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Research Article

Model Reference Control of Hyperchaotic Systems

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We have applied a famous engineering method, called model reference control, to control hyperchaos. We have proposed a general description of the hyperchaotic system and its reference system. By using the Lyapunov stability theorem, we have obtained the expression of the controller. Four examples for the both certain case and the uncertain case show that our method is very effective for controlling hyperchaotic systems with both certain parameters and uncertain parameters.

1. Introduction

Chaos has received increasing attentions in the last thirty years. Compared with the ordinary chaotic systems, the hyperchaotic systems hold at least two positive Lyapunov exponents and then possess more complicated attractors. Hyperchaotic systems have the characteristics of high capacity, high security, and high efficiency and have been studied in many fields, such as secure communication [1, 2], cellular neural network [3, 4], chemical processing [5], nonlinear circuit [6–8], and other fields [9, 10].

Controlling chaos (or hyperchaos) is very meaningful. The research has been started since the pioneering work of OGY method [11] was published. As the development of computational technique and controlling theory, many methods have been proposed for controlling chaos such as, LMI-based approach [12, 13], sliding control [14], active control [15], optimal control [16, 17], and passivity-based control [18, 19]. However, the hyperchaotic systems are more complex than the ordinary chaotic ones. To obtain the satisfying effect of control, we should focus on more advantage algorithms and ideas of controlling techniques. There have been some results on controlling hyperchaotic systems, such as feedback control [20, 21], adaptive control [22], backstepping control [23], and impulsive control [24].

In this paper, we will show that although a hyperchaotic system is complex, its dynamics can still be controlled along an expected trajectory. The controller is generated by an advantage control method, called model reference control (MRC). Nowadays, MRC is widely used in engineering, such as control of robots [25], mechanical oscillators [26], economic cycle [27], and disease spread [28]. Our aim is to control a hyperchaotic system to track with an expected trajectory with the aid of MRC. In the following section, the formulation of the problem will be presented. In Section 3, we will propose the framework of MRC on hyperchaotic systems with certain parameters and the systems with uncertain parameters. In Section 4, we will give two numerical examples to show the effectiveness of MRC on hyperchaotic systems with certain and uncertain parameters and also two examples for the hyperchaotic systems with uncertain parameters. Finally, the conclusion will be given in Section 5.

2. Problem Formulation

The first example of the hyperchaotic systems was presented by Rössler in 1979 [29]. Since then, other hyperchaotic systems have been reported [30], and many researchers are focusing on the discovery of new hyperchaotic systems and their control. In this section, we will describe the basic formulation of generalized hyperchaotic system and the reference system. Recently, nonlinear scientists are focusing on the control problem of chaotic and hyperchaotic systems with uncertain parameters [31–37]. Hence, we will give the formulation of both the hyperchaotic systems with certain parameters and the systems with uncertain parameters.

Since a hyperchaotic system with certain parameters should have quadratic terms at least, we may formulate the system as follows:

$$\dot{X} = X^T AX + BX + u, \tag{2.1}$$

where X is an n dimensional column vector, X^T is its transposed matrix, A is an $n^2 \times n$ matrix including n matrices $A_1, A_2, \ldots, A_n$ in the form of $n \times n$, B is an $n \times n$ matrix, and u is an n dimensional input control column vector. It is worth noting that $X^T AX$ is only the nominal expression of matrix multiplication, and it describes the quadratic terms in blocks as follows:

$$\left(X^T A_1 X, X^T A_2 X, \ldots, X^T A_n X\right)^T. \tag{2.2}$$

Equation (2.1) can cover a great many hyperchaotic systems, though it may not fit for the systems with higher order terms or even a fractional order term. The uncertain form of system (2.1) is as follows:

$$\dot{X} = X^T AX + BX + C(X)\alpha + u, \tag{2.3}$$

where $C(X)$ is an $n \times m$ matrix, uncertain parameter α is an m dimensional column vector, and other variables as above. Here, m might be equal to n, also smaller or bigger than n.

In this section, we will also introduce a reference system with only linear terms and constants. Our aim is to control the hyperchaotic system track along with the reference model system that exhibits asymptotic stability as follows

$$\dot{X}_r = \overline{B}X_r + \overline{u}, \tag{2.4}$$

where the matrix $\overline{B}$ is a known constant matrix with appropriate dimensions, the eigenvalues of the matrix $\overline{B}$ have negative real part such that the system is asymptotically stable. By letting the system (2.4) be the reference system, we will control the hyperchaotic system (2.1) track along with the system (2.4). Here, X_r is called state output vector of the reference model, and $\overline{u}$ the reference input vector.

Particularly, if the reference system has a two-order term, we may write it as

$$\dot{X}_r = X^T\overline{A}X + \overline{B}X_r + \overline{u}, \tag{2.5}$$

where $\overline{A}$ is a similar matrix as A.

3. Model Reference Control of Hyperchaotic Systems

In this section, MRC is applied to control a hyperchaotic system with both certain parameters and uncertain parameters. Our objective is to obtain the exact control law u such that the original system follows the dynamical behavior of the reference model. We will review the Lyapunov stability theorem for autonomous systems and then give the explicit expression of the controller.

Theorem 3.1 (Lyapunov stability theorem for autonomous systems). *Let $x = 0$ be an equilibrium point for a dynamical system described by*

$$\dot{x} = f(x), \tag{3.1}$$

where $f : D \rightarrow \mathbb{R}^n$ is a locally Lipschitz and $D \subset \mathbb{R}^n$ a domain that contains the origin. Let $V : D \rightarrow \mathbb{R}$ be a continuously differentiable, positive definite function in D. Then $x = 0$ is an asymptotically stable equilibrium point, if $\dot{V}(x)$ is negative definite. The scalar function V is a Lyapunov function if $\dot{V}(x)$ is negative semidefinite in the region $D : \dot{V}(x) \leq 0$.

