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Research Article

The Lie Group in Infinite Dimension

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A Lie group acting on finite-dimensional space is generated by its infinitesimal transformations and conversely, any Lie algebra of vector fields in finite dimension generates a Lie group (the first fundamental theorem). This classical result is adjusted for the infinite-dimensional case. We prove that the (local, C^∞ smooth) action of a Lie group on infinite-dimensional space (a manifold modelled on $\mathbb{R}^\infty$) may be regarded as a limit of finite-dimensional approximations and the corresponding Lie algebra of vector fields may be characterized by certain finiteness requirements. The result is applied to the theory of generalized (or higher-order) infinitesimal symmetries of differential equations.

1. Preface

In the symmetry theory of differential equations, the *generalized* (*or*: *higher-order*, *Lie-Bäcklund*) *infinitesimal symmetries*

$$Z = \sum z_i \frac{\partial}{\partial x_i} + \sum z_I^j \frac{\partial}{\partial w_I^j} \quad \left(i = 1, \dots, n;\ j = 1, \dots, m;\ I = i_1 \cdots i_n;\ i_1, \dots, i_n = 1, \dots, n\right), \tag{1.1}$$

where the coefficients

$$z_i = z_i\left(\dots, x_{i'}, w_{I'}^{j'}, \dots\right), \qquad z_I^j = z_I^j\left(\dots, x_{i'}, w_{I'}^{j'}, \dots\right) \tag{1.2}$$

are functions of independent variables x_i, dependent variables w^j and a finite number of jet variables $w_I^j = \partial^n w^j / \partial x_{i_1} \cdots \partial x_{i_n}$ belong to well-established concepts. However, in spite of

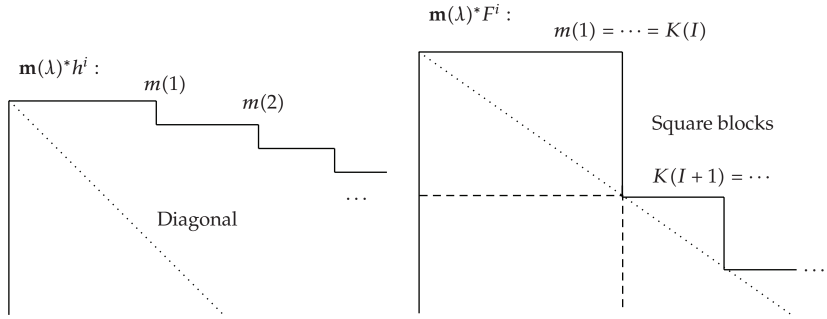

Figure 1

this matter of fact, they cause an unpleasant feeling. Indeed, such vector fields as a rule do not generate any one-parameter group of transformations

$$\overline{x}_i = G_i\left(\lambda; \ldots, x_{i'}, w_{I'}^{j'}, \ldots\right), \qquad \overline{w}_I^j = G_I^j\left(\lambda; \ldots, x_{i'}, w_{I'}^{j'}, \ldots\right) \tag{1.3}$$

in the underlying infinite-order jet space since the relevant Lie system

$$\frac{\partial G_i}{\partial \lambda} = z_i\left(\ldots, G_{i'}, G_{I'}^{j'}, \ldots\right), \quad \frac{\partial G_I^j}{\partial \lambda} = z_I^j\left(\ldots, G_{i'}, G_{I'}^{j'}, \ldots\right) \quad \left(G_i|_{\lambda=0} = x_i,\, G_I^j\Big|_{\lambda=0} = w_I^j\right) \tag{1.4}$$

need not have any reasonable (locally unique) solution. Then Z is a mere formal concept [1–7] not related to any true transformations and the term "infinitesimal symmetry Z" is misleading, no Z-symmetries of differential equations in reality appear.

In order to clarify the situation, we consider one-parameter groups of local transformations in $\mathbb{R}^\infty$. We will see that they admit "finite-dimensional approximations" and as a byproduct, the relevant infinitesimal transformations may be exactly characterized by certain "finiteness requirements" of purely algebraical nature. With a little effort, the multidimensional groups can be easily involved, too. This result was briefly discussed in [8, page 243] and systematically mentioned at several places in monograph [9], but our aim is to make some details more explicit in order to prepare the necessary tools for systematic investigation of *groups of generalized symmetries*. We intend to continue our previous articles [10–13] where the algorithm for determination of all *individual generalized symmetries* was already proposed.

For the convenience of reader, let us transparently describe the crucial approximation result. We consider transformations (2.1) of a local one-parameter group in the space $\mathbb{R}^\infty$ with coordinates $h^1, h^2, \ldots$. Equations (2.1) of transformations $\mathbf{m}(\lambda)$ can be schematically represented by Figure 1(a).

We prove that in appropriate new coordinate system $F^1, F^2, \ldots$ on $\mathbb{R}^\infty$, the same transformations $\mathbf{m}(\lambda)$ become block triangular as in Figure 1(b). It follows that a certain hierarchy of finite-dimensional subspaces of $\mathbb{R}^\infty$ is preserved which provides the "approximation" of $\mathbf{m}(\lambda)$. The infinitesimal transformation $Z = \mathrm{d}\mathbf{m}(\lambda)/\mathrm{d}\lambda|_{\lambda=0}$ clearly preserves the same hierarchy which provides certain algebraical "finiteness" of Z.

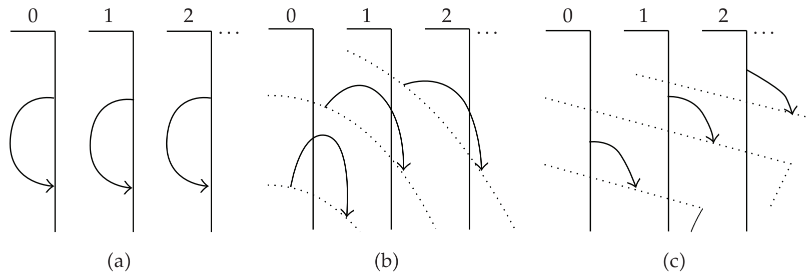

Figure 2

If the primary space $\mathbb{R}^\infty$ is moreover equipped with an appropriate structure, for example, the contact forms, it turns into the jet space and the results concerning the transformation groups on $\mathbb{R}^\infty$ become the theory of higher-order symmetries of differential equations. Unlike the common point symmetries which occupy a number of voluminous monographs (see, e.g., [14, 15] and extensive references therein) this higher-order theory was not systematically investigated yet. We can mention only the isolated article [16] which involves a direct proof of the "finiteness requirements" for one-parameter groups (namely, the result (ι) of Lemma 5.4 below) with two particular examples and monograph [7] involving a theory of generalized infinitesimal symmetries in the formal sense.

