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Review Article

The General Hybrid Approximation Methods for Nonexpansive Mappings in Banach Spaces

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We introduce two general hybrid iterative approximation methods (one implicit and one explicit) for finding a fixed point of a nonexpansive mapping which solving the variational inequality generated by two strongly positive bounded linear operators. Strong convergence theorems of the proposed iterative methods are obtained in a reflexive Banach space which admits a weakly continuous duality mapping. The results presented in this paper improve and extend the corresponding results announced by Marino and Xu (2006), Wangkeeree et al. (in press), and Ceng et al. (2009).

1. Introduction

Let C be a nonempty subset of a normed linear space E. Recall that a mapping $T : C \to C$ is called *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in E. \tag{1.1}$$

We use $F(T)$ to denote the set of fixed points of T; that is, $F(T) = \{x \in E : Tx = x\}$. A self-mapping $f : E \to E$ is a contraction on E if there exists a constant $\alpha \in (0,1)$ and $x, y \in E$ such that

$$\|f(x) - f(y)\| \le \alpha\|x - y\|. \tag{1.2}$$

One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping [1–3]. More precisely, take $t \in (0,1)$ and define a contraction $T_t : E \to E$ by

$$T_t x = tu + (1 - t)Tx, \quad \forall x \in E, \tag{1.3}$$

where $u \in E$ is a fixed point. Banach's contraction mapping principle guarantees that T_t has a unique fixed point x_t in E. It is unclear, in general, what is the behavior of x_t as $t \to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Browder [1] proved that if E is a Hilbert space, then x_t converges strongly to a fixed point of T. Reich [2] extended Browder's result to the setting of Banach spaces and proved that if E is a uniformly smooth Banach space, then $\{x_t\}$ converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from E onto $F(T)$. Xu [3] proved Reich's results hold in reflexive Banach spaces which have a weakly continuous duality mapping.

The iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, for example, [4–7] and the references therein. Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let A be a strongly positive bounded linear operator on H: that is, there is a constant $\overline{\gamma} > 0$ with property

$$\langle Ax, x \rangle \geq \overline{\gamma} \|x\|^2, \quad \forall x \in H. \tag{1.4}$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H

$$\min_{x \in F(T)} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle, \tag{1.5}$$

where T is a nonexpansive mapping on H and b is a given point in H. In 2003, Xu [6] proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily

$$x_{n+1} = (I - \lambda_n A) T x_n + \lambda_n u, \quad n \geq 0 \tag{1.6}$$

converges strongly to the unique solution of the minimization problem (1.5) provided the sequence $\{\lambda_n\}$ satisfies certain conditions. Using the viscosity approximation method, Moudafi [8] introduced the following iterative iterative process for nonexpansive mappings (see [9, 10] for further developments in both Hilbert and Banach spaces). Let f be a contraction on H. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \lambda_n) T x_n + \lambda_n f(x_n), \quad n \geq 0, \tag{1.7}$$

where $\{\lambda_n\}$ is a sequence in $(0, 1)$. It is proved [8, 10] that under certain appropriate conditions imposed on $\{\lambda_n\}$, the sequence $\{x_n\}$ generated by (1.7) strongly converges to the unique solution x^* in C of the variational inequality

$$\langle (I - f) x^*, x - x^* \rangle \geq 0, \quad x \in H. \tag{1.8}$$

Recently, Marino and Xu [11] mixed the iterative method (1.6) and the viscosity approximation method (1.7) and considered the following general iterative method:

$$x_{n+1} = (I - \lambda_n A)Tx_n + \lambda_n \gamma f(x_n), \quad n \geq 0, \tag{1.9}$$

where A is a strongly positive bounded linear operator on H. They proved that if the sequence $\{\lambda_n\}$ of parameters satisfies the following appropriate conditions: $\lim_{n\to\infty}\lambda_n = 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and either $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ or $\lim_{n\to\infty}(\lambda_n/\lambda_{n+1}) = 1$, then the sequence $\{x_n\}$ generated by (1.9) converges strongly to the unique solution x^* in H of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in H, \tag{1.10}$$

which is the optimality condition for the minimization problem: $\min_{x\in F(T)}(1/2)\langle Ax, x\rangle - h(x)$, where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Very recently, Wangkeeree et al. [12] extended Marino and Xu's result to the setting of Banach spaces and obtained the strong convergence theorems in a reflexive Banach space which admits a weakly continuous duality mapping. Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$. Let $T : E \to E$ be a nonexpansive mapping with $F(T) \neq \emptyset$, f a contraction with coefficient $0 < \alpha < 1$ and A a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}\varphi(1)/\alpha$. Define the net $\{x_t\}$ by

$$x_t = t\gamma f(x_t) + (I - tA)Tx_t. \tag{1.11}$$

It is proved in [12] that $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point $\tilde{x}$ of T which solves the variational inequality

$$\langle (A - \gamma f)\tilde{x}, J_\varphi(\tilde{x} - z) \rangle \leq 0, \quad z \in F(T). \tag{1.12}$$

On the other hand, Ceng et al. [13] introduced the iterative approximation method for solving the variational inequality generated by two strongly positive bounded linear operators on a real Hilbert space H. Let $f : H \to H$ be a contraction with coefficient $0 < \alpha < 1$, and let $A, B : H \to H$ be two strongly positive bounded linear operators with coefficient $\overline{\gamma} \in (0,1)$ and $\beta > 0$, respectively. Assume that $0 < \gamma\alpha < \beta$, $\{\lambda_n\}$ is a sequence in $(0,1)$, $\{\mu_n\}$ is a sequence in $(0, \min\{1, \|B\|^{-1}\})$. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \lambda_n A)Tx_n + \lambda_{n+1}\left[Tx_n - \mu_{n+1}\left(BTx_n - \gamma f(x_n)\right)\right], \quad n \geq 0. \tag{1.13}$$

It is proved in [13, Theorem 3.1] that if the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ satisfy the following conditions:

(C1) $\lim_{n\to\infty}\lambda_n = 0$,

(C2) $\sum_{n=1}^{\infty} \lambda_n = \infty$,

(C3) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ or $\lim_{n\to\infty}(\lambda_n/\lambda_{n+1}) = 1$,

(C4) $(1 - \overline{\gamma})/(\beta - \gamma\alpha) < \lim_{n\to\infty}\mu_n = \mu < (2 - \overline{\gamma})/(\beta - \gamma\alpha)$, then the sequence $\{x_n\}$ generated by (1.13) converges strongly to the unique solution $\widetilde{x}$ in H of the variational inequality

$$\langle (A - I + \mu(B - \gamma f))\widetilde{x}, \widetilde{x} - z \rangle \leq 0, \quad z \in F(T). \tag{1.14}$$

Observe that if $B = I$ and $\mu_n = 1$ for all $n \geq 1$, then algorithm (1.13) reduces to (1.9). Moreover, the variational inequality (1.14) reduces to (1.10). Furthermore, the applications of these results to constrained generalized pseudoinverse are studied.

In this paper, motivated by Marino and Xu [11], Wangkeeree et al. [12], and Ceng et al. [13], we introduce two general iterative approximation methods (one implicit and one explicit) for finding a fixed point of a nonexpansive mapping which solving the variational inequality generated by two strongly positive bounded linear operators. Strong convergence theorems of the proposed iterative methods are obtained in a reflexive Banach space which admits a weakly continuous duality mapping. The results presented in this paper improve and extend the corresponding results announced by Marino and Xu [11], Wangkeeree et al. [12], and Ceng et al. [13], and many others.