Assume that the error system (3.1) satisfies the conditions of Theorem 3.1, and its Lyapunov function has the form as

$$V(e) = e^T P e, \tag{3.2}$$

where P is a symmetric positive definite matrix. The derivative of $V(e)$ with respect to time is

$$\dot{V}(e) = \dot{e}^T P e + e^T P \dot{e}. \tag{3.3}$$

In the whole process of MRC, the output of reference model and that of the controlled system are compared, and the error vector e is

$$e = X_r - X. \tag{3.4}$$

If we want to control the system (2.1) to track along with system (2.4), we may have the following results. By using (2.1) and (2.4), we obtain the following error system:

$$\dot{e} = \dot{X}_r - \dot{X} = \overline{B}e - X^T AX + \left(\overline{B} - B\right)X + \overline{u} - u. \tag{3.5}$$

We will try to design a control vector u such that the objective equation

$$\lim_{t \to \infty} \|e\| = 0. \tag{3.6}$$

From (3.1) and (3.2), the MRC problem is converted to the asymptotic stability of zero vector of the error system (3.1). Here, we will use the Lyapunov stability theory to determine the proper control law u. The theorem is as follows.

Theorem 3.2. *Let P be any symmetric positive definite matrix, $\lambda_{\max}$ is the largest eigenvalue of the matrix $(\overline{B}^T P + P\overline{B})$, the controller $u = \overline{B}X + \overline{u} - X^T AX - BX - \lambda I(X_r - X)$. The hyperchaotic system* (2.1) *will asymptotically follow the desired dynamical system* (2.4)*, if $\lambda < -\lambda_{\max}/2$ and $\lambda \neq 0$.*

Proof. The fact that the hyperchaotic system (2.1) asymptotically follows the system (2.4) is equivalent to asymptotical stability of the error system. Hence, we will try to prove the inequality of $\dot{V}(e) \leq 0$.

For the symmetric property, the scalar equation (3.3) may have the following description:

$$\begin{aligned}
\dot{V}(e) &= \dot{e}^T Pe + e^T P\dot{e} \\
&= \left[e^T\overline{B}^T - X^T A^T X - X^T\left(B^T - \overline{B}^T\right) - u^T + \overline{u}^T\right]Pe \\
&\quad + e^T P\left(\overline{B}e - X^T AX - \left(B - \overline{B}\right)X - (u - \overline{u})\right) \\
&= e^T\left(\overline{B}^T P + P\overline{B}\right)e + 2e^T P\left[-X^T AX - \left(B - \overline{B}\right)X - (u - \overline{u})\right] \\
&= e^T\left(\overline{B}^T P + P\overline{B}\right)e + 2e^T(\lambda I)e \\
&= e^T\left(\overline{B}^T P + P\overline{B} + 2\lambda I\right)e.
\end{aligned} \tag{3.7}$$

For $\lambda < -\lambda_{\max}/2$, the matrix $(\overline{B}^T P + P\overline{B} + 2\lambda I)$ is negative definite. Also, the scalar $\dot{V} < 0$ is obtained.

Using the result of Theorem 3.1, it can be concluded that the error e will converge to 0 asymptotically. Equivalently, the controller u will make the hyperchaotic system (2.1) asymptotically follow the desired dynamical system (2.4). □

If we want to control the system (2.1) to track along with system (2.5), we may have the following results. By using (2.1) and (2.4), we obtain the following error system:

$$\dot{e} = \dot{X}_r - \dot{X} = X_r^T \overline{A} X_r + \overline{B} e - X^T A X + \left(\overline{B} - B \right) X + \overline{u} - u. \tag{3.8}$$

It is easy to obtain the control law in the following theorem with similar process of Theorem 3.2.

Theorem 3.3. *Let P be any symmetric positive definite matrix, $\lambda_{\max}$ is the largest eigenvalue of the matrix $(\overline{B}^T P + P\overline{B})$, and the controller $u = X_r^T \overline{A} X_r + \overline{B} X + \overline{u} - X^T A X - BX - \lambda I(X_r - X)$. The hyperchaotic system* (2.1) *will asymptotically follow the desired dynamical system* (2.4)*, if $\lambda < -\lambda_{\max}/2$ and $\lambda \neq 0$.*

We can omit the proof of Theorem 3.3, for it is similar to that of Theorem 3.2.

Theorems 3.2 and 3.3 are fit for the hyperchaotic systems with certain parameters. In the following, we will give two similar theorems for the systems with uncertain parameters. To simplify the control process, we will make $P = I$.

Theorem 3.4. *Let $\lambda_{\max}$ is the largest eigenvalue of the matrix $\overline{B}^T$, the controller $u = \overline{B} X + \overline{u} - X^T A X - BX - \lambda I(X_r - X) + C(X)\widehat{\alpha}$, where $\widehat{\alpha}$ is the estimation of uncertain parameter α, and satisfies the differential equations $\dot{\widehat{\alpha}} = C^T(X)e$. The hyperchaotic system* (2.3) *will asymptotically follow the desired dynamical system* (2.4)*, if $\lambda < -\lambda_{\max}$ and $\lambda \neq 0$.*

Proof. For the uncertain property of system (2.3), we should redefine the Lyapunov function as follows

$$V_1(e) = e^T e + (\alpha - \widehat{\alpha})^T (\alpha - \widehat{\alpha}). \tag{3.9}$$

Similar to the proof of Theorem 3.2, we have to prove the inequality of $\dot{V}_1(e) \leq 0$:

$$\begin{aligned}
\dot{V}_1(e) &= 2\dot{e}^T e - 2\dot{\widehat{\alpha}}^T (\alpha - \widehat{\alpha}) \\
&= 2\left(\overline{B} X_r + \overline{u} - X^T A X - BX - u - C(X)\alpha \right)^T e - 2e^T C(X)(\alpha - \widehat{\alpha}) \\
&= 2e^T \left(\overline{B}^T + \lambda I \right) e.
\end{aligned} \tag{3.10}$$

For $\lambda < -\lambda_{\max}$, the matrix $(\overline{B}^T + \lambda I)$ is negative definite. Also, the scalar $\dot{V} < 0$ is obtained.

Hence, the controller u will make the hyperchaotic system (2.3) asymptotically follow the desired dynamical system (2.4). □

If we want to control the system (2.3) to track along with system (2.5), we may have the following results. We have the following error system:

$$\dot{e} = X_r^T \overline{A} X_r + \overline{B} e - X^T A X + \left(\overline{B} - B \right) X + \overline{u} - u - C(X)\alpha. \tag{3.11}$$

It is easy to obtain the control law in the following theorem with similar process of Theorem 3.4.