Let us finally mention the intentions of this paper. In the *classical theory of point or Lie's contact-symmetries* of differential equations, the order of derivatives is preserved (Figure 2(a)). Then the common Lie's and Cartan's methods acting in finite dimensional spaces *given ahead of calculations* can be applied. On the other extremity, the *generalized symmetries* need not preserve the order (Figure 2(c)) and even any finite-dimensional space and then the common classical methods fail. For the favourable intermediate case of *groups of generalized symmetries*, the invariant finite-dimensional subspaces exist, however, they are *not known in advance* (Figure 2(b)). We believe that the classical methods can be appropriately adapted for the latter case, and this paper should be regarded as a modest preparation for this task.

2. Fundamental Approximation Results

Our reasonings will be carried out in the space $\mathbb{R}^\infty$ with coordinates $h^1, h^2, \ldots$ [9] and we introduce the structural family $\mathcal{F}$ of all real-valued, locally defined and C^∞-smooth functions $f = f(h^1, \ldots, h^{m(f)})$ depending on a finite number of coordinates. In future, such functions will contain certain C^∞-smooth real parameters, too.

We are interested in (local) groups of transformations $\mathbf{m}(\lambda)$ in $\mathbb{R}^\infty$ defined by formulae

$$\mathbf{m}(\lambda)^* h^i = H^i\left(\lambda; h^1, \ldots, h^{m(i)}\right), \quad -\varepsilon^i < \lambda < \varepsilon^i,\ \varepsilon^i > 0\ (i = 1, 2, \ldots), \tag{2.1}$$

where $H^i \in \mathcal{F}$ if the parameter λ is kept fixed. We suppose

$$\mathbf{m}(0) = \text{id.}, \qquad \mathbf{m}(\lambda + \mu) = \mathbf{m}(\lambda)\mathbf{m}(\mu) \tag{2.2}$$

whenever it makes a sense. An open and common definition domain for all functions H^i is tacitly supposed. (In more generality, a common definition domain for *every finite number* of functions H^i is quite enough and the germ and sheaf terminology would be more adequate for our reasonings, alas, it looks rather clumsy.)

Definition 2.1. For every $I = 1, 2, \ldots$ and $0 < \varepsilon < \min\{\varepsilon^1, \ldots, \varepsilon^I\}$, let $\mathcal{F}(I,\varepsilon) \subset \mathcal{F}$ be the subset of all composed functions

$$F = F\Big(\ldots, \mathbf{m}(\lambda_j)^* h^i, \ldots\Big) = F\Big(\ldots, H^i\Big(\lambda_j; h^1, \ldots, h^{m(i)}\Big), \ldots\Big), \tag{2.3}$$

where $i = 1, \ldots, I$; $-\varepsilon < \lambda_j < \varepsilon$; $j = 1, \ldots, J = J(I) = \max\{m(1), \ldots, m(I)\}$ and F is arbitrary C^∞-smooth function (of IJ variables). In functions $F \in \mathcal{F}(I,\varepsilon)$, variables $\lambda_1, \ldots, \lambda_J$ are regarded as mere parameters.

Functions (2.3) will be considered on open subsets of $\mathbb{R}^\infty$ where the rank of the Jacobi $(IJ \times J)$-matrix

$$\left(\frac{\partial}{\partial h^{j'}} H^i\Big(\lambda_j; h^1, \ldots, h^{m(i)}\Big)\right) \quad (i = 1, \ldots, I;\ j, j' = 1, \ldots, J) \tag{2.4}$$

of functions $H^i(\lambda_j; h^1, \ldots, h^{m(i)})$ locally attains the maximum (for appropriate choice of parameters). This rank and therefore the subset $\mathcal{F}(I,\varepsilon) \subset \mathcal{F}$ does not depend on ε as soon as $\varepsilon = \varepsilon(I)$ is close enough to zero. *This is supposed from now on and we may abbreviate* $\mathcal{F}(I) = \mathcal{F}(I,\varepsilon)$.

We deal with highly nonlinear topics. Then the definition domains cannot be kept fixed in advance. Our results will be true *locally*, near *generic points*, on certain *open everywhere dense subsets* of the underlying space $\mathbb{R}^\infty$. With a little effort, the subsets can be exactly characterized, for example, by locally constant rank of matrices, functional independence, existence of implicit function, and so like. We follow the common practice and as a rule omit such routine details from now on.

Lemma 2.2 (approximation lemma). *The following inclusion is true:*

$$\mathbf{m}(\lambda)^* \mathcal{F}(I) \subset \mathcal{F}(I). \tag{2.5}$$

Proof. Clearly

$$\mathbf{m}(\lambda)^* H^i(\lambda_j; \ldots) = \mathbf{m}(\lambda)^* \mathbf{m}(\lambda_j)^* h^i = \mathbf{m}(\lambda + \lambda_j)^* h^i = H^i(\lambda + \lambda_j; \ldots) \tag{2.6}$$

and therefore

$$\mathbf{m}(\lambda)^* F = F\Big(\ldots, H^i\Big(\lambda + \lambda_j; h^1, \ldots, h^{m(i)}\Big), \ldots\Big) \in \mathcal{F}(I). \tag{2.7}$$

□

Denoting by $K(I)$ the rank of matrix (2.4), there exist *basical functions*

$$F^k = F^k\left(\dots, H^i\left(\lambda_j; h^1, \dots, h^{m(i)}\right), \dots\right) \in \mathcal{F}(I) \quad (k = 1, \dots, K(I)) \tag{2.8}$$

such that $\operatorname{rank}(\partial F^k / \partial h^{j'}) = K(I)$. Then a function $f \in \mathcal{F}$ lies in $\mathcal{F}(I)$ if and only if $f = \overline{f}(F^1, \dots, F^{K(I)})$ is a composed function. In more detail

$$F = \overline{F}\left(\lambda_1, \dots, \lambda_J; F^1, \dots, F^{K(I)}\right) \in \mathcal{F}(I) \tag{2.9}$$

is such a composed function if we choose $f = F$ given by (2.3). Parameters $\lambda_1, \dots, \lambda_J$ occurring in (2.3) are taken into account here. It follows that

$$\frac{\partial F}{\partial \lambda_j} = \frac{\partial \overline{F}}{\partial \lambda_j}\left(\lambda_1, \dots, \lambda_J; F^1, \dots, F^{K(I)}\right) \in \mathcal{F}(I) \quad (j = 1, \dots, J) \tag{2.10}$$

and analogously for the higher derivatives.

In particular, we also have

$$H^i\left(\lambda; h^1, \dots, h^{m(i)}\right) = \overline{H}^i\left(\lambda; F^1, \dots, F^{K(I)}\right) \in \mathcal{F}(I) \quad (i = 1, \dots, I) \tag{2.11}$$

for the choice $F = H^i(\lambda; \dots)$ in (2.9) whence

$$\frac{\partial^r H^i}{\partial \lambda^r} = \frac{\partial^r \overline{H}^i}{\partial \lambda^r}\left(\lambda; F^1, \dots, F^{K(I)}\right) \in \mathcal{F}(I) \quad (i = 1, \dots, I;\ r = 0, 1, \dots). \tag{2.12}$$

The basical functions can be taken from the family of functions $H^i(\lambda; \dots)$ $(i = 1, \dots, I)$ for appropriate choice of *various* values of λ. Functions (2.12) are enough as well even for a *fixed* value λ, for example, for $\lambda = 0$, see Theorem 3.2 below.