2. Preliminaries

Throughout this paper, let E be a real Banach space and E^* its dual space. We write $x_n \rightharpoonup x$ (resp. $x_n \rightharpoonup^* x$) to indicate that the sequence $\{x_n\}$ weakly (resp. weak*) converges to x; as usual, $x_n \to x$ will symbolize strong convergence. Let $U_E = \{x \in E : \|x\| = 1\}$. A Banach space E is said to *uniformly convex* if, for any $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that, for any $x, y \in U_E$, $\|x - y\| \geq \varepsilon$ implies $\|(x+y)/2\| \leq 1 - \delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex (see also [14]). A Banach space E is said to be *smooth* if the limit $\lim_{t\to 0}(\|x + ty\| - \|x\|)/t$ exists for all $x, y \in U_E$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in U_E$.

By a gauge function φ, we mean a continuous strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Let E^* be the dual space of E. The duality mapping $J_\varphi : E \to 2^{E^*}$ associated to a gauge function φ is defined by

$$J_\varphi(x) = \{ f^* \in E^* : \langle x, f^* \rangle = \|x\|\varphi(\|x\|), \|f^*\| = \varphi(\|x\|) \}, \quad \forall x \in E. \tag{2.1}$$

In particular, the duality mapping with the gauge function $\varphi(t) = t$, denoted by J, is referred to as the normalized duality mapping. Clearly, there holds the relation $J_\varphi(x) = (\varphi(\|x\|)/\|x\|)J(x)$ for all $x \neq 0$ (see [15]). Browder [15] initiated the study of certain classes of nonlinear operators by means of the duality mapping J_φ. Following Browder [15], we say that a Banach space E has a *weakly continuous duality mapping* if there exists a gauge φ for which the duality mapping $J_\varphi(x)$ is single valued and continuous from the weak topology to the weak* topology; that is, for any $\{x_n\}$ with $x_n \rightharpoonup x$, the sequence $\{J_\varphi(x_n)\}$ converges weakly* to $J_\varphi(x)$. It is known that l^p has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$ for all $1 < p < \infty$. Set

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau, \quad \forall t \geq 0, \tag{2.2}$$

then

$$J_{\varphi}(x) = \partial\Phi(\|x\|), \quad \forall x \in E, \tag{2.3}$$

where ∂ denotes the subdifferential in the sense of convex analysis.

Now, we collect some useful lemmas for proving the convergence result of this paper.

The first part of the next lemma is an immediate consequence of the subdifferential inequality, and the proof of the second part can be found in [16].

Lemma 2.1 (see [16]). *Assume that a Banach space E has a weakly continuous duality mapping J_{φ} with gauge φ.*

(i) *For all $x, y \in E$, the following inequality holds:*

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, J_{\varphi}(x+y)\rangle. \tag{2.4}$$

In particular, for all $x, y \in E$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle. \tag{2.5}$$

(ii) *Assume that a sequence $\{x_n\}$ in E converges weakly to a point $x \in E$.*
Then, the following identity holds:

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y-x\|), \quad \forall x, y \in E. \tag{2.6}$$

Lemma 2.2 (see [7]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + b_n, \tag{2.7}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{b_n\}$ is a sequence such that

(a) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(b) $\limsup_{n\to\infty} b_n/\alpha_n \leq 0$ *or* $\sum_{n=1}^{\infty} |b_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

In a Banach space E having a weakly continuous duality mapping J_{φ} with a gauge function φ, an operator A is said to be *strongly positive* [12] if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, J_{\varphi}(x)\rangle \geq \overline{\gamma}\|x\|\varphi(\|x\|), \tag{2.8}$$

$$\|\alpha I - \beta A\| = \sup_{\|x\|\leq 1} |\langle(\alpha I - \beta A)x, J_{\varphi}(x)\rangle|, \quad \alpha \in [0,1], \ \beta \in [-1,1], \tag{2.9}$$

where I is the identity mapping. If $E := H$ is a real Hilbert space, then (2.8) reduces to (1.4). The next valuable lemma can be found in [12].

Lemma 2.3 (see [12, Lemma 3.1]). *Assume that a Banach space E has a weakly continuous duality mapping J_φ with gauge φ. Let A be a strongly positive bounded linear operator on E with coefficient $\overline{\gamma} > 0$ and $0 < \rho \le \varphi(1)\|A\|^{-1}$. Then, $\|I - \rho A\| \le \varphi(1)(1 - \rho\overline{\gamma})$.*

3. Main Results

Now, we are a position to state and prove our main results.

Lemma 3.1. *Let E be a Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$; that is, $T([0,1]) \subset [0,1]$. Let $T : E \to E$ be a nonexpansive mapping and $f : E \to E$ a contraction with coefficient $\alpha \in (0,1)$. Let A and B be two strongly positive bounded linear operators with coefficients $\overline{\gamma} > 0$ and $\beta > 0$, respectively. Let γ and μ be two constants satisfying the condition (C^*)*

$$(C^*) : 0 < \gamma < \frac{\beta\varphi(1)}{\alpha}, \qquad \frac{\varphi(1) - \varphi(1)\overline{\gamma}}{\varphi(1)\beta - \gamma\alpha} < \mu \le \min\left\{1, \varphi(1)\|B\|^{-1}, \frac{1 + \varphi(1) - \varphi(1)\overline{\gamma}}{\varphi(1)\beta - \gamma\alpha}\right\}. \tag{3.1}$$

Then, for any $\lambda \in (0, \min\{1, \varphi(1)\|A\|^{-1}\})$, the mapping $S_\lambda : E \to E$ defined by

$$S_\lambda(x) = (I - \lambda A)Tx + \lambda\left[Tx - \mu\left(BTx - \gamma f(x)\right)\right], \quad \forall x \in E \tag{3.2}$$

is a contraction with coefficient $1 - \lambda\tau$, where $\tau := \varphi(1)\overline{\gamma} - \varphi(1) + \mu(\varphi(1)\beta - \gamma\alpha)$.

Proof. Observe that

$$\begin{aligned}
\mu \le \frac{1 + \varphi(1) - \varphi(1)\overline{\gamma}}{\varphi(1)\beta - \gamma\alpha} &\Longleftrightarrow \mu\left(\varphi(1)\beta - \gamma\alpha\right) \le 1 + \varphi(1) - \varphi(1)\overline{\gamma} \\
&\Longleftrightarrow \varphi(1)\overline{\gamma} - \varphi(1) + \mu\left(\varphi(1)\beta - \gamma\alpha\right) \le 1, \\
\frac{\varphi(1) - \varphi(1)\overline{\gamma}}{\varphi(1)\beta - \gamma\alpha} < \mu &\Longleftrightarrow \varphi(1) - \varphi(1)\overline{\gamma} < \mu\left(\varphi(1)\beta - \gamma\alpha\right) \\
&\Longleftrightarrow 0 < \varphi(1)\overline{\gamma} - \varphi(1) + \mu\left(\varphi(1)\beta - \gamma\alpha\right).
\end{aligned} \tag{3.3}$$