Theorem 3.5. *Let $\lambda_{\max}$ is the largest eigenvalue of the matrix $\overline{B}^T$, the controller $u = \overline{B}X + \overline{u} - X^T AX - BX - \lambda I(X_r - X) + C(X)\widehat{\alpha}$, where $\widehat{\alpha}$ is the estimation of uncertain parameter α, and satisfies the differential equations $\dot{\widehat{\alpha}} = C^T(X)e$. The hyperchaotic system* (2.3) *will asymptotically follow the desired dynamical system* (2.5)*, if $\lambda < -\lambda_{\max}$ and $\lambda \neq 0$.*

We can also omit the proof of Theorem 3.5.

4. Numerical Examples

This section has two parts. The first part is the numerical examples for controlling hyperchaotic systems, where all the parameters are certain. We will use the result in Theorem 3.2 to control hyperchaotic Rössler system and use that of Theorem 3.3 to control the hyperchaotic Lorenz system. The second part is the numerical examples for controlling the systems with uncertain parameters. Especially, Example III has four uncertain parameters, and Example IV has six uncertain parameters.

The whole numerical results show that MRC is very suitable and efficient for controlling hyperchaotic system with both certain parameters and uncertain parameters.

4.1. Example I

The four-variable hyperchaotic Rössler system is described by

$$\begin{aligned}
\dot{x}_1 &= -x_2 - x_3, \\
\dot{x}_2 &= x_1 + 0.25x_2 + x_4, \\
\dot{x}_3 &= x_1x_3 + 3, \\
\dot{x}_4 &= -0.5x_3 + 0.05x_4.
\end{aligned} \tag{4.1}$$

According to (2.1), we have

$$\begin{gathered}
A_3 = \begin{bmatrix} 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0.25 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0.05 \end{bmatrix}, \\
u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 + 3 \\ u_4 \end{bmatrix}, \qquad A_1 = A_2 = A_4 = \operatorname{diag}[0, 0, 0, 0],
\end{gathered} \tag{4.2}$$

where u_1, u_2, u_3, and u_4 are equal to 0 in the initial state and need to be determined by Theorem 3.2.

The reference model with asymptotical stability is as follows:

$$\begin{aligned}
\dot{x}_{r1} &= -x_{r2} - x_{r3}, \\
\dot{x}_{r2} &= x_{r1} + 0.25x_{r2} + x_{r4}, \\
\dot{x}_{r3} &= 20x_{r1} + 10x_{r2} - 50x_{r3} + 100x_{r4} + 3, \\
\dot{x}_{r4} &= -0.5x_{r3} + 0.05x_{r4}.
\end{aligned} \tag{4.3}$$

According to (2.4), the matrix $\overline{B}$ and the control vector $\overline{u}$ of the 4-dimensional reference system (4.3) have the following description:

$$\overline{B} = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0.25 & 0 & 1 \\ 20 & 10 & -50 & 100 \\ 0 & 0 & -0.5 & 0.05 \end{bmatrix}, \qquad \overline{u} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}. \tag{4.4}$$

According to Theorem 3.2, we may let matrix $P = \mathrm{diag}[1, 1, 1, 1]$, then the matrix

$$\left(\overline{B}^T P + P\overline{B}\right) = \begin{bmatrix} 0 & 0 & 19 & 0 \\ 0 & 0.5 & 10 & 1 \\ 19 & 10 & -100 & 99.5 \\ 0 & 1 & 99.5 & 0.1 \end{bmatrix}, \tag{4.5}$$

and its largest eigenvalue is $\lambda_{\max} = 63.6297$. By considering the conditions $\lambda < -\lambda_{\max}/2$ and $\lambda \neq 0$, we should obtain $\lambda = -40$. Then, the controller should be determined by Theorem 3.2,

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 40x_{r1} - 40x_1 \\ 40x_{r2} - 40x_2 \\ (20 - x_3)x_1 + 10x_2 - 90x_3 + 100x_4 + 40x_{r3} + 3 \\ 40x_{r4} - 40x_4 \end{bmatrix}. \tag{4.6}$$

Figure 1 shows that the controller can make the hyperchaotic Rössler system (4.1) track along with its reference system (4.3). Here, the initials of these two systems are $[-10, 6, 0, 10]$ and $[10, -6, 10, -10]$, respectively.

4.2. Example II

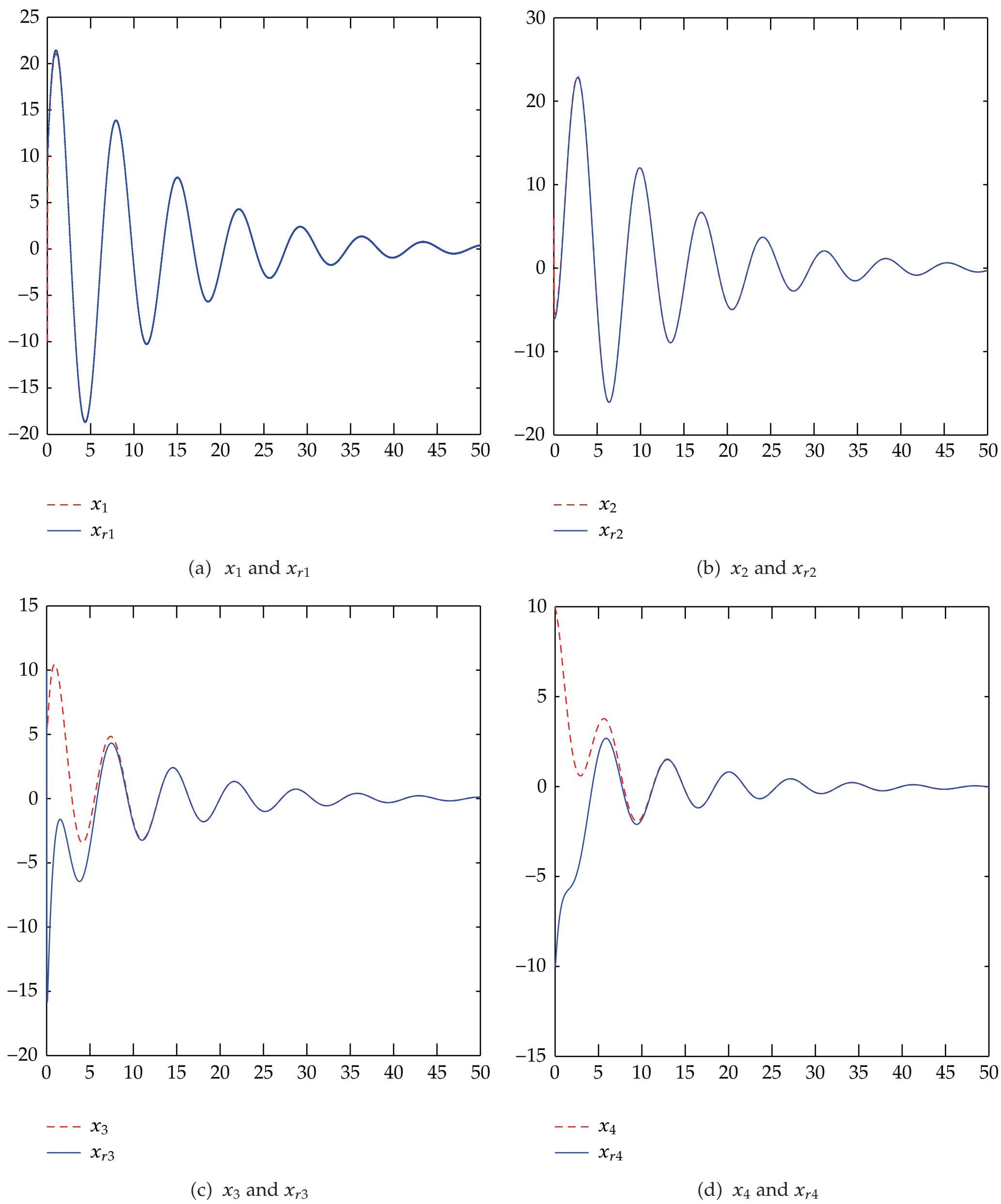
The four-variable hyperchaotic Lorenz system [30] is described by

