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Research Article

A Novel Control Method for Integer Orders Chaos Systems via Fractional-Order Derivative

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A fractional-order control method is obtained to stabilize the point in chaos attractor of integer orders chaos systems. The control law has simple structure and is designed easily. Two examples are also given to illustrate the effectiveness of the theoretical result.

1. Introduction

Chaotic phenomena has been observed in many areas of science and engineering such as mechanics, electronics, physics, medicine, ecology, biology, and economy. To avoid troubles arising from unusual behaviors of a chaotic system, chaos control has received a great deal of interest among scientists from various research fields in the past few decades [1–8]. On the other hand, derivatives and integrals of fractional order have been found in many applications in recent years in physics and engineering. Many systems [4–13] are known to display fractional-order dynamics, such as viscoelastic systems, dielectric polarization, electrode-electrolyte polarization, and electromagnetic waves.

In the recent years, emergence of effective methods in the differentiation and integration of non integer order equations makes fractional-order systems more and more attractive for the systems control community. It is verified that the fractional-order controllers can have better disturbance rejection ratios and less sensitivity to plant parameter variations compared to the traditional controllers [14]. However, few results on control the saddle point of index 2 in integer orders chaos system are presented via fractional-order derivative. A fractional-order control method [14] is presented to stabilize the unstable equilibrium of integer orders chaos systems, but the method cannot apply to the desired point in chaotic attractor. Motivated by that, in this paper, a very simple control method is presented for a class of integer orders chaotic systems via fractional-order derivative. The order of fractional-order derivative is only determined by the eigenvalues at the desired point in integer orders chaotic attractor. The control technique, based on stability theory of fractional-order systems, is simple and theoretically rigorous.

2. Control Chaos via Fractional-Order Derivative

Consider the following 3D integer orders chaos system:

$$\dot{X} = F(X), \tag{2.1}$$

where $X = (x_1, x_2, x_3)^T$ are real state variables and $F : R^3 \rightarrow R^3$ is differentiable function.

Let $X_0 = (x_{10} \quad x_{20} \quad x_{30})^T$ be the any point in the chaos attractor of integer orders system (2.1), A_0 the Jacobian matrix of $F(X)$ at point X_0, and $\lambda_\pm = a \pm jb$, $\lambda_3 = -c$ the eigenvalues of A_0, respectively. In this paper, we always assume that $a > 0$, $c > 0$, and $b \neq 0$, especially X_0 is called the saddle point of index 2 if it also is the equilibrium point of system (2.1). In order to stabilize the point X_0 via fractional-order derivative, we introduce the following controller:

$$V(t) = -F(X_0) - \frac{d^q X}{dt^q} + \dot{X}, \tag{2.2}$$

where $0 < q < 1$ is the fractional order and will be determined later. Then, we can construct the control system for (2.1) as follows:

$$\dot{X} = F(X) + V(t). \tag{2.3}$$

Hence, the problem of stabilization of X_0 in chaos attractor of system (2.1) is shifted into the stability of X_0 in system (2.3). So, the stabilization problem of X_0 in system (2.1) can be solved if q is suitably designed such that X_0 in system (2.3) is asymptotically stable.

Now, we state the main result in this paper as follows.

Theorem 2.1. *If* $0 < q < q_0 \triangleq (2/\pi) \arctan |b/a|$ *in* $V(t)$, *then the point* X_0 *is asymptotically stable in control system* (2.3)*; that is, the point* X_0 *in chaos system* (2.1) *can be stabilized via fractional-order derivative.*

Proof. According to system (2.3) and controller (2.2), we can obtain that

$$\frac{d^q X}{dt^q} = F(X) - F(X_0). \tag{2.4}$$

It can be seen that the point X_0 is one equilibrium point of system (2.4) and the Jacobian matrix at this point for system (2.4) is the same as the Jacobian matrix at this point for system (2.1); therefore, the eigenvalues of the Jacobian matrix of (2.4) at point X_0 are $\lambda_\pm = a \pm jb$, $\lambda_3 = -c$.