This shows that $\tau := \varphi(1)\overline{\gamma} - \varphi(1) + \mu(\varphi(1)\beta - \gamma\alpha) \in (0,1]$. Using Lemma 2.3, we obtain

$$\begin{aligned}
&\|S_\lambda(x) - S_\lambda(y)\| \\
&\quad = \|(I-\lambda A)Tx + \lambda[Tx - \mu(BTx - \gamma f(x))] - (I-\lambda A)Ty - \lambda[Ty - \mu(BTy - \gamma f(y))]\| \\
&\quad \le \|(I-\lambda A)Tx - (I-\lambda A)Ty\| + \lambda\|Tx - \mu(BTx - \gamma f(x)) - [Ty - \mu(BTy - \gamma f(y))]\| \\
&\quad \le \|I-\lambda A\|\|Tx - Ty\| + \lambda[\|(I-\mu B)Tx - (I-\mu B)Ty\| + \gamma\mu\|f(x) - f(y)\|] \\
&\quad \le \|I-\lambda A\|\|Tx - Ty\| + \lambda[\|I-\mu B\|\|Tx - Ty\| + \gamma\mu\|f(x) - f(y)\|] \\
&\quad \le \varphi(1)(1-\lambda\overline{\gamma})\|x-y\| + \lambda[\varphi(1)(1-\mu\beta)\|x-y\| + \gamma\mu\alpha\|x-y\|] \\
&\quad = [\varphi(1)(1-\lambda\overline{\gamma}) + \lambda[\varphi(1)(1-\mu\beta) + \gamma\mu\alpha]]\|x-y\| \\
&\quad = [\varphi(1)(1-\lambda\overline{\gamma}) + \lambda[\varphi(1) - \mu(\varphi(1)\beta - \gamma\alpha)]]\|x-y\| \\
&\quad = [\varphi(1) - \lambda[\varphi(1)\overline{\gamma} - \varphi(1) + \mu(\varphi(1)\beta - \gamma\alpha)]]\|x-y\| \\
&\quad = (\varphi(1) - \lambda\tau)\|x-y\| \\
&\quad \le (1-\lambda\tau)\|x-y\|.
\end{aligned} \tag{3.4}$$

Hence, S_λ is a contraction with coefficient $1-\lambda\tau$. □

Applying the Banach contraction principle to Lemma 3.1, there exists a unique fixed point x_λ of S_λ in E; that is,

$$x_\lambda = (I-\lambda A)Tx_\lambda + \lambda[Tx_\lambda - \mu(BTx_\lambda - \gamma f(x_\lambda))], \quad \forall\lambda \in (0,1). \tag{3.5}$$

Remark 3.2. For each $1 < p < \infty$, l^p space has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$ which is invariant on $[0,1]$.

Theorem 3.3. *Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$. Let $T : E \to E$ be a nonexpansive mapping with $F(T) \neq \emptyset$, $f : E \to E$ a contraction with coefficient $\alpha \in (0,1)$, and A, B two strongly positive bounded linear operators with coefficients $\overline{\gamma} > 0$ and $\beta > 0$, respectively. Let γ and μ be two constants satisfying the condition (C^*). Then, the net $\{x_\lambda\}$ defined by (3.5) converges strongly as $\lambda \to 0$ to a fixed point $\widetilde{x}$ of T which solves the variational inequality*

$$\langle (A - I + \mu(B - \gamma f))\widetilde{x}, J_\varphi(\widetilde{x} - z)\rangle \le 0, \quad z \in F(T). \tag{3.6}$$

Proof. We first show that the uniqueness of a solution of the variational inequality (3.6). Suppose that both $\widetilde{x} \in F(T)$ and $x^* \in F(T)$ are solutions to (3.6), then

$$\begin{aligned}
&\langle (A - I + \mu(B - \gamma f))\widetilde{x}, J_\varphi(\widetilde{x} - x^*)\rangle \le 0, \\
&\langle (A - I + \mu(B - \gamma f))x^*, J_\varphi(x^* - \widetilde{x})\rangle \le 0.
\end{aligned} \tag{3.7}$$

Adding (3.7), we obtain

$$\langle (A - I + \mu(B - \gamma f))\widetilde{x} - (A - I + \mu(B - \gamma f))x^*, J_\varphi(\widetilde{x} - x^*)\rangle \le 0. \tag{3.8}$$

On the other hand, we observe that

$$\begin{aligned}\frac{\varphi(1)-\varphi(1)\overline{\gamma}}{\varphi(1)\beta-\gamma\alpha}<\mu &\Longleftrightarrow \varphi(1)-\varphi(1)\overline{\gamma}<\mu(\varphi(1)\beta-\gamma\alpha)\\ &\Longleftrightarrow 1-\overline{\gamma}<\mu\left(\beta-\frac{\gamma\alpha}{\varphi(1)}\right)\\ &\Longleftrightarrow 0<\overline{\gamma}-1+\mu\left(\beta-\frac{\gamma\alpha}{\varphi(1)}\right).\end{aligned}\tag{3.9}$$

It then follows that for any $x, y \in E$,

$$\begin{aligned}&\langle (A-I+\mu(B-\gamma f))x-(A-I+\mu(B-\gamma f))y, J_\varphi(x-y)\rangle\\ &\quad=\langle A(x-y)-(x-y)+\mu[(B-\gamma f)x-(B-\gamma f)y], J_\varphi(x-y)\rangle\\ &\quad=\langle A(x-y), J_\varphi(x-y)\rangle-\langle x-y, J_\varphi(x-y)\rangle\\ &\qquad+\ \mu\langle (B-\gamma f)x-(B-\gamma f)y, J_\varphi(x-y)\rangle\\ &\quad\geq \overline{\gamma}\|x-y\|\varphi(\|x-y\|)-\|x-y\|\varphi(\|x-y\|)+\mu\langle B(x-y), J_\varphi(x-y)\rangle\\ &\qquad-\mu\gamma\langle f(x)-f(y), J_\varphi(x-y)\rangle\\ &\quad\geq \overline{\gamma}\|x-y\|\varphi(\|x-y\|)-\|x-y\|\varphi(\|x-y\|)+\mu\beta\|x-y\|\varphi(\|x-y\|)\\ &\qquad-\mu\gamma\|f(x)-f(y)\|\|J_\varphi(x-y)\|\\ &\quad\geq \overline{\gamma}\Phi(\|x-y\|)-\Phi(\|x-y\|)+\mu\beta\Phi(\|x-y\|)-\mu\gamma\alpha\Phi(\|x-y\|)\\ &\quad=(\overline{\gamma}-1+\mu\beta-\mu\gamma\alpha)\overline{\gamma}\Phi(\|x-y\|)\\ &\quad=(\overline{\gamma}-1+\mu(\beta-\gamma\alpha))\overline{\gamma}\Phi(\|x-y\|)\\ &\quad\geq\left(\overline{\gamma}-1+\mu\left(\beta-\frac{\gamma\alpha}{\varphi(1)}\right)\right)\overline{\gamma}\Phi(\|x-y\|)\geq 0.\end{aligned}\tag{3.10}$$

Applying (3.10) to (3.8), we obtain that $\tilde{x}=x^*$ and the uniqueness is proved. Below, we use $\tilde{x}$ to denote the unique solution of (3.6). Next, we will prove that $\{x_\lambda\}$ is bounded. Take a $p\in F(T)$, and denote the mapping S_λ by

$$S_\lambda := (I-\lambda A)T+\lambda[T-\mu(BT-\gamma f)], \quad \forall\lambda\in(0,1).\tag{3.11}$$

From Lemma 3.1, we have

$$\begin{aligned}\|x_\lambda-p\| &\leq \|S_\lambda x_\lambda - S_\lambda p\|+\|S_\lambda p-p\|\\ &\leq (1-\lambda\tau)\|x_\lambda-p\|+\|(I-\lambda A)Tp+\lambda[Tp-\mu(BTp-\gamma fp)]-p\|\\ &= (1-\lambda\tau)\|x_\lambda-p\|+\lambda\|-Ap+p-\mu(Bp-\gamma fp)\|\\ &\leq (1-\lambda\tau)\|x_\lambda-p\|+\lambda[\|I-A\|\|p\|+\mu\|Bp-\gamma fp\|],\end{aligned}\tag{3.12}$$

where $\tau := \varphi(1)\overline{\gamma} - \varphi(1) + \mu(\varphi(1)\beta - \gamma\alpha) \in (0,1]$. It follows that

$$\|x_\lambda - p\| \le \frac{1}{\tau}\left[\|I - A\|\|p\| + \mu\|Bp - \gamma f p\|\right]. \tag{3.13}$$