$$\dot{x}_1 = 10(x_2 - x_1), \tag{4.7}$$

$$\dot{x}_2 = 28x_1 - x_2 - x_1x_3 + x_4, \tag{4.8}$$

$$\dot{x}_3 = x_1x_2 - \frac{8}{3}x_3, \tag{4.9}$$

$$\dot{x}_4 = -5x_1. \tag{4.10}$$



(a) x_1 and x_{r1} (b) x_2 and x_{r2} (c) x_3 and x_{r3} (d) x_4 and x_{r4}

Figure 1: Comparison of the corresponding variables in the controlled hyperchaotic Rössler system and its reference system: (a) x_1 and x_{r1}; (b) x_2 and x_{r2}; (c) x_3 and x_{r3}; (d) x_4 and x_{r4}.

According to (2.1), we have

$$A_2 = \begin{bmatrix} 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} -10 & 10 & 0 & 0 \\ 28 & -1 & 0 & 1 \\ 0 & 0 & -\frac{8}{3} & 0 \\ -5 & 0 & 0 & 0 \end{bmatrix}, \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix},$$

$$A_1 = A_4 = \operatorname{diag}[0, 0, 0, 0], \tag{4.11}$$

where u_1, u_2, u_3, and u_4 are equal to 0 in the initial state and need to be determined by Theorem 3.3.

The reference model with asymptotical stability is as follows:

$$\begin{aligned} \dot{x}_{r1} &= 10(x_{r2} - x_{r1}) - 50x_{r2}, \\ \dot{x}_{r2} &= 28x_{r1} - x_{r2} - x_{r1}x_{r3} + x_{r4}, \\ \dot{x}_{r3} &= x_{r1}x_{r2} - \frac{8}{3}x_{r3}, \\ \dot{x}_{r4} &= -5x_{r1}. \end{aligned} \tag{4.12}$$

According to (2.5), the matrix A_i ($i = 1, \ldots, 4$) and $\overline{B}$ and the control vector $\overline{u}$ of the 4-dimensional reference system (4.12) have the following description:

$$\begin{gathered} \overline{A}_2 = \begin{bmatrix} 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \overline{A}_3 = \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ \overline{B} = \begin{bmatrix} -10 & -40 & 0 & 0 \\ 28 & -1 & 0 & 1 \\ 0 & 0 & -\frac{8}{3} & 0 \\ -5 & 0 & 0 & 0 \end{bmatrix}, \qquad \overline{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\ A_1 = A_4 = \operatorname{diag}[0, 0, 0, 0]. \end{gathered} \tag{4.13}$$

According to Theorem 3.3, we may let matrix $P = \operatorname{diag}[1, 1, 1]$, then the matrix

$$\left(\overline{B}^T P + P\overline{B}\right) = \begin{bmatrix} -20 & -12 & 0 & -5 \\ -12 & -2 & 0 & 1 \\ 0 & 0 & -\frac{16}{3} & 0 \\ -5 & 1 & 0 & 0 \end{bmatrix}, \tag{4.14}$$

and its largest eigenvalue is $\lambda_{\max} = 5.8387$. By considering the conditions $\lambda < -\lambda_{\max}/2$ and $\lambda \neq 0$, we should obtain $\lambda = -10$. Then, the controller should be determined by Theorem 3.3,

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} -10x_2 - 40x_{r2} \\ x_1x_3 - 9x_2 - x_4 + 9x_{r2} - x_{r1}x_{r3} + x_{r4} \\ -x_1x_2 - \dfrac{22}{3}x_3 + x_{r1}x_{r2} + \dfrac{22}{3}x_{r3} \\ 5x_1 - 10x_4 - 5x_{r4} + 10x_{r4} \end{bmatrix}. \tag{4.15}$$

Figure 2 shows that the controller can make the hyperchaotic Lorenz system (4.7)–(4.12) track along with its reference system (4.12). Here, the initials of these two systems are $[10, 0, 5, 8]$ and $[2, -10, 1, 10]$, respectively.

