\phi = \text{constant}$. The constant may be set to $\phi = \pi/2$ and the profile of the hypersurface is given by $\psi(r)$. Therefore, the induced (Euclidean) metric is

$$ds^2 = \frac{R^2}{r^2} \left[f dt^2 + \left(\frac{1}{f} + R^2 \cos^2 \psi \theta'^2 \right) dr^2 + R^2 d\psi^2 + R^2 \sin^2 \psi d\Omega_{d-3}^2 \right], \quad (30)$$

so that

$$\Gamma = \frac{\tau V_{d-3} R^d}{G_{d+2}} \int_{\epsilon}^{r_t} d\psi dr \frac{\cos^{d-3} \psi}{r^d} \sqrt{1 - \phi^2 + f R^2 \cos^2 \psi \theta'^2}. \quad (31)$$

Now the aim is to minimize Γ given in (28) or (31), which can be done by treating them as actions for θ . In what follows, we will mainly consider the first case, where Γ is given by (28) where unlike the previous cases, except for $d = 2$, the momentum conjugate of θ is not a constant of motion, and

therefore, one needs to directly solve the equation of θ which is

$$\begin{aligned} \frac{d}{dr} \left(\frac{\cos^{d-2} \theta}{r^d} \frac{f R^2 \theta'}{\sqrt{1 - \phi^2 + f R^2 \theta'^2}} \right) \\ + (d-2) \sin \theta \cos^{d-3} \\ \times \theta \frac{\sqrt{1 - \phi^2 + f R^2 \theta'^2}}{r^d} = 0. \end{aligned} \quad (32)$$

This equation may be solved with proper boundary conditions to find θ as a function of r_t . The corresponding boundary conditions could be $\theta(r \rightarrow 0) = 2\pi(\ell/R)$ and $\theta(r_t) = 0$. Then plugging the result into (28), one can find Γ as a function of ℓ .

Although it is not explicitly clear from the above equation, there is still a special point at $r = r_\phi$, where $\phi = 1$ and the minimization makes sense for $r \leq r_\phi$. Indeed, the situation is very similar to what we have considered in the previous section for the black brane. In particular, for $r_+ \leq r_\phi$, the function Γ may also be minimized by a disconnected hypersurface which in the present case is given by

$$\Gamma^{\text{diss}} = \frac{V_{d-2} R^{d-1}}{G_{d+1}} \int_{\epsilon}^{r_H} dr \frac{\cos^{d-2} \theta_0}{r^d} \sqrt{1 - \phi^2}, \quad (33)$$

where $\theta_0 = \theta(r = 0)$. It is then natural to look for $\Delta\Gamma$ as a function of ℓ .

To proceed, let us first consider $d = 2$ case in which the momentum conjugate of θ is, indeed, a constant of motion:

$$\frac{R\theta'}{\sqrt{1 - \phi^2 + f R^2 \theta'^2}} = \left(\frac{r}{r_t} \right)^d \frac{f_t^{1/2}}{f}, \quad (34)$$

where r_t is the turning point, so that

$$\ell = \frac{1}{\pi} \int_0^{r_t} dr \frac{(f_t/f^2)^{1/2} (r/r_t)^2 \sqrt{1 - \phi^2}}{\sqrt{1 - (r/r_t)^4} (f_t/f)}, \quad (35)$$

$$\Gamma = \frac{\tau R^2}{G_4} \int_{\epsilon}^{r_t} dr \frac{\sqrt{1 - \phi^2}}{r^2 \sqrt{1 - (r/r_t)^4} (f_t/f)}, \quad (36)$$

which have essentially the same form as the corresponding expressions we have found in the previous section, though the function f is different. Therefore, one expects that the system may exhibit the same behavior as in the black brane. In particular, one can show that as long as we are in the range of the parameters where $r_+ \leq r_\phi$, the corresponding width, ℓ , vanishes at both $r_t = 0$ and $r_t = r_+$ points, while for $r_\phi < r_+$ although the width vanishes at r_t , it tends to a nonzero constant as $r_t \rightarrow r_\phi$. Moreover, $r_t = r_\phi$ is not a turning point and the hypersurface can cross the point of $r = r_\phi$ to reach the horizon which is, indeed, the turning point in this case.

In order to calculate Γ , one distinguishes two different cases depending on whether $r_+ \leq r_\phi$ or $r_+ > r_\phi$. Indeed,

for sufficiently small charges, that is, $Q \leq Q_c$, where we are in the region of $r_+ \leq r_\phi$, the main contributions come from the metric and the effects of the charge are only due to the location of the horizon which is encoded in the metric's components. Indeed, in this case, the behavior of Γ is the same as the holographic geometric entropy.

On the other hand, as one increases the background charge so that $Q > Q_c$, one reaches the region $r_\phi < r_+$, where the effects of the background charge become important. In this region since there is no place where the hypersurface can end, the minimization procedure does not lead to the disconnected solution.