Hence, $\{x_\lambda\}$ is bounded, so are $\{f(x_\lambda)\}$, $\{AT(x_\lambda)\}$ and $\{BT(x_\lambda)\}$. The definition of $\{x_\lambda\}$ implies that

$$\|x_\lambda - Tx_\lambda\| = \lambda\left\|Tx_\lambda - \mu\left(BTx_\lambda - \gamma f(x_\lambda)\right) - ATx_\lambda\right\| \longrightarrow 0, \quad \text{as } \lambda \longrightarrow 0. \tag{3.14}$$

If follows from reflexivity of E and the boundedness of sequence $\{x_\lambda\}$ that there exists $\{x_{\lambda_n}\}$ which is a subsequence of $\{x_\lambda\}$ converging weakly to $w \in E$ as $n \to \infty$. Since J_φ is weakly sequentially continuous, we have by Lemma 2.1 that

$$\limsup_{n\to\infty} \Phi(\|x_{\lambda_n} - x\|) = \limsup_{n\to\infty} \Phi(\|x_{\lambda_n} - w\|) + \Phi(\|x - w\|), \quad \forall x \in E. \tag{3.15}$$

Let

$$H(x) = \limsup_{n\to\infty} \Phi(\|x_{\lambda_n} - x\|), \quad \forall x \in E. \tag{3.16}$$

It follows that

$$H(x) = H(w) + \Phi(\|x - w\|), \quad \forall x \in E. \tag{3.17}$$

Since

$$\|x_{\lambda_n} - Tx_{\lambda_n}\| = \lambda_n\left\|Tx_{\lambda_n} - \mu\left(BTx_{\lambda_n} - \gamma f(x_{\lambda_n})\right) - ATx_{\lambda_n}\right\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.18}$$

We obtain

$$\begin{aligned} H(Tw) &= \limsup_{n\to\infty} \Phi(\|x_{\lambda_n} - Tw\|) = \limsup_{n\to\infty} \Phi(\|Tx_{\lambda_n} - Tw\|) \\ &\le \limsup_{n\to\infty} \Phi(\|x_{\lambda_n} - w\|) = H(w). \end{aligned} \tag{3.19}$$

On the other hand, however,

$$H(Tw) = H(w) + \Phi(\|T(w) - w\|). \tag{3.20}$$

It follows from (3.19) and (3.20) that

$$\Phi(\|T(w) - w\|) = H(Tw) - H(w) \le 0, \tag{3.21}$$

which gives us, $Tw = w$. Next, we show that $x_{\lambda_n} \to w$ as $n \to \infty$. In fact, since $\Phi(t) = \int_0^t \varphi(\tau)d\tau$, $\forall\ t \geq 0$ and $\varphi : [0,\infty) \to [0,\infty)$ is a gauge function, then for $1 \geq k \geq 0$, $\varphi(kx) \leq \varphi(x)$ and

$$\Phi(kt) = \int_0^{kt} \varphi(\tau)d\tau = k\int_0^t \varphi(kx)dx \leq k\int_0^t \varphi(x)dx = k\Phi(t). \tag{3.22}$$

Following Lemma 2.1, we have

$$\begin{aligned}
&\Phi(\|x_{\lambda_n} - w\|)\\
&\quad= \Phi\left(\left\|(I-\lambda_n A)Tx_{\lambda_n} + \lambda_n\left[Tx_{\lambda_n} - \mu\left(BTx_{\lambda_n} - \gamma f(x_{\lambda_n})\right)\right] - (I-\lambda_n A)w - \lambda_n Aw\right\|\right)\\
&\quad\leq \Phi(\|(I-\lambda_n A)Tx_{\lambda_n} - (I-\lambda_n A)w\|)\\
&\qquad+\ \lambda_n\left\langle Tx_{\lambda_n} - \mu\left(BTx_{\lambda_n} - \gamma f(x_{\lambda_n})\right) - Aw, J_\varphi(x_{\lambda_n} - w)\right\rangle\\
&\quad\leq \Phi\left(\varphi(1)(1-\lambda_n\overline{\gamma})\|x_{\lambda_n} - w\|\right)\\
&\qquad+\ \lambda_n\left\langle (I-\mu B)Tx_{\lambda_n} + \mu\gamma f(x_{\lambda_n}) - Aw, J_\varphi(x_{\lambda_n} - w)\right\rangle\\
&\quad\leq \varphi(1)(1-\lambda_n\overline{\gamma})\Phi(\|x_{\lambda_n} - w\|)\\
&\qquad+\ \lambda_n\left\langle (I-\mu B)Tx_{\lambda_n} - (I-\mu B)w + \mu\gamma f(x_{\lambda_n}) - \mu\gamma f(w), J_\varphi(x_{\lambda_n} - w)\right\rangle\\
&\qquad+\ \lambda_n\left\langle (I-\mu B)w + \mu\gamma f(w) - Aw, J_\varphi(x_{\lambda_n} - w)\right\rangle\\
&\quad\leq \varphi(1)(1-\lambda_n\overline{\gamma})\Phi(\|x_{\lambda_n} - w\|) + \lambda_n\left\langle (I-\mu B)Tx_{\lambda_n} - (I-\mu B)w, J_\varphi(x_{\lambda_n} - w)\right\rangle\\
&\qquad+\ \lambda_n\mu\gamma\left\langle f(x_{\lambda_n}) - f(w), J_\varphi(x_{\lambda_n} - w)\right\rangle + \lambda_n\left\langle (I-\mu B)w + \mu\gamma f(w) - Aw, J_\varphi(x_{\lambda_n} - w)\right\rangle\\
&\quad\leq \varphi(1)(1-\lambda_n\overline{\gamma})\Phi(\|x_{\lambda_n} - w\|) + \lambda_n\left\|(I-\mu B)Tx_{\lambda_n} - (I-\mu B)w\right\|\left\|J_\varphi(x_{\lambda_n} - w)\right\|\\
&\qquad+\ \lambda_n\mu\gamma\left\|f(x_{\lambda_n}) - f(w)\right\|\left\|J_\varphi(x_{\lambda_n} - w)\right\| + \lambda_n\left\langle (I-\mu B)w + \mu\gamma f(w) - Aw, J_\varphi(x_{\lambda_n} - w)\right\rangle\\
&\quad\leq \varphi(1)(1-\lambda_n\overline{\gamma})\Phi(\|x_{\lambda_n} - w\|) + \lambda_n\varphi(1)(1-\mu\beta)\|x_{\lambda_n} - w\|\left\|J_\varphi(x_{\lambda_n} - w)\right\|\\
&\qquad+\ \lambda_n\mu\gamma\alpha\|x_{\lambda_n} - w\|\left\|J_\varphi(x_{\lambda_n} - w)\right\| + \lambda_n\left\langle (I-\mu B)w + \mu\gamma f(w) - Aw, J_\varphi(x_{\lambda_n} - w)\right\rangle\\
&\quad= \left[\varphi(1)(1-\lambda_n\overline{\gamma}) + \lambda_n\left(\varphi(1)(1-\mu\beta) + \mu\gamma\alpha\right)\right]\Phi(\|x_{\lambda_n} - w\|)\\
&\qquad+\ \lambda_n\left\langle (I-\mu B)w + \mu\gamma f(w) - Aw, J_\varphi(x_{\lambda_n} - w)\right\rangle\\
&\quad= \left[\varphi(1) - \lambda_n\left(\varphi(1)\overline{\gamma} - \varphi(1) + \mu\left(\varphi(1)\beta - \gamma\alpha\right)\right)\right]\Phi(\|x_{\lambda_n} - w\|)\\
&\qquad+\ \lambda_n\left\langle (I-\mu B)w + \mu\gamma f(w) - Aw, J_\varphi(x_{\lambda_n} - w)\right\rangle\\
&\quad\leq \left[1 - \lambda_n\left(\varphi(1)\overline{\gamma} - \varphi(1) + \mu\left(\varphi(1)\beta - \gamma\alpha\right)\right)\right]\Phi(\|x_{\lambda_n} - w\|)\\
&\qquad+\ \lambda_n\left\langle (I-\mu B)w + \mu\gamma f(w) - Aw, J_\varphi(x_{\lambda_n} - w)\right\rangle.
\end{aligned} \tag{3.23}$$

