THE PROBABILITY THAT TWO ELEMENTS COMMUTE IN SOME 2-GENERATOR 2-GROUPS OF NILPOTENCY CLASS 2

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A dissertation submitted in partial fulfilment of the
requirements for the award of the degree of
Master of Science (Mathematics)

Faculty of Science
Universiti Teknologi Malaysia

OCTOBER, 2005

ACKNOWLEDGEMENT

I would like to extend my gratitude to many people for the successful completion of this dissertation in due course of time. Firstly I would like to thank my supervisor, PM Dr Nor Haniza Sarmin for giving me all the necessary and valuable guidance in conducting this dissertation. Moreover, I would like to express my gratitude to PM Dr. Ramli Salleh, PM Dr. Abdullah Tahir Hj. Othman and Dr. Nor'aini Aris for some comments on this work.

My friends should also be recognised for their support. My sincere appreciation also extends to Azhana Ahmad and Nor Muhainiah Mohd. Ali for their relentless help in providing me with all the necessary information and their perseverance in helping me to solve some problems encountered while doing this dissertation.

Last but not the least, I would like to thank my family and those person who are involved in completing this dissertation.

ABSTRACT

Erdos and Turan have introduced the determination of the abelianness of a finite group for symmetric groups, finite groups and finite rings in 1968. Basic probability theory will be used in studying its connection with group theory. Some basic concepts related with $\mathrm{P}(G)$, the probability a random pair of elements in a group commutes, will be presented. This research will focus on the 2-generator 2-groups of nilpotency class 2, based on the classification that have been done by Kappe *et.al* in 1999. Finally, some properties of $\mathrm{P}(G)$ for some 2-generator 2-groups of nilpotency class 2 will be determined.

ABSTRAK

Erdos dan Turan telah memperkenalkan penentuan darjah keabelanan bagi suatu kumpulan terhingga untuk kumpulan simetri, kumpulan terhingga dan gelanggang terhingga pada tahun 1968. Asas bagi teori kebarangkalian akan digunakan untuk melihat perkaitannya dengan teori kumpulan. Beberapa konsep asas yang berkaitan dengan $\mathrm{P}(G)$, iaitu kebarangkalian suatu pasangan tertib unsur rawak dalam suatu kumpulan kalis tukar tertib akan diterangkan. Kajian ini difokuskan kepada kumpulan-2 dengan 2-penjana yang mempunyai nilpoten 2 berdasarkan klasifikasi yang telah diperolehi oleh Kappe dan rakan-rakan pada tahun 1999. Beberapa ciri $\mathrm{P}(G)$ bagi beberapa kumpulan-2 dengan 2-penjana yang mempunyai nilpoten kelas 2 ditentukan pada akhir kajian.

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LIST OF SYMBOLS / NOTATIONS

1	identity element
a^{-1}	inverse of *a*
$<a>$	cyclic subgroup generated by *a*
$[a,b]$	commutator of *a* and *b* (ie: $a^{-1}b^{-1}ab$)
$[a,b,c]$	commutator of $[a,b]$ and *c* (ie: $[[a,b],c]$)
$\mathbb{N}$	natural numbers
C_n	the cyclic group of order *n*
$C(a)$	the centralizer of *a*
$\mathrm{cl}(a)$	conjugacy class of *a*
D_n	the dihedral group of order 2*n*
G	a finite group
$\lvert G\rvert$	the order of *G*
$G \cong H$	the groups *G* and *H* are isomorphic
$\lvert G:H\rvert$	index of *H* in *G*
$H \leq G$	*H* is a subgroup of *G*
$H \lhd G$	*H* is a normal subgroup of *G*
$k(G)$	the number of conjugacy classes of *G*
$k_{\mathrm{LB}}(G)$	the lower bound of the number of conjugacy classes in *G*
$\mathrm{P}(G)$	the probability that two elements commute in *G*
$\mathrm{P}_{\mathrm{LB}}(G)$	the lower bound of the probability that two elements commute in *G*

Q	the quaternion group of order 8
$t \mid s$	t divides s
$Z(G)$	the center of G
□	integers
$\in$	element of
$\notin$	not an element of
$\ni$	such that
$\forall$	for every

CHAPTER 1

PROBLEM FORMULATION

1.1 Background of the Problem

In recent years probabilistic methods have proved useful in the solution of several difficult problems in group theory. In some cases the probabilistic nature of the problem has been apparent from its formulation, but in other cases the use of probability seems surprising, and cannot be anticipated by the nature of the problem. The roots of the subject lie in a series of papers by Erdos and Turan [1] in which they study the properties of random permutations, and develop a statistical theory for the symmetric group.