4.3. Example III

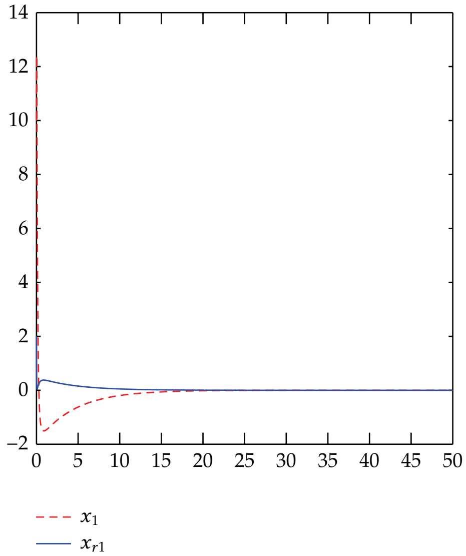
The four-variable hyperchaotic Rössler system with uncertain parameters is described by

$$\begin{aligned} \dot{x}_1 &= a(x_2 + x_3), \\ \dot{x}_2 &= x_1 + bx_2 + cx_4, \\ \dot{x}_3 &= x_1x_3 + 3, \\ \dot{x}_4 &= -0.5x_3 + dx_4. \end{aligned} \tag{4.16}$$

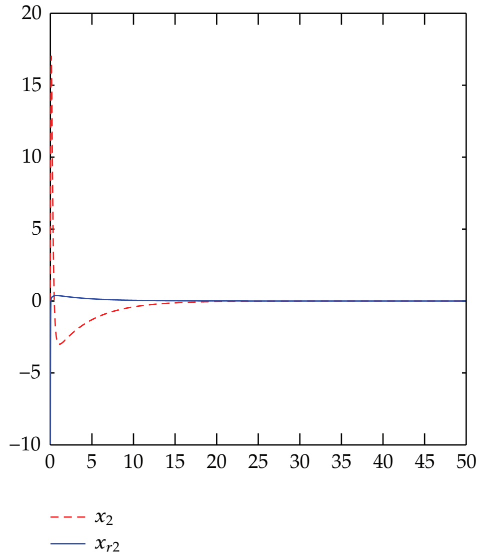
According to (2.3), we have

$$\begin{gathered} A_3 = \begin{bmatrix} 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0 \end{bmatrix}, \\ u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 + 3 \\ u_4 \end{bmatrix}, \qquad C(X) = \begin{bmatrix} x_2 + x_3 & 0 & 0 & 0 \\ 0 & x_2 & x_4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_4 \end{bmatrix}, \\ A_1 = A_2 = A_4 = \operatorname{diag}[0, 0, 0, 0], \end{gathered} \tag{4.17}$$

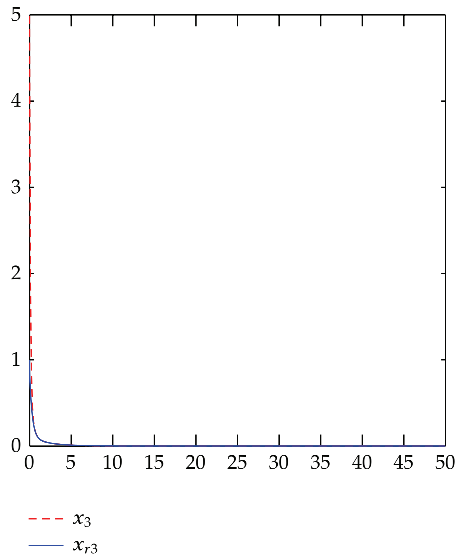
where u_1, u_2, u_3, and u_4 are equal to 0 in the initial state and need to be determined by Theorem 3.4. In Figure 3, $\widehat{\alpha}$ is the estimation value of the column vector α, where $\widehat{\alpha} = (\widehat{a}, \widehat{b}, \widehat{c}, \widehat{d})^T$.