Thus,

$$\Phi(\|x_{\lambda_n} - w\|) \leq \frac{1}{\varphi(1)\overline{\gamma} - \varphi(1) + \mu(\varphi(1)\beta - \gamma\alpha)} \langle (I - \mu B)w + \mu\gamma f(w) - Aw, J_\varphi(x_{\lambda_n} - w) \rangle. \tag{3.24}$$

Now, observing that $x_{\lambda_n} \rightharpoonup w$ implies $J_\varphi(x_{\lambda_n} - w) \rightharpoonup 0$, we conclude from the last inequality that

$$\Phi(\|x_{\lambda_n} - w\|) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.25}$$

Hence, $x_{\lambda_n} \to w$ as $n \to \infty$. Next, we prove that w solves the variational inequality (3.6). For any $z \in F(T)$, we observe that

$$\begin{aligned}
\langle (I - T)x_\lambda - (I - T)z, J_\varphi(x_\lambda - z) \rangle &= \langle x_t - z, J_\varphi(x_\lambda - z) \rangle + \langle Tx_t - Tz, J_\varphi(x_\lambda - z) \rangle \\
&= \Phi(\|x_\lambda - z\|) - \langle Tz - Tx_t, J_\varphi(x_\lambda - z) \rangle \\
&\geq \Phi(\|x_\lambda - z\|) - \|Tz - Tx_t\| \|J_\varphi(x_\lambda - z)\| \\
&\geq \Phi(\|x_\lambda - z\|) - \|z - x_t\| \|J_\varphi(x_\lambda - z)\| \\
&= \Phi(\|x_\lambda - z\|) - \Phi(\|x_\lambda - z\|) = 0.
\end{aligned} \tag{3.26}$$

Since

$$x_\lambda = (I - \lambda_n A)Tx_{\lambda_n} + \lambda_n [Tx_{\lambda_n} - \mu(BTx_{\lambda_n} - \gamma f(x_{\lambda_n}))], \tag{3.27}$$

we can derive that

$$\begin{aligned}
&\lambda_n [Ax_{\lambda_n} - (I - \mu B)x_{\lambda_n}] \\
&\quad = (I - \lambda_n A)Tx_{\lambda_n} - (I - \lambda_n A)x_{\lambda_n} + \lambda_n (I - \mu B)Tx_{\lambda_n} - \lambda_n (I - \mu B)x_{\lambda_n} + \lambda_n \gamma f(x_{\lambda_n}).
\end{aligned} \tag{3.28}$$

That is

$$[A - I + \mu(B - \gamma f)]x_{\lambda_n} = -\frac{1}{\lambda_n} [(I - \lambda_n A)(I - T)x_{\lambda_n} + \lambda_n (I - \mu B)(I - T)x_{\lambda_n}]. \tag{3.29}$$

Using (3.26), for each $p \in F(T)$, we have

$$\begin{aligned}
&\left\langle [A - I + \mu(B - \gamma f)] x_{\lambda_n}, J_\varphi(x_{\lambda_n} - p) \right\rangle \\
&\quad = -\frac{1}{\lambda_n}\left[\left\langle (I - \lambda_n A)(I - T)x_{\lambda_n}, J_\varphi(x_{\lambda_n} - p)\right\rangle + \lambda_n \left\langle (I - \mu B)(I - T)x_{\lambda_n}, J_\varphi(x_{\lambda_n} - p)\right\rangle\right] \\
&\quad = -\frac{1}{\lambda_n}\left\langle (I - T)x_{\lambda_n} - (I - T)p, J_\varphi(x_{\lambda_n} - p)\right\rangle + \left\langle A(I - T)x_{\lambda_n}, J_\varphi(x_{\lambda_n} - p)\right\rangle \\
&\qquad - \left\langle (I - T)x_{\lambda_n} - (I - T)p, J_\varphi(x_{\lambda_n} - p)\right\rangle + \mu\left\langle B(I - T)x_{\lambda_n}, J_\varphi(x_{\lambda_n} - p)\right\rangle \\
&\quad \le \left\langle A(I - T)x_{\lambda_n}, J_\varphi(x_{\lambda_n} - p)\right\rangle + \left\langle B(I - T)x_{\lambda_n}, J_\varphi(x_{\lambda_n} - p)\right\rangle \\
&\quad \le \|A\| \|x_{\lambda_n} - Tx_{\lambda_n}\| \left\| J_\varphi(x_{\lambda_n} - p)\right\| + \mu\|B\| \|x_{\lambda_n} - Tx_{\lambda_n}\| \left\| J_\varphi(x_{\lambda_n} - p)\right\| \\
&\quad \le \|x_{\lambda_n} - Tx_{\lambda_n}\| M,
\end{aligned} \tag{3.30}$$

where M is a constant satisfying $M \ge \sup_{n\ge1}\{\|A\|\|J_\varphi(x_{\lambda_n} - p)\|, \mu\|B\|\|J_\varphi(x_{\lambda_n} - p)\|\}$. Noticing that

$$x_{\lambda_n} - Tx_{\lambda_n} \longrightarrow w - T(w) = w - w = 0. \tag{3.31}$$

It follows from (3.30) that

$$\left\langle (A - I + \mu(B - \gamma f))w, J_\varphi(w - p) \right\rangle \le 0. \tag{3.32}$$

So, $w \in F(T)$ is a solution of the variational inequality (3.6), and hence, $w = \tilde{x}$ by the uniqueness. In a summary, we have shown that each cluster point of $\{x_\lambda\}$(at $\lambda \to 0$) equals $\tilde{x}$. Therefore, $x_\lambda \to \tilde{x}$ as $\lambda \to 0$. This completes the proof. □

According to the definition of strongly positive operator A in a Banach space E having a weakly continuous duality mapping J_φ with a gauge function φ, an operator A is said to be *strongly positive* [12] if there exists a constant $\overline{\gamma} > 0$ with the property

$$\begin{gathered}
\left\langle Ax, J_\varphi(x) \right\rangle \ge \overline{\gamma}\|x\|\varphi(\|x\|), \\
\left\|\alpha I - \beta A\right\| = \sup_{\|x\|\le1}\left|\left\langle (\alpha I - \beta A)x, J_\varphi(x)\right\rangle\right|, \quad \alpha \in [0,1],\ \beta \in [-1,1],
\end{gathered} \tag{3.33}$$

where I is the identity mapping. We may assume, without loss of generality, that $\overline{\gamma} < 1$. Therefore, if $0 < \gamma < \overline{\gamma}\varphi(1)/\alpha$, then we have the Corollary 3.4 immediately. Indeed, putting $B = I$ and $\beta = 1$, we have

$$\frac{\varphi(1) - \varphi(1)\overline{\gamma}}{\varphi(1)\beta - \gamma\alpha} = \frac{\varphi(1) - \varphi(1)\overline{\gamma}}{\varphi(1) - \gamma\alpha} < 1 < \frac{1 + \varphi(1) - \varphi(1)\overline{\gamma}}{\varphi(1) - \gamma\alpha} = \frac{1 + \varphi(1) - \varphi(1)\overline{\gamma}}{\varphi(1)\beta - \gamma\alpha}. \tag{3.34}$$

Taking $\mu \equiv 1$ in Theorem 3.3, we obtain the following result.