Lemma 2.3. *For any basical function, one has*

$$\mathbf{m}(\lambda)^* F^k = \overline{F}^k\left(\lambda; F^1, \dots, F^{K(I)}\right) \quad (k = 1, \dots, K(I)). \tag{2.13}$$

Proof. $F^k \in \mathcal{F}(I)$ implies $\mathbf{m}(\lambda)^* F^k \in \mathcal{F}(I)$ and (2.9) may be applied with the choice $F = \mathbf{m}(\lambda)^* F^k$ and $\lambda_1 = \cdots = \lambda_J = \lambda$. □

Summary 1. Coordinates $h^i = H^i(0; \dots)$ $(i = 1, \dots, I)$ were included into the subfamily $\mathcal{F}(I) \subset \mathcal{F}$ which is transformed into itself by virtue of (2.13). So we have a one-parameter group acting on $\mathcal{F}(I)$. One can even choose $F^1 = h^1, \dots, F^I = h^I$ here and then, if I is large enough, formulae (2.13) provide a "finite-dimensional approximation" of the primary mapping $\mathbf{m}(\lambda)$. The block-triangular structure of the infinite matrix of transformations $\mathbf{m}(\lambda)$ mentioned in Section 1 appears if $I \to \infty$ and the system of functions $F^1, F^2, \dots$ is succesively completed.

3. The Infinitesimal Approach

We introduce the vector field

$$Z = \sum z^i \frac{\partial}{\partial h^i} = \left.\frac{\mathrm{d}\mathbf{m}(\lambda)}{\mathrm{d}\lambda}\right|_{\lambda=0} \quad \left(z^i = \frac{\partial H^i}{\partial \lambda}\left(0; h^1, \dots, h^{m(i)}\right);\ i = 1, 2, \dots \right), \tag{3.1}$$

the *infinitesimal transformation* ($\mathcal{IT}$) of group $\mathbf{m}(\lambda)$. Let us recall the celebrated Lie system

$$\begin{aligned} \frac{\partial}{\partial \lambda}\mathbf{m}(\lambda)^* h^i &= \frac{\partial H^i}{\partial \lambda}(\lambda; \dots) = \left.\frac{\partial H^i}{\partial \mu}(\lambda + \mu; \dots)\right|_{\mu=0} \\ &= \left.\frac{\partial}{\partial \mu}\mathbf{m}(\lambda+\mu)^* h^i\right|_{\mu=0} = \mathbf{m}(\lambda)^* \left.\frac{\partial}{\partial \mu}\mathbf{m}(\mu)^* h^i\right|_{\mu=0} = \mathbf{m}(\lambda)^* Z h^i = \mathbf{m}(\lambda)^* z^i. \end{aligned} \tag{3.2}$$

In more explicit (and classical) transcription

$$\frac{\partial H^i}{\partial \lambda}\left(\lambda; h^1, \dots, h^{m(i)}\right) = z^i\left(H^1\left(\lambda; h^1, \dots, h^{m(1)}\right), \dots, H^{m(i)}\left(\lambda; h^1, \dots, h^{m(m(i))}\right)\right). \tag{3.3}$$

One can also check the general identity

$$\frac{\partial^r}{\partial \lambda^r}\mathbf{m}(\lambda)^* f = \mathbf{m}(\lambda)^* Z^r f \quad (f \in \mathcal{F};\ r = 0, 1, \dots) \tag{3.4}$$

by a mere routine induction on r.

Lemma 3.1 (finiteness lemma). *For all $r \in \mathbb{N}$, $Z^r \mathcal{F}(I) \subset \mathcal{F}(I)$.*

Proof. Clearly

$$ZF = \mathbf{m}(\lambda)^* ZF|_{\lambda=0} = \left.\frac{\partial}{\partial \lambda}\mathbf{m}(\lambda)^* F\right|_{\lambda=0} \in \mathcal{F}(I) \tag{3.5}$$

for any function (2.3) by virtue of (2.10): induction on r. □

Theorem 3.2 (finiteness theorem). *Every function $F \in \mathcal{F}(I)$ admits (locally, near generic points) the representation*

$$F = \widetilde{F}\left(\dots, \frac{\partial^r H^i}{\partial \lambda^r}\left(0; h^1, \dots, h^{m(i)}\right), \dots\right) \tag{3.6}$$

in terms of a composed function where $i = 1, \dots, I$ and $\widetilde{F}$ is a $\mathbb{C}^\infty$-smooth function of a finite number of variables.

Proof. Let us temporarily denote

$$H_r^i = \frac{\partial^r H^i}{\partial \lambda^r}(\lambda; \ldots) = \frac{\partial^r}{\partial \lambda^r}\mathbf{m}(\lambda)^* h^i, \qquad h_r^i = H_r^i(0; \ldots) = Z^r h^i, \tag{3.7}$$

where the second equality follows from (3.4) with $f = h^i$, $\lambda = 0$. Then

$$H_r^i = \mathbf{m}(\lambda)^* h_r^i = \mathbf{m}(\lambda)^* Z^r h^i \tag{3.8}$$

by virtue of (3.4) with general λ.

If $j = j(i)$ is large enough, there does exist an identity $h_{j+1}^i = G^i(h_0^i, \ldots, h_j^i)$. Therefore

$$\frac{\partial^{j+1} H^i}{\partial \lambda^{j+1}} = H_{j+1}^i = G^i\left(H_0^i, \ldots, H_j^i\right) = G^i\left(H^i, \ldots, \frac{\partial^j H^i}{\partial \lambda^j}\right) \tag{3.9}$$

by applying $\mathbf{m}(\lambda)^*$. This may be regarded as ordinary differential equation with initial values

$$H^i\Big|_{\lambda=0} = h_0^i, \ldots, \frac{\partial^j H^i}{\partial \lambda^j}\Big|_{\lambda=0} = h_j^i. \tag{3.10}$$

The solution $H^i = \widetilde{H}^i(\lambda; h_0^i, \ldots, h_j^i)$ expressed in terms of initial values reads

$$H^i\left(\lambda; h^1, \ldots, h^{m(i)}\right) = \widetilde{H}^i\left(\lambda; H^i\left(0; h^1, \ldots, h^{m(i)}\right), \ldots, \frac{\partial^j H^i}{\partial \lambda^j}\left(0; h^1, \ldots, h^{m(i)}\right)\right) \tag{3.11}$$

in full detail. If λ is kept fixed, this is exactly the identity (3.6) for the particular case $F = H^i(\lambda; h^1, \ldots, h^{m(i)})$. The general case follows by a routine. □

Definition 3.3. Let $\mathbb{G}$ be the set of (local) vector fields

$$Z = \sum z^i \frac{\partial}{\partial h^i} \quad \left(z^i \in \mathcal{F}, \text{ infinite sum}\right) \tag{3.12}$$

such that every family of functions $\{Z^r h^i\}_{r \in \mathbb{N}}$ (i fixed but arbitrary) can be expressed in terms of a finite number of coordinates.