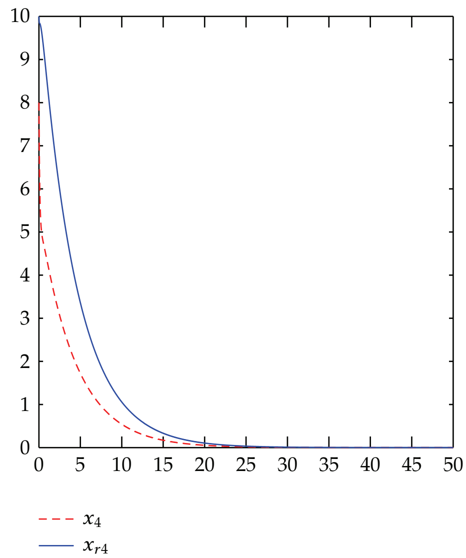
(a) x_1 and x_{r1}



(b) x_2 and x_{r2}



(c) x_3 and x_{r3}

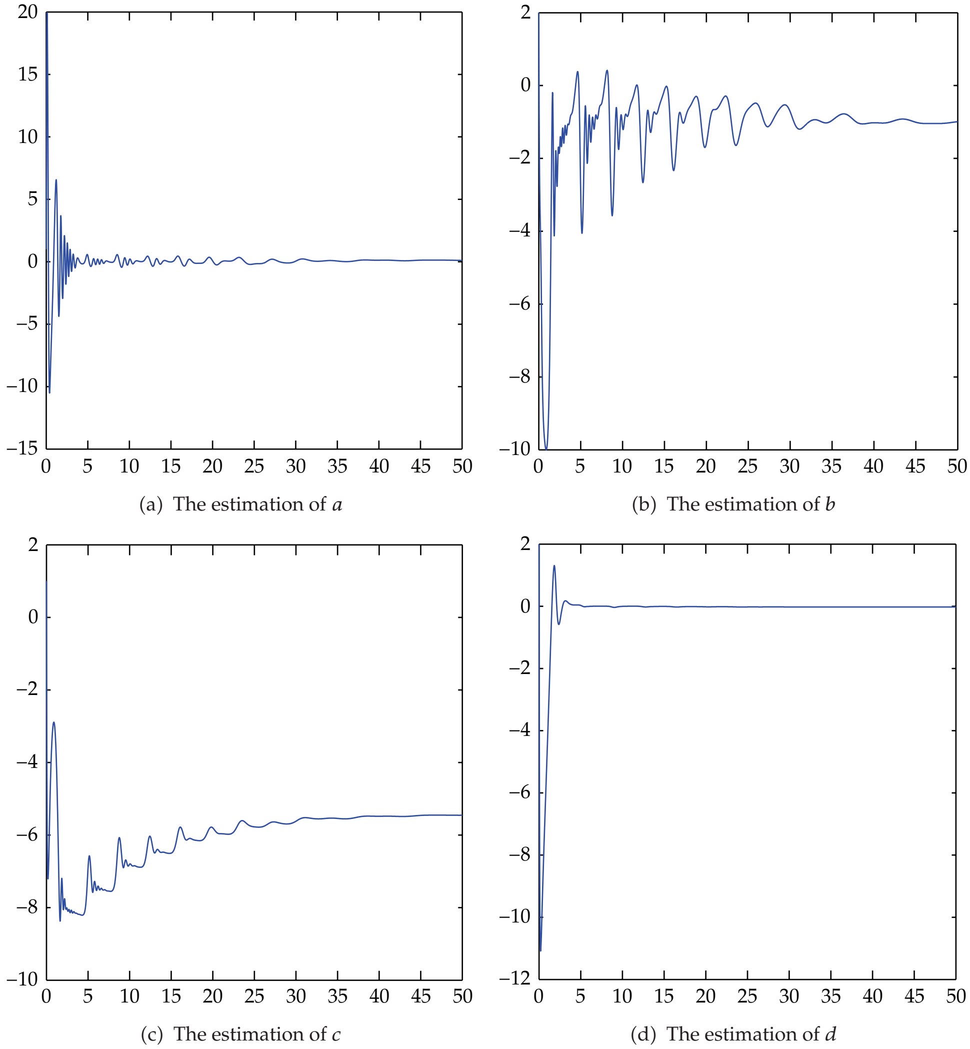


(d) x_4 and x_{r4}

Figure 2: Comparison of the corresponding variables in the controlled hyperchaotic Lorenz system and its reference system: (a) x_1 and x_{r1}; (b) x_2 and x_{r2}; (c) x_3 and x_{r3}; (d) x_4 and x_{r4}.

The estimation system of uncertain parameters is

$$\begin{aligned}\dot{\hat{a}} &= (x_2 + x_3)(x_{r1} - x_1),\\ \dot{\hat{b}} &= x_2(x_{r2} - x_2),\\ \dot{\hat{c}} &= x_4(x_{r2} - x_2),\\ \dot{\hat{d}} &= x_4(x_{r4} - x_4).\end{aligned} \tag{4.18}$$



(a) The estimation of a

(b) The estimation of b

(c) The estimation of c

(d) The estimation of d

Figure 3: The estimation of the uncertain parameters: (a) $\widehat{a}$ is the estimation of a; (b) $\widehat{b}$ is the estimation of b; (c) $\widehat{c}$ is the estimation of c; (d) $\widehat{d}$ is the estimation of d.

The reference model is the same as Example I,

$$\begin{aligned}
\dot{x}_{r1} &= -x_{r2} - x_{r3}, \\
\dot{x}_{r2} &= x_{r1} + 0.25x_{r2} + x_{r4}, \\
\dot{x}_{r3} &= 20x_{r1} + 10x_{r2} - 50x_{r3} + 100x_{r4} + 3, \\
\dot{x}_{r4} &= -0.5x_{r3} + 0.05x_{r4}, \\
\overline{B} &= \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0.25 & 0 & 1 \\ 20 & 10 & -50 & 100 \\ 0 & 0 & -0.5 & 0.05 \end{bmatrix}, \qquad \overline{u} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}.
\end{aligned} \tag{4.19}$$

We have the largest eigenvalue $\lambda_{\max} = -0.9675$. By considering the conditions $\lambda < -\lambda_{\max}$ and $\lambda \neq 0$, we should obtain $\lambda = -1$.

Figure 4 shows that the controller can make the hyperchaotic Rössler system with four uncertain parameters (4.1) track along with its reference system (4.3). Here, the initials of the hyperchaotic system, reference system, and estimation system of uncertain parameters are $[-10, 6, 0, 10]$, $[10, -6, 10, -10]$, and $[1, 2, 1, 2]$, respectively.

Comparing with Figures 1(c), 1(d), 4(c), and 4(d), we find that our method has even a better performance in the uncertain case than the certain case. In Figure 4(a), we can see some waves in time domain $[0, 2]$, and it is not better than the certain case. In Figures 4(b) and 1(b), the controlling performance is similar.

4.4. Examples IV

The four-variable hyperchaotic Lorenz system with uncertain parameters is described by

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1), \\ \dot{x}_2 &= bx_1 - cx_2 - x_1x_3 + dx_4, \\ \dot{x}_3 &= x_1x_2 - fx_3, \\ \dot{x}_4 &= gx_1, \end{aligned} \tag{4.20}$$

where

$$\begin{gathered} A_2 = \begin{bmatrix} 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}, \qquad \alpha = \begin{bmatrix} a \\ b \\ c \\ d \\ f \\ g \end{bmatrix}, \\ C(X) = \begin{bmatrix} x_2 - x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_1 & -x_2 & x_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_1 \end{bmatrix}, \\ A_1 = A_4 = \operatorname{diag}[0, 0, 0, 0], \end{gathered} \tag{4.21}$$

where u_1, u_2, u_3, and u_4 are equal to 0 in the initial state and need to be determined by Theorem 3.5. Figure 5 shows the estimation of α, and $\hat{\alpha}$ is the estimation value of the column vector α, where $\hat{\alpha} = (\hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{f}, \hat{g})^T$.