Corollary 3.4 (see [12, Lemma 3.3]). *Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$. Let $T : E \to E$ be a nonexpansive mapping with $F(T) \neq \emptyset$, $f : E \to E$ a contraction with coefficient $\alpha \in (0,1)$, and A a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}\varphi(1)/\alpha$. Then, the net $\{x_\lambda\}$ defined by*

$$x_\lambda = (I - \lambda A)Tx_\lambda + \lambda\gamma f(x_\lambda), \tag{3.35}$$

converges strongly as $\lambda \to 0$ to a fixed point $\widetilde{x}$ of T which solves the variational inequality:

$$\left\langle (A - \gamma f)\widetilde{x}, J_\varphi(\widetilde{x} - z)\right\rangle \leq 0, \quad z \in F(T). \tag{3.36}$$

Corollary 3.5 (see [11, Theorem 3.6]). *Let H be a real Hilbert space. Let $T : H \to H$ be a nonexpansive mapping with $F(T) \neq \emptyset$, $f : H \to H$ a contraction with coefficient $\alpha \in (0,1)$, and A a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$. Then, the net $\{x_\lambda\}$ defined by*

$$x_\lambda = (I - \lambda A)Tx_\lambda + \lambda\gamma f(x_\lambda), \tag{3.37}$$

converges strongly as $\lambda \to 0$ to a fixed point $\widetilde{x}$ of T which solves the variational inequality

$$\left\langle (A - \gamma f)\widetilde{x}, \widetilde{x} - z\right\rangle \leq 0, \quad z \in F(T). \tag{3.38}$$

Theorem 3.6. *Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$. Let $T : E \to E$ be a nonexpansive mapping with $F(T) \neq \emptyset$, $f : E \to E$ a contraction with coefficient $\alpha \in (0,1)$, and A and B two strongly positive bounded linear operators with coefficients $\overline{\gamma} > 0$ and $\beta > 0$, respectively. Let $x_0 \in E$ be arbitrary and the sequence $\{x_n\}$ generated by the following iterative scheme:*

$$x_{n+1} = (I - \lambda_n A)Tx_n + \lambda_n\left[Tx_n - \mu(BTx_n - \gamma f(x_n))\right], \quad \forall n \geq 0, \tag{3.39}$$

where γ and μ are two constants satisfying the condition (C*) *and $\{\lambda_n\}$ is a real sequence in* $(0,1)$ *satisfying the following conditions:*

- (C1) $\lim_{n\to\infty}\lambda_n = 0$ *and* $\sum_{n=1}^{\infty}\lambda_n = \infty$,
- (C2) $\sum_{n=1}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty$ *or* $\lim_{n\to\infty}(\lambda_n/\lambda_{n+1}) = 1$.

Then, the sequence $\{x_n\}$ defined by (3.39) *converges strongly to a fixed point $\widetilde{x}$ of T that is obtained by Theorem 3.3.*

Proof. We first prove that $\{x_n\}$ is bounded. Take a $p \in F(T)$, and denote

$$S_{\lambda_n} := (I - \lambda_n A)T + \lambda_n\left[T - \mu(BT - \gamma f)\right]. \tag{3.40}$$

Using Lemma 3.1, we have

$$\begin{aligned}\|x_{n+1}-p\| &\le \|S_{\lambda_n}x_n - S_{\lambda_n}p\| + \|S_{\lambda_n}p-p\| \\ &\le (1-\lambda_n\tau)\|x_n-p\| + \|(I-\lambda_nA)Tp+\lambda_n[Tp-\mu(BTp-\gamma fp)]-p\| \\ &= (1-\lambda_n\tau)\|x_n-p\| + \lambda_n\|-Ap+p-\mu(Bp-\gamma fp)\| \\ &\le (1-\lambda_n\tau)\|x_n-p\| + \lambda_n[\|I-A\|\|p\|+\mu\|Bp-\gamma fp\|] \\ &= (1-\lambda_n\tau)\|x_n-p\| + \lambda_n\tau\frac{[\|I-A\|\|p\|+\mu\|Bp-\gamma fp\|]}{\tau} \\ &\le \max\left\{\|x_n-p\|, \frac{\tau[\|I-A\|\|p\|+\mu\|Bp-\gamma fp\|]}{\tau}\right\},\end{aligned} \tag{3.41}$$

where $\tau := \varphi(1)\overline{\gamma} - \varphi(1) + \mu(\varphi(1)\beta - \gamma\alpha) \in (0,1]$. By induction, it is easy to see that

$$\|x_n-p\| \le \max\left\{\|x_0-p\|, \frac{\tau[\|I-A\|\|p\|+\mu\|Bp-\gamma fp\|]}{\tau}\right\}, \quad \forall n \ge 0. \tag{3.42}$$

Thus, $\{x_n\}$ is bounded, and hence so are $\{y_n\}$, $\{ATx_n\}$, $\{BTx_n\}$, and $\{f(x_n)\}$. Now, we show that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.43}$$

From the definition of $\{x_n\}$, it is easily seen that

$$\begin{aligned}S_{\lambda_{n+1}}x_n - S_{\lambda_n}x_n &= (I-\lambda_{n+1}A)Tx_n + \lambda_{n+1}[Tx_n - \mu(BTx_n-\gamma f(x_n))] \\ &\quad - (I-\lambda_nA)Tx_n - \lambda_n[Tx_n-\mu(BTx_n-\gamma f(x_n))] \\ &= (\lambda_n-\lambda_{n+1})ATx_n + (\lambda_{n+1}-\lambda_n)Tx_n + \mu(\lambda_n-\lambda_{n+1})(BTx_n-\gamma f(x_n)) \\ &= (\lambda_{n+1}-\lambda_n)(I-A)Tx_n + \mu(\lambda_n-\lambda_{n+1})(BTx_n-\gamma f(x_n)).\end{aligned} \tag{3.44}$$

It follows that

$$\begin{aligned}\|x_{n+2}-x_{n+1}\| &= \|S_{\lambda_{n+1}}x_{n+1} - S_{\lambda_n}x_n\| \\ &\le \|S_{\lambda_{n+1}}x_{n+1} - S_{\lambda_{n+1}}x_n\| + \|S_{\lambda_{n+1}}x_n - S_{\lambda_n}x_n\| \\ &\le (1-\lambda_{n+1}\tau)\|x_{n+1}-x_n\| + |\lambda_{n+1}-\lambda_n|\|(I-A)Tx_n\| \\ &\quad + \mu|\lambda_n-\lambda_{n+1}|\|BTx_n-\gamma f(x_n)\| \\ &\le (1-\lambda_{n+1}\tau)\|x_{n+1}-x_n\| + (1+\mu)|\lambda_{n+1}-\lambda_n|M \\ &= (1-\lambda_{n+1}\tau)\|x_{n+1}-x_n\| + (1+\mu)\lambda_{n+1}\tau\frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}\tau}M,\end{aligned} \tag{3.45}$$

where M is a constant satisfying $M \geq \sup\{\|(I-A)Tx_n\|, \|BTx_n - \gamma f(x_n)\|\}$. From condition (C2), we deduce that either $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| M < \infty$ or $\lim_{n\to\infty}((\lambda_{n+1} - \lambda_n)/\lambda_{n+1}\tau)M = 0$. Therefore, it follows from Lemma 2.2 that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. It then follows that