All groups considered will be assumed to be finite. We will denote by $\mathrm{P}(G)$, the probability that two elements of the group *G*, chosen randomly with replacement, commute. This will loosely be called the "probability of *G*".

We will only consider about the nonabelian group, since for Abelian groups, this probability is clearly equal to 1.

The classification of 2-generator 2-groups of nilpotency class 2 has been introduced by Kappe *et. al*. [2]. This classification will be used throughout this research.

The probability that a random pair of elements in 2-generator 2-groups of nilpotency class 2 has not been done yet. Therefore in this research, $\mathrm{P}(G)$ for some 2-generator 2-groups of nilpotency class 2 will be computed.

1.2 Statement of the Problem

What is the probability that two elements in some 2-generator 2-groups of nilpotency class 2 commute?

1.3 Objectives of the Researh

The research objectives are:

(a) To study the classification of 2-generator 2-groups of nilpotency class 2.

(b) To determine the probability that two random elements in some 2-generator 2-groups of nilpotency class 2 commute.

(c) To investigate the properties of $\mathrm{P}(G)$ that are suitable for some 2-generator 2-groups of nilpotency class 2.

1.4 Scope of the Study

This research will focus only on 2-generator 2-groups of nilpotency class 2 up to order 256.

1.5 Literature Review

An earlier example of a probabilistic statement about groups describes how commutative a nonabelian group can be. The idea of computing $\mathrm{P}(G)$, which is the probability that a random pair of elements in a group commutes, was introduced in 1968 by Erdos and Turan [1] who explored this concept mainly for symmetric groups.

The concepts of this probability and research on this topic have been done by several authors, including Erdos and Turan [1], Sarmin and Sapiri [3], Sarmin and Mohd Seran [4], Belcastro and Sherman [5], Rusin [6], MacHale [7] and Sherman [8]. In 1974, MacHale found that $\mathrm{P}(G)$ is equal to the number of conjugacy classes divides the order of that group.

It is also found in 1974 by MacHale [7] that for any finite nonabelian group, the probability that two elements chosen at random from G commute is at most $5/8$. The same concept was then used by Sherman [5] to find the probability for finite groups in general and showed that this probability cannot be arbitrarily close to 1 if G is finite nonabelian group. In fact, it will always be less than or equal to 5/8.

$\mathrm{P}(G)$ of a group is the number of conjugacy classes divides the order of the group, therefore the determination of the number of conjugacy classes is very important. The problem of estimating the number of conjugacy classes, $k(G)$ in a finite group *G*, has been around since the turn of the century. In year 1979, using Landau's technique, Sherman [5] proved that each group have the lower bound of number of conjugacy classes.

The classification of 2-generator 2-groups of nilpotency class 2 has been done by Kappe *et. al*. [2]. There are four types in the classification. Later, Magidin modified the classification into presentation form and combined them into three types only. This modified version is needed in computing the number of conjugacy class.

The probability that a random pair of elements, $\mathrm{P}(G)$ in 2-generator 2-groups of nilpotency class 2 commute, have not been done yet. Therefore, in this research, $\mathrm{P}(G)$ for the 2-generator 2-groups of nilpotency class 2 will be computed.

1.6 Theoretical Framework

This chapter is the introduction chapter that includes the background of the problem, statement of the problem, objectives of the research, scope of the study, literature review and theoretical framework.

In Chapter 2, some basic definitions and theorems that will be used throughout the dissertation are included.

Chapters 3 discusses the classification of 2-generator 2-group of class 2 up to isomorphism. The proof for the classification is omitted and details can be referred from a paper by Kappe, *et. al* [2]. Besides that, some basic definitions and theorems about the probability that two elements commute in a group are presented. Some results of the theorems are applied and proved by some examples that are given in the fourth chapter.

Chapter 4 presents specific examples of group of order 8, 16 and 32. Groups Algorithms and Programming (GAP) software is used to identify the groups that fulfill the classification and to confirm the number of conjugacy classes of a group, and then to get the probability $\mathrm{P}(G)$ of that group.

Some properties of $\mathrm{P}(G)$ that are suitable for 2-generator 2-groups of nilpotency class 2 are investigated in Chapter 5. Moreover, some results will be stated. These results will help to investigate the properties of $\mathrm{P}(G)$ that are suitable for 2-generator 2-group class 2.

Finally, Chapter 6 includes the conclusion and suggestion for further research.

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