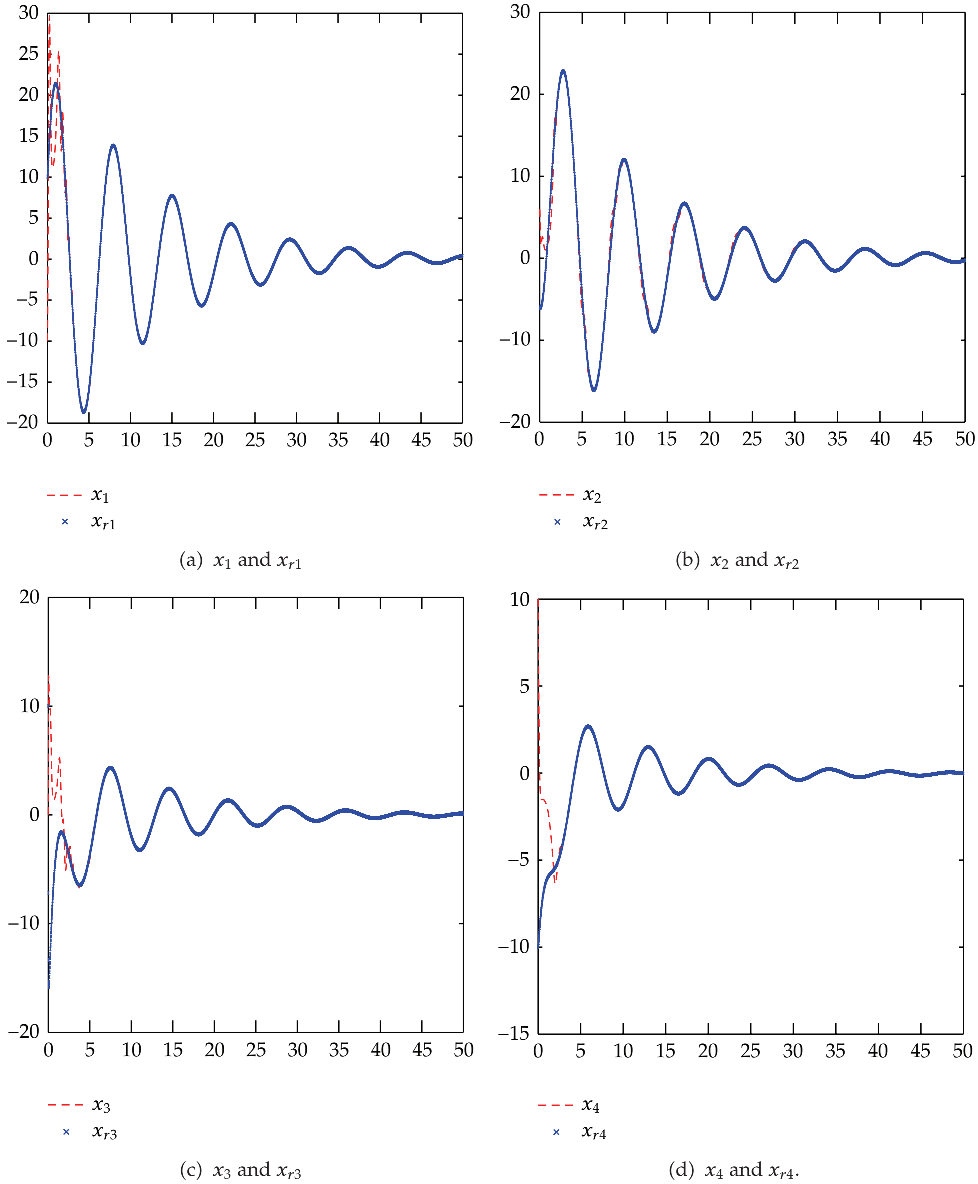


Figure 4: Comparison of the corresponding variables in the controlled hyperchaotic Rössler system and its reference system: (a) x_1 and x_{r1}; (b) x_2 and x_{r2}; (c) x_3 and x_{r3}; (d) x_4 and x_{r4}.

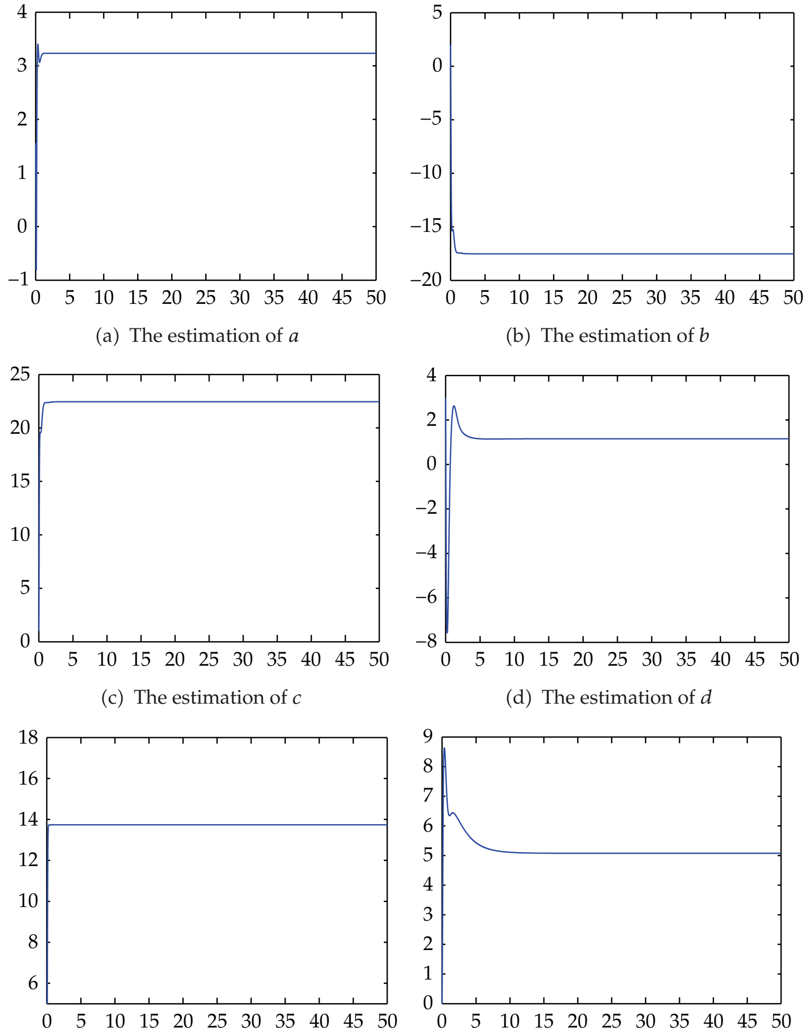
The estimation system of uncertain parameters is