It is worth to mention that for $d \geq 3$ using the expression of Γ given in (31), we get exactly the same behavior as that in $d = 2$ discussed above which is, indeed, the same as what we have found in the previous section displayed in Figures 1 and 3.

On the other hand, using the expression (28) for $d \geq 3$, although qualitatively, we get the same behavior; a new feature appears when we change the ratio of R/r_+ . Of course as far as the effect of the gauge field is concerned, the situation remains unchanged. Namely, $\phi = 1$ sets a scale which controls the effects of the gauge field as before.

In order to explore the new feature, let us consider the situation where $0 \leq Q^2 \leq 1/2d(d-1)$ which corresponds to the case of $r_+ \leq r_\phi$. Note that in this region, the effect of the background field is irrelevant and indeed we could have done the same for the geometric entropy. To proceed, it is useful to study the behavior of $\Delta\Gamma$ which we will do by using a numerical method. By making use of a scaling, one may set $r_+ = 1$. It is important to note that unlike the black brane case where $\Delta\Gamma$ depends on R just through a trivial overall factor, in the present case, it appears in the function f , and therefore it may affect the behavior of the order parameter. To find the corresponding behavior numerically, we will fix the dimension and the charge, and therefore, we are left with a free parameter R which controls the behavior of the order parameter. Indeed, one observes that for R of order r_+ or bigger, the model undergoes a phase transition; However, for a sufficiently small R/r_+ , it exhibits no phase transition. More precisely, there is a critical R/r_+ that indicates whether the system exhibits a phase transition. In Figure 5, we have summarized the above discussions by plotting $\Delta\Gamma$ as a function of ℓ for different values of R .

4. Discussions

In this paper, we have introduced a quantity which is sensitive to the background fractionalized charge not only due to its effects in the components of the metric, but also directly from the gauge field. To explore its properties, we have explicitly computed the quantity for the RN black brane and a black hole in an asymptotically AdS geometry.

For sufficiently small charges, the metric plays the essential roles; while as one increases the charge, one would expect to see the effects of the gauge field. Indeed, following our definition in the quantity (6), there is natural scale over which the direct effects of gauge field become significant.

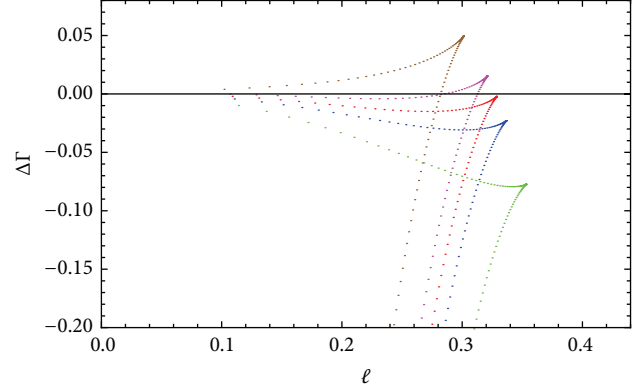


FIGURE 5: $\Delta\Gamma$ as a function of ℓ for $Q = 0.3$, $r_+ = 1$ and $R = 0.6, 0.66, 0.69, 0.72, 0.8$ which are shown by green, blue, red, magenta, and brown, respectively. Note that as far as $0 \leq Q \leq 0.5$, where $r_+ \leq r_\phi$, one gets qualitatively the same behavior.

To elaborate this point, it is illustrative to study the induced metric in more detail. To proceed, it is useful to recall the following identity:

$$\sqrt{\det(\tilde{g} + RF_{ab})} = [\det(\tilde{g}) \det(G)]^{1/4}, \quad (37)$$

where

$$G_{\mu\nu} = \tilde{g} + R^2 F_{\mu\rho} \tilde{g}^{\rho\sigma} F_{\sigma\nu}. \quad (38)$$

In our case, using the explicit expression for x' (e.g. obtained from (15)) the induced (Euclidean) metric may be recast in to the following form:

$$ds_{\text{ind}}^2 = \frac{R^2}{r^2} \left[f dt^2 + \left(\frac{f - f_t \phi^2 (r/r_t)^{2d}}{f - f_t (r/r_t)^{2d}} \right) \times \frac{dr^2}{f} + \sum_{i=1}^{d-2} dx_i^2 \right], \quad (39)$$

which shows that there is a horizon at $r = r_H$, as expected. On the other hand, for the metric $G_{\mu\nu}$, one finds

$$ds_{\text{open}}^2 = \frac{R^2}{r^2} \left\{ \frac{f(1-\phi^2)}{f - f_t \phi^2 (r/r_t)^{2d}} \times \left[f dt^2 + \left(\frac{f - f_t \phi^2 (r/r_t)^{2d}}{f - f_t (r/r_t)^{2d}} \right) \frac{dr^2}{f} \right] + \sum_{i=1}^{d-2} dx_i^2 \right\}, \quad (40)$$

which indicates a possibility of having a natural scale at $r = r_\phi$ where $\phi = 1$, though the original geometry is smooth at this point. Indeed, for $Q \leq Q_c$, one has $r_H \leq r_\phi$. Therefore, the scale r_ϕ is behind the horizon and does not play an essential role indicating that the main contributions come from the

metric. In fact in this case, the effect of the charge is only through the components of the metric which in turn may fix the position of the horizon, and indeed, qualitatively, the function Γ has the same behavior as the geometric entropy.