For $\lambda_3 = -c < 0$, we have

$$\arg(\lambda_3) = \pi > \frac{q\pi}{2}. \tag{2.5}$$

For $\lambda_\pm = a \pm jb$, we also derive that

$$\arg(\lambda_\pm) = \arctan\left|\frac{b}{a}\right| = \frac{q_0\pi}{2} > \frac{q\pi}{2}. \tag{2.6}$$

In term of (2.5) and (2.6), for any eigenvalue λ of A_0, we obtain that

$$|\arg\lambda| > \frac{\pi q}{2}, \tag{2.7}$$

which implies that the equilibrium point X_0 of system (2.4) is asymptotically stable [15, 16]; that is, the point X_0 in the chaos attractor of integer orders chaos system (2.1) can be stabilized via fractional-order derivative; the proof is completed. □

Remark 2.2. Especially, if X_0 is the saddle point of index 2 of system (2.1), then the control law obtained in this paper is similar with the controller obtained in [14]. So, Theorem 2.1 proposed by this paper extends the controller proposed by [14].

Remark 2.3. By Theorem 2.1 proposed in this paper, we know that it is possible to stabilize the closed-loop system without applying any changes on the places of the poles.

Remark 2.4. If X_0 is the same saddle point for hyperchaotic systems, which implies that the eigenvalues of A_0 are $\lambda_\pm = a \pm jb$ $(a > 0, b \neq 0)$, $\lambda_3 = -c_1$ $(c_1 > 0)$, and $\lambda_4 = -c_2$ $(c_2 > 0)$, respectively, then this point X_0 in hyperchaotic systems can be stabilized via the proposed controller.

3. Applications

Now, we take Lü chaos system [17] and hyperchaotic Chen system [18] for numerical simulation, respectively. All the numerical simulation of fractional-order system in this paper is based on [19]. It is a direct time-domain approximation numerical simulation. Reference [20] has shown that using frequency-domain approximation in the numerical simulations of fractional systems may result in wrong consequences. This mistake has occurred in the recent literature that found the lowest-order chaotic systems among fractional-order systems.

We introduce the numerical solution of fractional differential equations in [19]. There have been several definitions of fractional derivatives. In the following, we introduce the most common one of them

$$\frac{d^q f(t)}{dt^q} = J^{m-q} f^{(m)}(t), \quad q > 0, \tag{3.1}$$

where m is the first integer which is not less than q, $f^{(m)}(t)$ is the m-order derivative in the usual sense, and J^q $(q > 0)$ is the q-order Riemann-Liouville integral operator with the expression

$$J^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau) d\tau. \tag{3.2}$$

Here, $\Gamma(\cdot)$ stands for gamma function, and d^q/dt^q is generally called "q-order Caputo differential operator."

Consider the following fractional-order system:

$$\frac{d^{q_1}x}{dt^{q_1}} = f(x,y), \quad \frac{d^{q_2}y}{dt^{q_2}} = g(x,y), \quad 0 < q_1,\ q_2 < 1, \tag{3.3}$$

with initial condition (x_0, y_0). Now, set $h = T/N$, $t_n = nh$ $(n = 0, 1, 2 \ldots, N)$. The above system can be discretized as follows:

$$\begin{aligned} x_{n+1} &= x_0 + \frac{h^{q_1}}{\Gamma(q_1+2)} \left[f\left(x_{n+1}^p, y_{n+1}^p\right) + \sum_{j=0}^{n} \alpha_{1,j,n+1} f(x_j, y_j) \right], \\ y_{n+1} &= y_0 + \frac{h^{q_2}}{\Gamma(q_2+2)} \left[g\left(x_{n+1}^p, y_{n+1}^p\right) + \sum_{j=0}^{n} \alpha_{2,j,n+1} g(x_j, y_j) \right], \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} x_{n+1}^p &= x_0 + \frac{1}{\Gamma(q_1)} \sum_{j=0}^{n} \beta_{1,j,n+1} f(x_j, y_j), \\ y_{n+1}^p &= y_0 + \frac{1}{\Gamma(q_2)} \sum_{j=0}^{n} \beta_{2,j,n+1} g(x_j, y_j), \end{aligned} \tag{3.5}$$

and for $i = 1, 2$,

$$\alpha_{i,j,n+1} = \begin{cases} n^{q_i+1} - (n-q_i)(n+1)^{q_i}, & j = 0, \\ (n-j+2)^{q_i+1} + (n-j)^{q_i+1} - 2(n-j+1)^{q_i+1}, & 1 \le j \le n, \\ 1, & j = n+1, \end{cases} \tag{3.6}$$

$$\beta_{i,j,n+1} = \frac{h^{q_i}}{q_i} \left[(n-j+1)^{q_i} - (n-j)^{q_i} \right], \quad 0 \le j \le n. \tag{3.7}$$