Remark 3.4. Neither $\mathbb{G} + \mathbb{G} \subset \mathbb{G}$ nor $[\mathbb{G}, \mathbb{G}] \subset \mathbb{G}$ as follows from simple examples. However, $\mathbb{G}$ is a *conical set* (over $\mathcal{F}$): if $Z \in \mathbb{G}$ then $fZ \in \mathbb{G}$ for any $f \in \mathcal{F}$. Easy direct proof may be omitted here.

Summary 2. If Z is $\mathcal{IC}$ of a group then all functions $Z^r h^i$ ($i = 1, \ldots, I$; $r = 0, 1, \ldots$) are included into family $\mathcal{F}(I)$ hence $Z \in \mathbb{G}$. The converse is clearly also true: every vector field $Z \in \mathbb{G}$ generates a local Lie group since the Lie system (3.3) admits finite-dimensional approximations in spaces $\mathcal{F}(I)$.

Let us finally reformulate the last sentence in terms of basical functions.

Theorem 3.5 (approximation theorem). *Let $Z \in \mathbb{G}$ be a vector field locally defined on $\mathbb{R}^\infty$ and $F^1, \ldots, F^{K(I)} \in \mathcal{F}$ be a maximal functionally independent subset of the family of all functions*

$$Z^r h^i \quad (i = 1, \ldots, I;\ r = 0, 1, \ldots). \tag{3.13}$$

Denoting $ZF^k = \overline{F}^k(F^1, \ldots, F^{K(I)})$, then the system

$$\frac{\partial}{\partial \lambda} \mathbf{m}(\lambda)^* F^k = \mathbf{m}(\lambda)^* Z F^k = \overline{F}^k\left(\mathbf{m}(\lambda)^* F^1, \ldots, \mathbf{m}(\lambda)^* F^{K(I)}\right) \quad (k = 1, \ldots, K(I)) \tag{3.14}$$

may be regarded as a "finite-dimensional approximation" to the Lie system (3.3) *of the one-parameter local group* $\mathbf{m}(\lambda)$ *generated by Z.*

In particular, assuming $F^1 = h^1, \ldots, F^I = h^I$, then the the initial portion

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \mathbf{m}(\lambda)^* F^i = \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathbf{m}(\lambda)^* h^i = \frac{\mathrm{d}}{\mathrm{d}\lambda} H^i = z^i\left(H^1, \ldots, H^{m(i)}\right) \quad (i = 1, \ldots, I) \tag{3.15}$$

of the above system transparently demonstrates the approximation property.

4. On the Multiparameter Case

The following result does not bring much novelty and we omit the proof.

Theorem 4.1. *Let $Z_1, \ldots, Z_d$ be commuting local vector fields in the space $\mathbb{R}^\infty$. Then $Z_1, \ldots, Z_d \in \mathbb{G}$ if and only if the vector fields $Z = a_1 Z_1 + \cdots + a_d Z_d$ $(a_1, \ldots, a_d \in \mathbb{R})$ locally generate an abelian Lie group.*

In full non-Abelian generality, let us consider a (local) multiparameter group formally given by the same equations (2.1) as above where $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d$ are parameters close to the zero point $0 = (0, \ldots, 0) \in \mathbb{R}^d$. The rule (2.2) is generalized as

$$\mathbf{m}(0) = \mathrm{id.}, \qquad \mathbf{m}(\varphi(\lambda, \mu)) = \mathbf{m}(\lambda)\mathbf{m}(\mu), \tag{4.1}$$

where $\lambda = (\lambda_1, \ldots, \lambda_d)$, $\mu = (\mu_1, \ldots, \mu_d)$ and $\varphi = (\varphi_1, \ldots, \varphi_d)$ determine the composition of parameters. Appropriately adapting the space $\mathcal{F}(I)$ and the concept of basical functions $F^1, \ldots, F^{K(I)}$, Lemma 2.2 holds true without any change.

Passing to the infinitesimal approach, we introduce vector fields $Z_1, \ldots, Z_d$ which are $\mathcal{IT}$ of the group. We recall (without proof) the Lie equations [17]

$$\frac{\partial}{\partial \lambda_j} \mathbf{m}(\lambda)^* f = \sum a_i^j(\lambda) \mathbf{m}(\lambda)^* Z_j f \quad (f \in \mathcal{F};\ j = 1, \ldots, d) \tag{4.2}$$

with the initial condition $\mathbf{m}(0) = \mathrm{id}$. Assuming $Z_1, \ldots, Z_d$ linearly independent over $\mathbb{R}$, coefficients $a_i^j(\lambda)$ may be arbitrarily chosen and the solution $\mathbf{m}(\lambda)$ always is a group transformation (the first fundamental theorem). If basical functions $F^1,\ldots,F^{K(I)}$ are inserted for f, we have a finite-dimensional approximation which is self-contained in the sense that

$$Z_jF^k=\widetilde{F}_j^k\left(F^1,\ldots,F^{K(I)}\right)\quad (j=1,\ldots,d;\ k=1,\ldots,K(I)) \tag{4.3}$$

are composed functions in accordance with the definition of the basical functions.

Let us conversely consider a Lie algebra of local vector fields $Z=a_1Z_1+\cdots+a_dZ_d$ ($a_i\in\mathbb{R}$) on the space $\mathbb{R}^\infty$. Let moreover $Z_1,\ldots,Z_d\in\mathbb{G}$ *uniformly* in the sense that there is a *universal space* $\mathcal{F}(I)$ with $\mathcal{L}_{Z_i}\mathcal{F}(I)\subset\mathcal{F}(I)$ for all $i=1,\ldots,d$. Then the Lie equations may be applied and we obtain reasonable finite-dimensional approximations.

Summary 3. Theorem 4.1 holds true even in the non-Abelian and multidimensional case if the inclusions $Z_1,\ldots,Z_d\in\mathbb{G}$ are uniformly satisfied.

As yet we have closely simulated the primary one-parameter approach, however, the results are a little misleading: the uniformity requirement in Summary 3 may be completely omitted. This follows from the following result [9, page 30] needless here and therefore stated without proof.

Theorem 4.2. *Let $\mathcal{K}$ be a finite-dimensional submodule of the module of vector fields on $\mathbb{R}^\infty$ such that $[\mathcal{K},\mathcal{K}]\subset\mathcal{K}$. Then $\mathcal{K}\subset\mathbb{G}$ if and only if there exist generators (over $\mathcal{F}$) of submodule $\mathcal{K}$ that are lying in $\mathbb{G}$.*

5. Symmetries of the Infinite-Order Jet Space

The previous results can be applied to the groups of generalized symmetries of *partial differential equations*. Alas, some additional technical tools cannot be easily explained at this place, see the concluding Section 11 below. So we restrict ourselves to the *trivial differential equations*, that is, to the groups of generalized symmetries in the total *infinite-order jet space* which do not require any additional preparations.