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \\ &= \|x_n - x_{n+1}\| + \lambda_n \|Tx_n - \mu(BTx_n - \gamma f(x_n)) - ATx_n\| \longrightarrow 0. \end{aligned} \tag{3.46}$$

Next, we prove that

$$\limsup_{n\to\infty} \langle -(A - I + \mu(B - \gamma f))\tilde{x}, J_\varphi(x_n - \tilde{x}) \rangle \leq 0. \tag{3.47}$$

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{k\to\infty} \langle -(A - I + \mu(B - \gamma f))\tilde{x}, J_\varphi(x_{n_k} - \tilde{x}) \rangle = \limsup_{n\to\infty} \langle -(A - I + \mu(B - \gamma f))\tilde{x}, J_\varphi(x_n - \tilde{x}) \rangle. \tag{3.48}$$

If follows from reflexivity of E and the boundedness of a sequence $\{x_{n_k}\}$ that there exists $\{x_{n_{k_i}}\}$ which is a subsequence of $\{x_{n_k}\}$ converging weakly to $w \in E$ as $i \to \infty$. Since J_φ is weakly continuous, we have by Lemma 2.1 that

$$\limsup_{n\to\infty} \Phi\left(\left\|x_{n_{k_i}} - x\right\|\right) = \limsup_{n\to\infty} \Phi\left(\left\|x_{n_{k_i}} - w\right\|\right) + \Phi(\|x - w\|), \quad \forall x \in E. \tag{3.49}$$

Let

$$H(x) = \limsup_{n\to\infty} \Phi\left(\left\|x_{n_{k_i}} - x\right\|\right), \quad \forall x \in E. \tag{3.50}$$

It follows that

$$H(x) = H(w) + \Phi(\|x - w\|), \quad \forall x \in E. \tag{3.51}$$

From (3.46), we obtain

$$\begin{aligned} H(Tw) &= \limsup_{i\to\infty} \Phi\left(\left\|x_{n_{k_i}} - Tw\right\|\right) = \limsup_{i\to\infty} \Phi\left(\left\|Tx_{n_{k_i}} - Tw\right\|\right) \\ &\leq \limsup_{i\to\infty} \Phi\left(\left\|x_{n_{k_i}} - w\right\|\right) = H(w). \end{aligned} \tag{3.52}$$

On the other hand, however,

$$H(Tw) = H(w) + \Phi(\|T(w) - w\|). \tag{3.53}$$

It follows from (3.52) and (3.53) that

$$\Phi(\|T(w) - w\|) = H(Tw) - H(w) \leq 0. \tag{3.54}$$

This implies that $Tw = w$. Since the duality map J_φ is single valued and weakly continuous, we get that

$$\begin{aligned}
\limsup_{n\to\infty}\langle -(A - I + \mu(B - \gamma f))\tilde{x}, J_\varphi(x_n - \tilde{x})\rangle &= \lim_{k\to\infty}\langle -(A - I + \mu(B - \gamma f))\tilde{x}, J_\varphi(x_{n_k} - \tilde{x})\rangle \\
&= \lim_{i\to\infty}\left\langle -(A - I + \mu(B - \gamma f))\tilde{x}, J_\varphi\left(x_{n_{k_i}} - \tilde{x}\right)\right\rangle \\
&= \langle -(A - I + \mu(B - \gamma f))\tilde{x}, J_\varphi(w - \tilde{x})\rangle \\
&= \langle (A - I + \mu(B - \gamma f))\tilde{x}, J_\varphi(\tilde{x} - w)\rangle \leq 0
\end{aligned} \tag{3.55}$$

as required. Finally, we show that $x_n \to \tilde{x}$ as $n \to \infty$

$$\begin{aligned}
&\Phi(\|x_{n+1} - \tilde{x}\|) \\
&\quad= \Phi\left(\left\|(I - \lambda_n A)Tx_n + \lambda_n\left[Tx_n - \mu(BTx_n - \gamma f(x_n))\right] - (I - \lambda_n A)\tilde{x} - \lambda_n A\tilde{x}\right\|\right) \\
&\quad\leq \Phi(\|(I - \lambda_n A)Tx_n - (I - \lambda_n A)\tilde{x}\|) \\
&\qquad+ \lambda_n\langle Tx_n - \mu(BTx_n - \gamma f(x_n)) - A\tilde{x}, J_\varphi(x_{n+1} - \tilde{x})\rangle \\
&\quad\leq \varphi(1)(1 - \lambda_n\overline{\gamma})\Phi(\|x_n - \tilde{x}\|) + \lambda_n\langle (I - \mu B)Tx_n + \gamma\mu f(x_n) - A\tilde{x}, J_\varphi(x_{n+1} - \tilde{x})\rangle \\
&\quad= \varphi(1)(1 - \lambda_n\overline{\gamma})\Phi(\|x_n - \tilde{x}\|) \\
&\qquad+ \lambda_n\left[\langle (I - \mu B)Tx_n + \gamma\mu f(x_n) - (I - \mu B)Tx_{n+1} - \gamma\mu f(x_{n+1}), J_\varphi(x_{n+1} - \tilde{x})\rangle\right. \\
&\qquad\quad+ \langle (I - \mu B)Tx_{n+1} + \gamma\mu f(x_{n+1}) - (I - \mu B)\tilde{x} - \gamma\mu f(\tilde{x}), J_\varphi(x_{n+1} - \tilde{x})\rangle \\
&\qquad\quad\left.+\langle (I - \mu B)\tilde{x} + \gamma\mu f(\tilde{x}) - A(\tilde{x}), J_\varphi(x_{n+1} - \tilde{x})\rangle\right] \\
&\quad= \varphi(1)(1 - \lambda_n\overline{\gamma})\Phi(\|x_n - \tilde{x}\|) \\
&\qquad+ \lambda_n\left[\langle (I - \mu B)(Tx_n - Tx_{n+1}), J_\varphi(x_{n+1} - \tilde{x})\rangle + \gamma\mu\langle f(x_n) - f(x_{n+1}), J_\varphi(x_{n+1} - \tilde{x})\rangle\right. \\
&\qquad\quad+ \langle (I - \mu B)(Tx_{n+1} - \tilde{x}), J_\varphi(x_{n+1} - \tilde{x})\rangle + \gamma\mu\langle f(x_{n+1}) - f(\tilde{x}), J_\varphi(x_{n+1} - \tilde{x})\rangle \\
&\qquad\quad\left.+\langle I - A - \mu(B - \gamma f)\tilde{x}, J_\varphi(x_{n+1} - \tilde{x})\rangle\right]
\end{aligned}$$