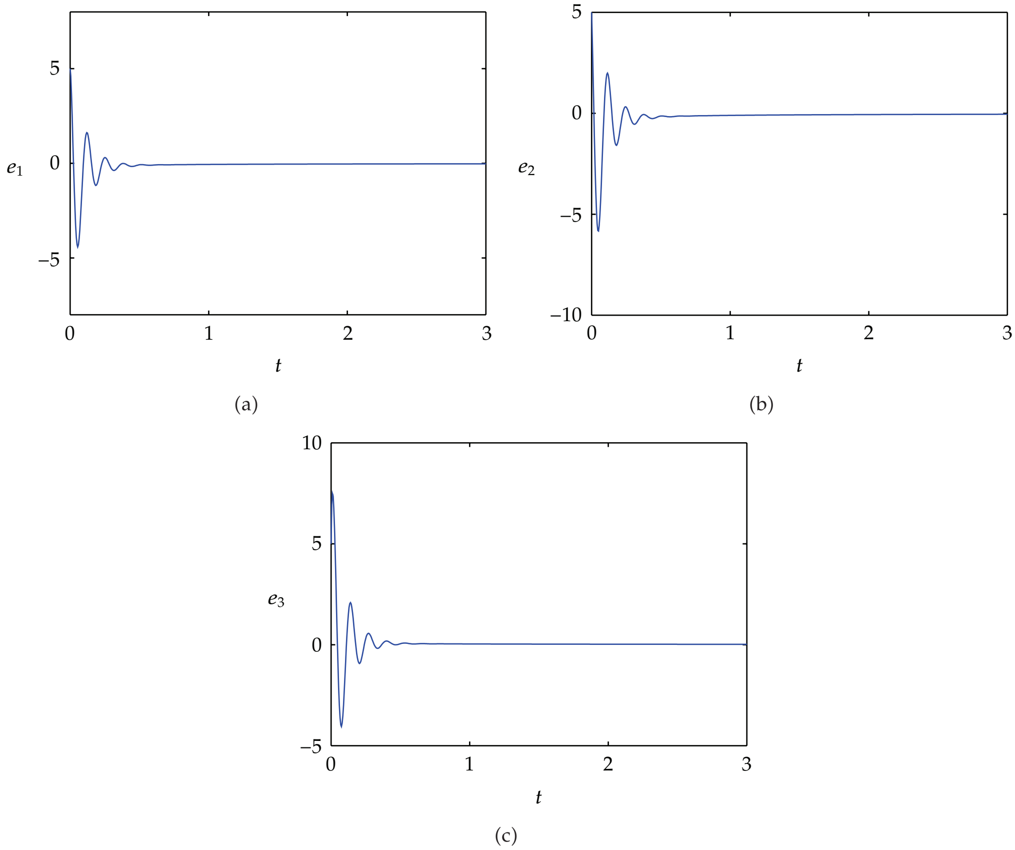

Figure 1: The control simulation result for (10, 9, 15) when $q = 0.7$.

The error of this approximation is described as follows:

$$
\begin{aligned}
|x(t_n) - x_n| &= o\left(h^{p_1}\right), \quad p_1 = \min(2, 1 + q_1), \\
\left|y(t_n) - y_n\right| &= o\left(h^{p_2}\right), \quad p_2 = \min(2, 1 + q_2).
\end{aligned}
\tag{3.8}
$$

Now, we choose Lü chaos system for numerical simulation. The Lü chaos system [17] is

$$
\begin{aligned}
\frac{dx_1}{dt} &= 36(x_2 - x_1), \\
\frac{dx_2}{dt} &= -x_1 x_3 + 20 x_2, \\
\frac{dx_3}{dt} &= x_1 x_2 - 3 x_3.
\end{aligned}
\tag{3.9}
$$

Now, we choose $(x_{10}, x_{20}, x_{30}) = (10, 9, 15)$ in chaos attractor of chaos system (3.9). Since the eigenvalues of the Jacobian matrix at X_0 for system (3.9) are $\lambda_{\pm} = 4.5554 \pm 14.2605j$,