Let $\mathbf{M}(m,n)$ be the jet space of n-dimensional submanifolds in $\mathbb{R}^{m+n}$ [9–13]. We recall the familiar (local) jet coordinates

$$x_i,w_I^j\quad (I=i_1\ldots i_r;\ i,i_1,\ldots,i_r=1,\ldots,n;\ r=0,1,\ldots;\ j=1,\ldots,m). \tag{5.1}$$

Functions $f=f(\ldots,x_i,w_I^j,\ldots)$ on $\mathbf{M}(m,n)$ are C^∞-smooth and depend on a finite number of coordinates. The jet coordinates serve as a mere technical tool. The true jet structure is given just by the *module* $\Omega(m,n)$ *of contact forms*

$$\omega=\sum a_I^j\omega_I^j\quad \left(\text{finite sum},\omega_I^j=\mathrm{d}w_I^j-\sum w_{Ii}^j\mathrm{d}x_i\right) \tag{5.2}$$

or, equivalently, by the *"orthogonal" module* $\mathcal{H}(m,n)=\Omega^\perp(m,n)$ *of formal derivatives*

$$D=\sum a_iD_i\left(D_i=\frac{\partial}{\partial x_i}+\sum w_{Ii}^j\frac{\partial}{\partial w_I^j};\ i=1,\ldots,n;D\rfloor\omega_I^j=\omega_I^j(D)=0\right). \tag{5.3}$$

Let us state useful formulae

$$\mathrm{d}f = \sum D_i f\,\mathrm{d}x_i + \sum \frac{\partial f}{\partial w_I^j}\omega_I^j, \qquad D_i\rfloor \mathrm{d}\omega_I^j = \omega_{Ii}^j, \qquad \mathcal{L}_{D_i}\omega_I^j = \omega_{Ii}^j, \tag{5.4}$$

where $\mathcal{L}_{D_i} = D_i\rfloor \mathrm{d} + \mathrm{d}D_i\rfloor$ denotes the Lie derivative.

We are interested in (local) one-parameter groups of transformations $\mathbf{m}(\lambda)$ given by certain formulae

$$\mathbf{m}(\lambda)^* x_i = G_i\left(\lambda; \dots, x_{i'}, w_{I'}^{j'}, \dots\right), \qquad \mathbf{m}(\lambda)^* w_I^j = G_I^j\left(\lambda; \dots, x_{i'}, w_{I'}^{j'}, \dots\right) \tag{5.5}$$

and in vector fields

$$Z = \sum z_i\left(\dots, x_{i'}, w_{I'}^{j'}, \dots\right)\frac{\partial}{\partial x_i} + \sum z_I^j\left(\dots, x_{i'}, w_{I'}^{j'}, \dots\right)\frac{\partial}{\partial w_I^j} \tag{5.6}$$

locally defined on the jet space $\mathbf{M}(m,n)$; see also (1.1) and (1.2).

Definition 5.1. We speak of a *group of morphisms* (5.5)of the jet structure if the inclusion $\mathbf{m}(\lambda)^*\Omega(m,n) \subset \Omega(m,n)$ holds true. We speak of a *(universal) variation* (5.6) of the jet structure if$\mathcal{L}_Z\Omega(m,n) \subset \Omega(m,n)$. If a variation (5.6) moreover generates a group, speaks of a *(generalized or higher-order) infinitesimal symmetry* of the jet structure.

So we intentionally distinguish between true infinitesimal transformations generating a group and the formal concepts; this point of view and the terminology are not commonly used in the current literature.

Remark 5.2. A few notes concerning this unorthodox terminology are useful here. In actual literature, the vector fields (5.6) are as a rule decomposed into the "trivial summand D" and the so-called "evolutionary form V" of the vector field Z, explicitly

$$Z = D + \mathrm{V} \quad \left(D = \sum z_i D_i \in \mathcal{H}(m,n), V = \sum Q_I^j \frac{\partial}{\partial w_I^j},\; Q_I^j = z_I^j - \sum w_{Ii}^j z_i\right). \tag{5.7}$$

The summand D is usually neglected in a certain sense [3–7] and the "essential" summand V is identified with the evolutional system

$$\frac{\partial w_I^j}{\partial \lambda} = Q_I^j\left(\dots, x_{i'}, w_{I'}^{j'}, \dots\right) \quad \left(w_I^j = \frac{\partial^n w^j}{\partial x_{i_1}\cdots\partial x_{i_n}}(\lambda, x_1, \dots, x_n)\right) \tag{5.8}$$

of partial differential equations (the finite subsystem with $I = \phi$ empty is enough here since the remaining part is a mere prolongation). This evolutional system is regarded as a "virtual flow" on the "space of solutions" $w^j = w^j(x_1, \dots, x_n)$, see [7, especially page 11]. In more generality, some differential constraints may be adjoint. However, in accordance with the ancient classical tradition, functions $\delta w^j = \partial w^j/\partial\lambda$ are just the *variations*. (There is only one novelty: in classical theory, δw^j are introduced only *along a given solution* while the vector fields Z are "universally" defined on the space.) In this "evolutionary approach", the properties of the primary vector field Z are utterly destroyed. It seems that the true sense of this approach lies in the applications to the topical soliton theory. However, then the evolutional system is always completed with boundary conditions and embedded into some normed functional spaces in order to ensure the existence of global "true flows". This is already quite a different story and we return to our topic.

In more explicit terms, morphisms (5.5) are characterized by the (implicit) recurrence

$$\sum G^j_{Ii} D_{i'} G_i = D_{i'} G^j_I \quad (i' = 1, \ldots, n), \tag{5.9}$$

where $\det(D_{i'}G_i) \neq 0$ is supposed and vector field (5.6) is a variation if and only if

$$z^j_{Ii} = D_i z^j_I - \sum w^j_{Ii'} D_i z_{i'}. \tag{5.10}$$

Recurrence (5.9) easily follows from the inclusion $\mathbf{m}(\lambda)^* \omega^j_I \in \Omega(m,n)$ and we omit the proof. Recurrence (5.10) follows from the identity

$$\begin{aligned} \mathcal{L}_Z \omega^j_I &= \mathcal{L}_Z \left(\mathrm{d}w^j_I - \sum w^j_{Ii} \mathrm{d}x_i \right) = \mathrm{d}z^j_I - \sum z^j_{Ii} \mathrm{d}x_i - \sum w^j_{Ii} \mathrm{d}z_i \\ &\cong \left(\sum D_{i'} z^j_I - \sum z^j_{Ii'} - \sum w^j_{Ii} D_{i'} z_i \right) \mathrm{d}x_{i'} \ (\mathrm{mod}\ \Omega(m,n)) \end{aligned} \tag{5.11}$$

and the inclusion $\mathcal{L}_Z \omega^j_I \in \Omega(m,n)$. The obvious formula

$$\mathcal{L}_Z \omega^j_I = \sum \left(\frac{\partial z^j_I}{\partial w^{j'}_{I'}} - \sum w^j_{Ii} \frac{\partial z_i}{\partial w^{j'}_{I'}} \right) \omega^{j'}_{I'} \tag{5.12}$$

appearing on this occasion also is of a certain sense, see Theorem 5.5 and Section 10 below. It follows that the initial functions G_i, G^j, z_i, z^j (empty $I = \phi$) may be in principle arbitrarily prescribed in advance. This is the familiar *prolongation procedure* in the jet theory.