$$
\begin{aligned}
&= \varphi(1)(1-\lambda_n\overline{\gamma})\Phi(\|x_n-\widetilde{x}\|) \\
&\quad + \lambda_n\big[\varphi(1)(1-\mu\beta)\|x_n-x_{n+1}\|\big\|J_\varphi(x_{n+1}-\widetilde{x})\big\| + \gamma\mu\alpha\|x_n-x_{n+1}\|\big\|J_\varphi(x_{n+1}-\widetilde{x})\big\| \\
&\qquad + \varphi(1)(1-\mu\beta)\|x_{n+1}-\widetilde{x}\|\big\|J_\varphi(x_{n+1}-\widetilde{x})\big\| + \gamma\mu\alpha\|x_{n+1}-\widetilde{x}\|\big\|J_\varphi(x_{n+1}-\widetilde{x})\big\| \\
&\qquad + \langle I-A-\mu(B-\gamma f)\widetilde{x}, J_\varphi(x_{n+1}-\widetilde{x})\rangle\big] \\
&\le \varphi(1)(1-\lambda_n\overline{\gamma})\Phi(\|x_n-\widetilde{x}\|) \\
&\quad + \lambda_n\big[\varphi(1)(1-\mu\beta)\|x_n-x_{n+1}\|M' + \gamma\mu\alpha\|x_n-x_{n+1}\|M' \\
&\qquad + \langle I-A-\mu(B-\gamma f)\widetilde{x}, J_\varphi(x_{n+1}-\widetilde{x})\rangle\big] \\
&\quad + \lambda_n\big[\varphi(1)-\mu(\varphi(1)\beta-\gamma\alpha)\big]\Phi(\|x_{n+1}-\widetilde{x}\|),
\end{aligned}
\tag{3.56}
$$

where M' is a constant satisfying $M' \ge \sup_{n\ge 0}\|J_\varphi(x_{n+1}-\widetilde{x})\|$. It then follows that

$$
\begin{aligned}
&\Phi(\|x_{n+1}-\widetilde{x}\|) \\
&\quad\le \frac{\varphi(1)(1-\lambda_n\overline{\gamma})}{1-\lambda_n[\varphi(1)-\mu(\varphi(1)\beta-\gamma\alpha)]}\Phi(\|x_n-\widetilde{x}\|) \\
&\qquad + \lambda_n\Bigg[\frac{\varphi(1)(1-\mu\beta)}{1-\lambda_n[\varphi(1)-\mu(\varphi(1)\beta-\gamma\alpha)]}\|x_n-x_{n+1}\|M' \\
&\qquad\quad + \frac{\gamma\mu\alpha}{1-\lambda_n[\varphi(1)-\mu(\varphi(1)\beta-\gamma\alpha)]}\|x_n-x_{n+1}\|M' \\
&\qquad\quad + \frac{1}{1-\lambda_n[\varphi(1)-\mu(\varphi(1)\beta-\gamma\alpha)]}\langle I-A-\mu(B-\gamma f)\widetilde{x}, J_\varphi(x_{n+1}-\widetilde{x})\rangle\Bigg] \\
&\quad= \left(1-\lambda_n\frac{[\varphi(1)\overline{\gamma}-(\varphi(1)-\mu(\varphi(1)\beta-\gamma\alpha))]}{1-\lambda_n[\varphi(1)-\mu(\varphi(1)\beta-\gamma\alpha)]}\right)\Phi(\|x_n-\widetilde{x}\|) \\
&\qquad + \lambda_n\Bigg[\frac{\varphi(1)(1-\mu\beta)}{1-\lambda_n[\varphi(1)-\mu(\varphi(1)\beta-\gamma\alpha)]}\|x_n-x_{n+1}\|M' \\
&\qquad\quad + \frac{\gamma\mu\alpha}{1-\lambda_n[\varphi(1)-\mu(\varphi(1)\beta-\gamma\alpha)]}\|x_n-x_{n+1}\|M' \\
&\qquad\quad + \frac{1}{1-\lambda_n[\varphi(1)-\mu(\varphi(1)\beta-\gamma\alpha)]}\langle -A-I+\mu(B-\gamma f)\widetilde{x}, J_\varphi(x_{n+1}-\widetilde{x})\rangle\Bigg].
\end{aligned}
\tag{3.57}
$$

Put

$$\gamma_n = \lambda_n \frac{[\varphi(1)\overline{\gamma} - (\varphi(1) - \mu(\varphi(1)\beta - \gamma\alpha))]}{1 - \lambda_n[\varphi(1) - \mu(\varphi(1)\beta - \gamma\alpha)]},$$

$$\begin{aligned}\delta_n = {} & \frac{1 - \lambda_n[\varphi(1) - \mu(\varphi(1)\beta - \gamma\alpha)]}{[\varphi(1)\overline{\gamma} - (\varphi(1) - \mu(\varphi(1)\beta - \gamma\alpha))]} \\ & \times \left[\frac{\varphi(1)(1-\mu\beta)}{1 - \lambda_n[\varphi(1) - \mu(\varphi(1)\beta - \gamma\alpha)]}\|x_n - x_{n+1}\|M' \right. \\ & \quad + \frac{\gamma\mu\alpha}{1 - \lambda_n[\varphi(1) - \mu(\varphi(1)\beta - \gamma\alpha)]}\|x_n - x_{n+1}\|M' \\ & \quad \left. + \frac{1}{1 - \lambda_n[\varphi(1) - \mu(\varphi(1)\beta - \gamma\alpha)]}\langle -A - I + \mu(B - \gamma f)\tilde{x}, J_\varphi(x_{n+1} - \tilde{x})\rangle \right].\end{aligned} \tag{3.58}$$

It follows that from condition (C1), $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and (3.47) that

$$\lim_{n\to\infty}\gamma_n = 0, \qquad \sum_{n=1}^{\infty}\gamma_n = \infty, \qquad \limsup_{n\to\infty}\delta_n \le 0. \tag{3.59}$$

The inequality (3.57) reduces to the following:

$$\Phi(\|x_{n+1} - \tilde{x}\|) \le (1 - \gamma_n)\Phi(\|x_n - \tilde{x}\|) + \gamma_n\delta_n. \tag{3.60}$$

Applying Lemma 2.2, we conclude that $\Phi(\|x_{n+1} - \tilde{x}\|) \to 0$ as $n \to \infty$; that is, $x_n \to \tilde{x}$ as $n \to \infty$. This completes the proof. □

Remark 3.7. In comparison to the results in [13, Theorem 3.1], the strong convergence in a real Hilbert space is extended to the strong convergence in a reflexive Banach space which admits a weakly continuous duality mapping.

Setting $B \equiv I$, and $\mu \equiv 1$ in Theorem 3.6, we obtain the following result.

Corollary 3.8. *Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$. Let $T : E \to E$ be a nonexpansive mapping with $F(T) \neq \emptyset$, $f : E \to E$ a contraction with coefficient $\alpha \in (0,1)$ and A a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}\varphi(1)/\alpha$. Let $x_0 \in E$ be arbitrary, and let the sequence $\{x_n\}$ be generated by the following iterative scheme:*

$$x_{n+1} = (I - \lambda_n A)Tx_n + \lambda_n\gamma f(x_n), \quad \forall n \ge 0, \tag{3.61}$$

where $\{\lambda_n\}$ is a real sequence in $(0,1)$ satisfying the following conditions:

(C1) $\lim_{n\to\infty}\lambda_n = 0$ *and* $\sum_{n=1}^{\infty}\lambda_n = \infty$,

(C2) $\sum_{n=1}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty$ *or* $\lim_{n\to\infty}(\lambda_n/\lambda_{n+1}) = 1$.

Then, the sequence $\{x_n\}$ *converges strongly to a fixed point* $\widetilde{x}$ *of* T *which solves the variational inequality*

$$\left\langle (A-\gamma f)\widetilde{x}, J_{\varphi}(\widetilde{x}-z) \right\rangle \leq 0, \quad z \in F(T). \tag{3.62}$$

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