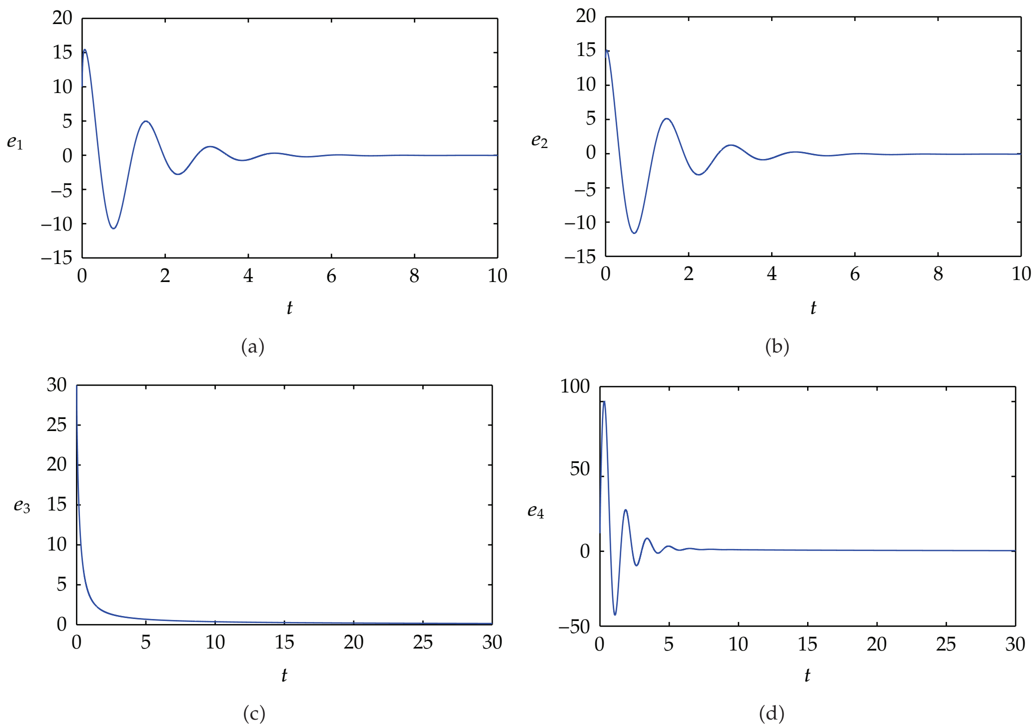

Figure 2: The control simulation result for (0, 0, 19, 0) when $q = 0.8$.

$\lambda_3 = -28.1108$, we have $q_0 \overset{\Delta}{=} (2/\pi)\arctan|b/a| = 0.8032$. According to Theorem 2.1, in order to stabilize this point, we choose $q < q_0$ for controller (2.2).

For example, the simulation results are shown in Figure 1 for $q = 0.7$, and the initial conditions are $(15, 14, 20)^T$, where $e_1 = x_1 - x_{10}$, $e_2 = x_2 - x_{20}$, and $e_3 = x_3 - x_{30}$.

Finally, we choose hyperchaotic Chen system for numerical simulation. The hyperchaotic Chen system [18] is

$$\begin{aligned} \frac{dx_1}{dt} &= 35(x_2 - x_1) + x_4, \\ \frac{dx_2}{dt} &= 7x_1 - x_1x_3 + 12x_2, \\ \frac{dx_3}{dt} &= x_1x_2 - 3x_3, \\ \frac{dx_4}{dt} &= x_2x_3 + 0.5x_4. \end{aligned} \tag{3.10}$$

Now, we choose $(x_{10}, x_{20}, x_{30}, x_{40}) = (0, 0, 19, 0)$ in chaos attractor of the hyperchaotic Chen system (3.10). Since the eigenvalues of the Jacobian matrix at X_0 for system (3.10) are $\lambda_\pm = 0.4537 \pm 3.0878j$, $\lambda_3 = -23.4074$, and $\lambda_4 = -3$, respectively, we have $q_0 \overset{\Delta}{=} (2/\pi)\arctan|b/a| = 0.9071$. According to Theorem 2.1, in order to stabilize this point, we choose $q < q_0$ for controller (2.2).

For example, the simulation results are shown in Figure 2 for $q = 0.8$ and the initial conditions are $(10, 14, 49, 12)^T$, where $e_1 = x_1 - x_{10}$, $e_2 = x_2 - x_{20}$, $e_3 = x_3 - x_{30}$, and $e_4 = x_4 - x_{40}$.

4. Conclusion

We construct a novel control law for integer orders chaos system via fractional-order derivative. Any desired point in chaos attractor of integer orders chaos system can be stabilized via the fractional-order derivative. The order of fractional-order derivative of chaos system is only determined by the eigenvalues at the point. By comparison with the traditional controllers, we know that this controller without causing any changes in the eigenvalues of the system at the desired fixed points. The proposed controller is employed to control Lü chaotic system and hyperchaotic Chen system, and some numerical simulation results are obtained. Theoretical analysis and simulation results show that the control method in this paper is effective.

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