Remark 5.3. Recurrence (5.10) for the variation Z can be succintly expressed by $\omega^j_{Ii}(Z) = D_i \omega^j_I(Z)$. This remarkable formula admits far going generalizations, see concluding Examples 11.3 and 11.4 below.

Let us recall that a vector field (5.6) generates a group (5.5) if and only if $Z \in \mathbb{G}$ hence if and only if every family

$$\{Z^r x_i\}_{r\in\mathbb{N}}, \qquad \left\{Z^r w^j_I\right\}_{r\in\mathbb{N}} \tag{5.13}$$

can be expressed in terms of a finite number of jet coordinates. We conclude with simple but practicable remark: due to jet structure, the *infinite number of conditions* (5.13) can be replaced by a *finite number of requirements* if Z is a variation.

Lemma 5.4. *Let* (5.6) *be a variation of the jet structure. Then the inclusion $Z \in \mathbb{G}$ is equivalent to any of the requirements*

(ι) *every family of functions*

$$\{Z^r x_i\}_{r\in\mathbb{N}}, \quad \left\{Z^r w^j\right\}_{r\in\mathbb{N}} \quad (i = 1,\ldots,n;\ j = 1,\ldots,m) \tag{5.14}$$

can be expressed in terms of a finite number of jet coordinates,

($\iota\iota$) *every family of differential forms*

$$\{\mathcal{L}_Z^r \mathrm{d}x_i\}_{r\in\mathbb{N}}, \quad \left\{\mathcal{L}_Z^r \mathrm{d}w^j\right\}_{r\in\mathbb{N}} \quad (i = 1,\ldots,n;\ j = 1,\ldots,m) \tag{5.15}$$

involves only a finite number of linearly independent terms,

($\iota\iota\iota$) *every family of differential forms*

$$\{\mathcal{L}_Z^r \mathrm{d}x_i\}_{r\in\mathbb{N}}, \quad \left\{\mathcal{L}_Z^r \mathrm{d}w_I^j\right\}_{r\in\mathbb{N}} \quad (i = 1,\ldots,n;\ j = 1,\ldots,m;\ \text{arbitrary } I) \tag{5.16}$$

involves only a finite number of linearly independent terms.

Proof. Inclusion $Z \in \mathbb{G}$ is defined by using the families (5.13) and this trivially implies (ι) where only the empty multi-indice $I = \phi$ is involved. Then (ι) implies ($\iota\iota$) by using the rule $\mathcal{L}_Z \mathrm{d}f = \mathrm{d}Zf$. Assuming ($\iota\iota$), we may employ the commutative rule

$$[D_i, Z] = D_i Z - Z D_i = \sum a_i^{i'} D_{i'} \quad \left(a_i^{i'} = D_i z_{i'}\right) \tag{5.17}$$

in order to verify identities of the kind

$$\mathcal{L}_Z \mathrm{d}w_i^j = \mathcal{L}_Z \mathrm{d}D_i w^j = \mathcal{L}_Z \mathcal{L}_{D_i} \mathrm{d}w^i = \mathcal{L}_{D_i} \mathcal{L}_Z \mathrm{d}w^i - \sum a_i^{i'} \mathcal{L}_{D_{i'}} w^j \tag{5.18}$$

and in full generality identities of the kind

$$\mathcal{L}_Z^k \mathrm{d}w_I^j = \sum a_{I,k}^{I'} \mathcal{L}_{D_{I'}} \mathcal{L}_Z^{k'} \mathrm{d}w^j \quad \left(\text{sum with } k' \le k,\ \left|I'\right| \le |I|\right) \tag{5.19}$$

with unimportant coefficients, therefore ($\iota\iota\iota$) follows. Finally ($\iota\iota\iota$) obviously implies the primary requirement on the families (5.13). □

This is not a whole story. The requirements can be expressed only in terms of the structural contact forms. With this final result, the algorithms [10–13] for determination of all *individual morphisms* can be closely simulated in order to obtain the algorithm for the determination of all *groups* $\mathbf{m}(\lambda)$ *of morphisms*, see Section 10 below.

Theorem 5.5 (technical theorem). *Let* (5.6) *be a variation of the jet space. Then* $Z \in \mathbb{G}$ *if and only if every family*

$$\left\{ \mathcal{L}_Z^r \omega^j \right\}_{r\in\mathbb{N}} \quad (j = 1, \ldots, m) \tag{5.20}$$

involves only a finite number of linearly independent terms.

Some nontrivial preparation is needful for the proof. Let Θ be a finite-dimensional module of 1-forms (on the space $\mathbf{M}(m,n)$ but the underlying space is irrelevant here). Let us consider vector fields X such that $\mathcal{L}_{fX}\Theta \subset \Theta$ for all functions f. Let moreover Adj Θ be the module of all forms φ satisfying $\varphi(X) = 0$ for all such X. Then Adj Θ has a basis consisting of total differentials of certain functions $f^1, \ldots, f^K$ (the Frobenius theorem), and there is a basis of module Θ which can be expressed in terms of functions $f^1, \ldots, f^K$. Alternatively saying, (an appropriate basis of) the Pfaffian system $\vartheta = 0$ ($\vartheta \in \Theta$) can be expressed only in terms of functions $f^1, \ldots, f^K$. This result frequently appears in Cartan's work, but we may refer only to [9, 18, 19] and to the appendix below for the proof.

Module Adj Θ is intrinsically related to Θ: if a mapping $\mathbf{m}$ preserves Θ then $\mathbf{m}$ preserves Adj Θ. In particular, assuming

$$\mathbf{m}(\lambda)^*\Theta \subset \Theta, \quad \text{then } \mathbf{m}(\lambda)^*\mathrm{Adj}\;\Theta \subset \mathrm{Adj}\;\Theta \tag{5.21}$$

is true for a group $\mathbf{m}(\lambda)$. In terms of $\mathcal{IT}$ of the group $\mathbf{m}(\lambda)$, we have equivalent assertion

$$\mathcal{L}_Z\Theta \subset \Theta \text{ implies } \mathcal{L}_Z\mathrm{Adj}\;\Theta \subset \mathrm{Adj}\;\Theta \tag{5.22}$$

and therefore $\mathcal{L}_Z^r\mathrm{Adj}\;\Theta \subset \mathrm{Adj}\;\Theta$ for all r. The preparation is done.