$$\dot{\hat{a}} = (x_2 - x_1)(x_{r1} - x_1),$$

$$\dot{\hat{b}} = x_1(x_{r2} - x_2),$$

$$\dot{\hat{c}} = -x_2(x_{r2} - x_2),$$

$$\dot{\hat{d}} = x_4(x_{r2} - x_2),$$



(a) The estimation of a

(b) The estimation of b

(c) The estimation of c

(d) The estimation of d

(e) The estimation of f

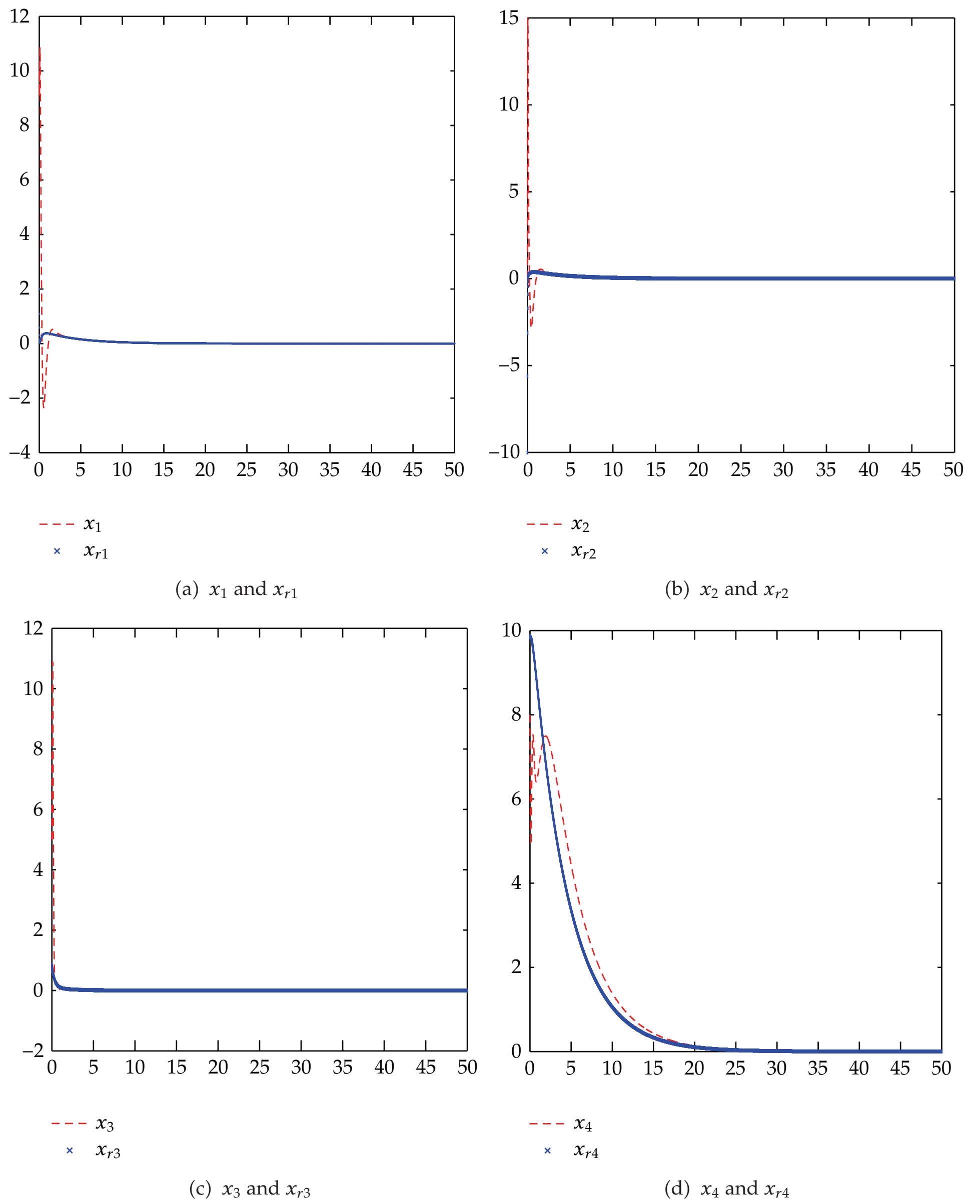
(f) The estimation of g

Figure 5: The estimation of the uncertain parameters: (a) $\hat{a}$ is the estimation of a; (b) $\hat{b}$ is the estimation of b; (c) $\hat{c}$ is the estimation of c; (d) $\hat{d}$ is the estimation of d; (e) $\hat{f}$ is the estimation of f; (f) $\hat{g}$ is the estimation of g.

$$\begin{aligned} \dot{\hat{f}} &= -x_3(x_{r3} - x_{r3}), \\ \dot{\hat{g}} &= x_1(x_{r4} - x_4). \end{aligned} \tag{4.22}$$

The reference model is the same as Example II with asymptotical stability is as follows:

$$\begin{aligned} \dot{x}_{r1} &= 10(x_{r2} - x_{r1}) - 50x_{r2}, \\ \dot{x}_{r2} &= 28x_{r1} - x_{r2} - x_{r1}x_{r3} + x_{r4}, \end{aligned}$$



(a) x_1 and x_{r1} (b) x_2 and x_{r2} (c) x_3 and x_{r3} (d) x_4 and x_{r4}

Figure 6: Comparison of the corresponding variables in the controlled hyperchaotic Lorenz system and its reference system: (a) x_1 and x_{r1}; (b) x_2 and x_{r2}; (c) x_3 and x_{r3}; (d) x_4 and x_{r4}.

$$\dot{x}_{r3} = x_{r1}x_{r2} - \frac{8}{3}x_{r3},$$
$$\dot{x}_{r4} = -5x_{r1}, \tag{4.23}$$

and the largest eigenvalue of $\overline{B}^T$ is $\lambda_{\max} = 0.1767$. By considering the conditions $\lambda < -\lambda_{\max}$ and $\lambda \neq 0$, we should obtain $\lambda = -1$.

Figure 6 shows that the controller can make the hyperchaotic Lorenz system with uncertain parameters (4.20) track along with its reference system (4.23). Here, the initials of the hyperchaotic system, the reference system, and the estimation system of uncertain parameters are $[10, 0, 5, 8]$, $[2, -10, 1, 10]$, and $[1, 2, 1, 3, 1, 0]$, respectively.

The performance in Figures 6(a)-6(b) is much better than the one in Figures 2(a)-2(b). Other parts of the two figures have almost the same performance.

5. Conclusion

In this paper, we have used an MRC technique to control the hyperchaotic system to track with an expected trajectory. The expression of the controller has been given. Four numerical examples show that the MRC method is very effective for controlling both the hyperchaotic system with all certain parameters and the systems with uncertain parameters. By comparing the results in the corresponding figures, we find that our method does not only fit for controlling hyperchaotic systems with certain parameters, but also is a robust for the systems with uncertain parameters.

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