On the other hand, in the opposite limit when $Q > Q_c$ where one has $r_\phi \leq r_H$, the effect of background gauge field is important, and for a generic value of r_t , the solution is well defined if $0 \leq r_t < r_\phi$. Note that in this case, one gets a “bubble solution,” and therefore, the horizon cannot be probed. In this case, the behavior of the function Γ is still qualitatively the same as the geometric entropy, though since we are in the large charge limit, for fixed r_H , the corresponding dual theory should be at low temperature and therefore, it does not exhibit a phase transition.

Note also that for the special value of $r_t = r_\phi$, the metric (40) is well defined at $r = r_\phi$ and, indeed, it has a horizon at $r = r_H$.

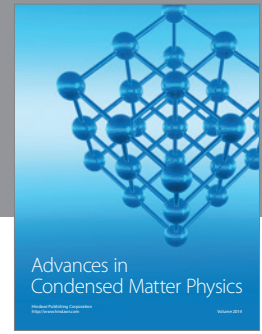
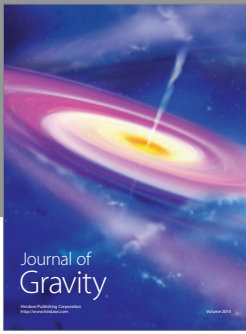
Probably the most interesting, but rather difficult, aspect of our study, is to find an interpretation for the quantity defined by (6) from dual field theory point of view. Of course we should admit that we do not have a good answer to this question, and indeed in this paper, we have considered this quantity as a parameter which could probe the system. It would be very interesting to find the corresponding interpretation from field theory point of view.

Acknowledgments

The authors would like to thank A. Davodi, M. M. Mohammadi Mozaffar, A. Mollabashi, A. E. Mosaffa, M. R. Tanhayi, and A. Vahedi for the useful discussions. They would also like to thank D. Tong for the comments on the draft of the paper.

References

- [1] J. Maldacena, “The large N limit of superconformal field theories and supergravity,” *Advances in Theoretical and Mathematical Physics*, vol. 2, no. 2, pp. 231–252, 1998.
- [2] E. Witten, “Anti-de Sitter space, thermal phase transition, and confinement in gauge theories,” *Advances in Theoretical and Mathematical Physics*, vol. 2, no. 3, pp. 505–532, 1998.
- [3] S. A. Hartnoll, “Horizons, holography and condensed matter,” Submitted, <http://arxiv.org/abs/1106.4324>.
- [4] T. Senthil, S. Sachdev, and M. Vojta, “Fractionalized fermi liquids,” *Physical Review Letters*, vol. 90, no. 21, Article ID 216403, 4 pages, 2003.
- [5] S. Sachdev, “Holographic metals and the fractionalized fermi liquid,” *Physical Review Letters*, vol. 105, no. 15, Article ID 151602, 2010.
- [6] L. Huijse and S. Sachdev, “Fermi surfaces and gauge-gravity duality,” *Physical Review D*, vol. 84, no. 2, Article ID 026001, 2011.
- [7] S. A. Hartnoll and D. Radicevic, “Holographic order parameter for charge fractionalization,” *Physical Review D*, vol. 86, Article ID 066001, 2012.
- [8] S. Ryu and T. Takayanagi, “Holographic derivation of entanglement entropy from the anti-de Sitter space/conformal field theory correspondence,” *Physical Review Letters*, vol. 96, no. 18, Article ID 181602, 4 pages, 2006.
- [9] S.-J. Rey and J.-T. Yee, “Macroscopic strings as heavy quarks: large- N gauge theory and anti-de Sitter supergravity,” *The European Physical Journal C*, vol. 22, no. 2, pp. 379–394, 2001.
- [10] J. Maldacena, “Wilson loops in large N field theories,” *Physical Review Letters*, vol. 80, no. 22, pp. 4859–4862, 1998.
- [11] M. Fujita, T. Nishioka, and T. Takayanagi, “Geometric entropy and Hagedorn/deconfinement transition,” *Journal of High Energy Physics*, vol. 2008, no. 9, article 18, 2008.
- [12] I. Bah, L. A. Pando Zayas, and C. A. Terrero-Escalante, “Holographic geometric entropy at finite temperature from black holes in global anti de sitter spaces,” *International Journal of Modern Physics A*, vol. 27, Article ID 1250048, 2012.
- [13] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, “Charged AdS black holes and catastrophic holography,” *Physical Review D*, vol. 60, no. 6, Article ID 064018, 17 pages, 1999.



Hindawi

Submit your manuscripts at
<http://www.hindawi.com>