Proof. Let Θ be the module generated by all differential forms $\mathcal{L}_Z^r\omega^j$ ($j = 1, \ldots, m$; $r = 0, 1, \ldots$). Assuming finite dimension of module Θ, we have module Adj Θ and clearly $\mathcal{L}_Z\Theta \subset \Theta$ whence $\mathcal{L}_Z^r\mathrm{Adj}\;\Theta \subset \mathrm{Adj}\;\Theta$ ($r = 0, 1, \ldots$). However Adj Θ involves both the differentials $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ (see below) and the forms $\omega^1, \ldots, \omega^m$. Point ($u$) of previous Lemma 5.4 implies $Z \in \mathbb{G}$. The converse is trivial.

In order to finish the proof, let us on the contrary assume that Adj Θ *does not* contain all differentials $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$. Alternatively saying, the Pfaffian system $\vartheta = 0$ ($\vartheta \in \Theta$) can be expressed in terms of certain functions $f^1, \ldots, f^K$ such that $\mathrm{d}f^1 = \cdots = \mathrm{d}f^K = 0$ does not imply $\mathrm{d}x_1 = \cdots = \mathrm{d}x_n = 0$. On the other hand, it follows clearly that maximal solutions of the Pfaffian system can be expressed only in terms of functions $f^1, \ldots, f^K$ and therefore we *do not need all* independent variables $x_1, \ldots, x_n$. This is however a contradiction: the Pfaffian system consists of contact forms and involves the equations $\omega^1 = \cdots = \omega^n = 0$. All independent variables are needful if we deal with the common classical solutions $w^j = w^j(x_1, \ldots, x_n)$. ☐

The result can be rephrased as follows.

Theorem 5.6. *Let* $\Omega_0 \subset \Omega(m,n)$ *be the submodule of all zeroth-order contact forms* $\omega = \sum a^j\omega^j$ *and* Z *be a variation of the jet structure. Then* $Z \in \mathbb{G}$ *if and only if* $\dim \oplus\mathcal{L}_Z^r\Omega_0 < \infty$.

6. On the Multiparameter Case

Let us temporarily denote by $\mathbb{V}$ the family of all infinitesimal variations (5.6) of the jet structure. Then $\mathbb{V} + \mathbb{V} \subset \mathbb{V}$, $c\mathbb{V} \subset \mathbb{V}$ $(c \in \mathbb{R})$, $[\mathbb{V}, \mathbb{V}] \subset \mathbb{V}$, and it follows that $\mathbb{V}$ is an infinite-dimensional Lie algebra (coefficients in $\mathbb{R}$). On the other hand, if $Z \in \mathbb{V}$ and $fZ \in \mathbb{V}$ for certain $f \in \mathcal{F}$ then $f \in \mathbb{R}$ is a constant. (Briefly saying: *the conical variations of the total jet space do not exist*. We omit easy direct proof.) It follows that only the common Lie algebras over $\mathbb{R}$ are engaged if we deal with morphisms of the jet spaces $\mathbf{M}(m, n)$.

Theorem 6.1. *Let $\mathcal{G} \subset \mathbb{V}$ be a finite-dimensional Lie subalgebra. Then $\mathcal{G} \subset \mathbb{G}$ if and only if there exists a basis of $\mathcal{G}$ that is lying in $\mathbb{G}$.*

The proof is elementary and may be omitted. Briefly saying, Theorem 4.2 (coefficients in $\mathcal{F}$) turns into quite other and much easier Theorem 6.1 (coefficients in $\mathbb{R}$).

7. The Order-Preserving Groups in Jet Space

Passing to particular examples from now on, we will briefly comment some well-known classical results for the sake of completeness.

Let $\Omega_l \subset \Omega(m, n)$ be the submodule of all contact forms $\omega = \sum a_I^j \omega_I^j$ (sum with $|I| \le l$) of the order l at most. A morphism (5.5) and the infinitesimal variation (5.6) are called *order preserving* if

$$\mathbf{m}(\lambda)^* \Omega_l \subset \Omega_l, \qquad \mathcal{L}_Z \Omega_l \subset \Omega_l, \tag{7.1}$$

respectively, for a certain $l = 0, 1, \ldots$ (equivalently: for all $l \in \mathbb{N}$, see Lemmas 9.1 and 9.2 below). Due to the fundamental Lie-Bäcklund theorem [1, 3, 6, 10–13], this is possible only in the *pointwise case* or in the *Lie's contact transformation case*. In quite explicit terms: assuming (7.1) then either functions G_i, G^j, z_i, z^j (empty $I = \phi$) in formulae (5.5) and (5.6) are functions only of the zeroth-order jet variables $x_{i'}$, $w^{j'}$ or, in the second case, we have $m = 1$ and all functions G_i, G^1, G_i^1, z_i, z^1, z_i^1 contain only the zeroth- and first-order variables $x_{i'}$, w^1, $w_{i'}^1$.

A somewhat paradoxically, short proofs of this fundamental result are not easily available in current literature. We recall a tricky approach here already applied in [10–13], to the case of the order-preserving morphisms. The approach is a little formally improved and appropriately adapted to the infinitesimal case.

Theorem 7.1 (infinitesimal Lie-Bäcklund). *Let a variation Z preserve a submodule $\Omega_l \subset \Omega(m, n)$ of contact forms of the order l at most for a certain $l \in \mathbb{N}$. Then $Z \in \mathbb{G}$ and either Z is an infinitesimal point transformation or $m = 1$ and Z is the infinitesimal Lie's contact transformation.*

Proof. We suppose $\mathcal{L}_Z \Omega_l \subset \Omega_l$. Then $\mathcal{L}_Z^r \Omega_0 \subset \mathcal{L}_Z^r \Omega_l \subset \Omega_l$ therefore $Z \in \mathbb{G}$ by virtue of Theorem 5.5. Moreover $\mathcal{L}_Z \Omega_{l-1} \subset \Omega_{l-1}, \ldots, \mathcal{L}_Z \Omega_0 \subset \Omega_0$ by virtue of Lemma 9.2 below. So we have

$$\mathcal{L}_Z \omega^j = \sum a^{jj'} \omega^{j'} \quad (j, j' = 1, \ldots, m). \tag{7.2}$$

Assuming $m = 1$, then (7.2) turns into the classical definition of Lie's infinitesimal contact transformation. Assume $m \geq 2$. In order to finish the proof we refer to the following result which implies that Z is indeed an infinitesimal point transformation. □

Lemma 7.2. *Let Z be a vector field on the jet space $\mathbf{M}(m,n)$ satisfying* (7.2) *and $m \geq 2$. Then*

$$Zx_i = z_i\left(\dots, x_{i'}, w^{j'}, \dots\right), \quad Zw^j = z^j\left(\dots, x_{i'}, w^{j'}, \dots\right) \quad (i = 1, \dots, n;\ j = 1, \dots, m) \tag{7.3}$$

are functions only of the point variables.

Proof. Let us introduce module Θ of $(m+2n)$-forms generated by all forms of the kind

$$\begin{aligned} &\omega^1 \wedge \cdots \wedge \omega^m \wedge \left(\mathrm{d}\omega^{j_1}\right)^{n_1} \wedge \left(\mathrm{d}\omega^{j_k}\right)^{n_k} \\ &\quad = \mathrm{d}w^1 \wedge \cdots \mathrm{d}w^m \wedge \mathrm{d}x_1 \wedge \cdots \mathrm{d}x_n \wedge \sum \pm\, \mathrm{d}w^{j'_1}_{i_1} \wedge \cdots \wedge \mathrm{d}w^{j'_n}_{i_n}, \end{aligned} \tag{7.4}$$

where $\sum n_k = n$. Clearly $\Theta = (\Omega_0)^m \wedge (\mathrm{d}\Omega_0)^n$. The inclusions

$$\mathcal{L}_Z\Omega_0 \subset \Omega_0, \qquad \mathcal{L}_Z\mathrm{d}\Omega_0 = \mathrm{d}\mathcal{L}_Z\Omega_0 + \Omega_0 \subset \mathrm{d}\Omega_0 + \Omega_0 \tag{7.5}$$

are true by virtue of (7.2) and imply $\mathcal{L}_Z\Theta \subset \Theta$.

Module Θ vanishes when restricted to certain hyperplanes, namely, just to the hyperplanes of the kind

$$\vartheta = \sum a_i \mathrm{d}x_i + \sum a^j \mathrm{d}w^j = 0 \tag{7.6}$$

(use $m \geq 2$ here). This is expressed by $\Theta \wedge \vartheta = 0$ and it follows that

$$0 = \mathcal{L}_Z(\Theta \wedge \vartheta) = \mathcal{L}_Z\Theta \wedge \vartheta + \Theta \wedge \mathcal{L}_Z\vartheta = \Theta \wedge \mathcal{L}_Z\vartheta. \tag{7.7}$$

Therefore $\mathcal{L}_Z\vartheta$ again is such a hyperplane: $\mathcal{L}_Z\vartheta \cong 0$ (mod all $\mathrm{d}x_i$ and $\mathrm{d}w^j$). On the other hand,

$$\mathcal{L}_Z\vartheta \cong \sum a_i \mathrm{d}z_i + \sum a^j \mathrm{d}z^j \quad \left(\text{mod all } \mathrm{d}x_i \text{ and } \mathrm{d}w^j\right) \tag{7.8}$$

and it follows that $\mathrm{d}z_i$, $\mathrm{d}z^j \cong 0$. □

There is a vast literature devoted to the pointwise transformations and symmetries so that any additional comments are needless. On the other hand, the contact transformations are more involved and less popular. They explicitly appear on rather peculiar and dissimilar occasions in actual literature [20, 21]. However, in reality the groups of Lie contact transformations are latently involved in the classical calculus of variations and provide the core of the Hilbert-Weierstrass extremality theory of variational integrals.

8. Digression to the Calculus of Variations

We establish the following principle.

Theorem 8.1 (metatheorem). *The geometries of nondegenerate local one-parameter groups of Lie contact transformations ($\mathcal{CT}$) and of nondegenerate first-order one-dimensional variational integrals ($\mathcal{VI}$) are identical. In particular, the orbits of a given $\mathcal{CT}$ group are extremals of appropriate $\mathcal{VI}$ and conversely.*

Proof. The $\mathcal{CT}$ groups act in the jet space $\mathbf{M}(1,n)$ equipped with the contact module $\Omega(1,n)$. Then the abbreviations

$$w_I = w_I^1, \quad \omega_I = \omega_I^1 = \mathrm{d}w_I - \sum w_{Ii}\mathrm{d}x_i \quad Z = \sum z_i \frac{\partial}{\partial x_i} + \sum z_I^1 \frac{\partial}{\partial w_I} \tag{8.1}$$

are possible. Let us recall the classical approach [22, 23]. The Lie contact transformations defined by certain formulae

$$\mathbf{m}^* x_i = G_i(\cdot), \quad \mathbf{m}^* w = G^1(\cdot), \quad \mathbf{m}^* w_i = G_i^1(\cdot) \quad ((\cdot) = (x_1,\dots,x_n,w,w_1,\dots,w_n)) \tag{8.2}$$

preserve the Pfaffian equation $\omega = \mathrm{d}w - \sum w_i \mathrm{d}x_i = 0$ or (equivalently) the submodule $\Omega_0 \subset \Omega(1,n)$ of zeroth-order contact forms. Explicit formulae are available in literature. We are interested in one-parameter local $\mathcal{CT}$ groups of transformations $\mathbf{m}(\lambda)(-\varepsilon < \lambda < \varepsilon)$ which are "nondegenerate" in a sense stated below and then the explicit formulae are not available yet. On the other hand, our $\mathcal{VI}$ with smooth Lagrangian Ł

$$\int Ł(t, y_1,\dots,y_n,y_1',\dots,y_n')dt \qquad \left(y_i = y_i(t),\ ' = \frac{d}{dt},\ \det\left(\frac{\partial^2 Ł}{\partial y_i' \partial y_j'}\right) \neq 0\right) \tag{8.3}$$

to appear later, involves variables from quite other jet space $\mathbf{M}(n,1)$ with coordinates denoted t (the independent variable), $y_1,\dots,y_n$ (the dependent variables) and higher-order jet variables like y_i', y_i'' and so on.

We are passing to the topic proper. Let us start in the space $\mathbf{M}(1,n)$ with $\mathcal{CT}$ groups. One can check that vector field (5.6) is infinitesimal $\mathcal{CT}$ if and only if

$$Z = -\sum Q_{w_i} \frac{\partial}{\partial x_i} + \left(Q - \sum w_i Q_{w_i}\right)\frac{\partial}{\partial w} + \sum (Q_{x_i} + w_i Q_w)\frac{\partial}{\partial w_i} + \cdots, \tag{8.4}$$

where the function $Q = Q(x_1,\dots,x_n,w,w_1,\dots,w_n)$ may be arbitrarily chosen.

"Hint: we have, by definition

$$\mathcal{L}_Z \omega = Z \rfloor \mathrm{d}\omega + \mathrm{d}\omega(Z) = \sum (z_i \omega_i - \omega_i(Z)\mathrm{d}x_i) + \mathrm{d}Q \in \Omega_0, \tag{8.5}$$

where $Q = Q(x_1,\dots,x_n,w,w_1,\dots,w_n,\dots) = \omega(Z) = z^1 - \sum w_i z_i$,

$$\mathrm{d}Q = \sum D_i Q \mathrm{d}x_i + \frac{\partial Q}{\partial w}\omega + \sum \frac{\partial Q}{\partial w_i}\omega_i \tag{8.6}$$

whence immediately $z_i = -\partial Q/\partial w_i$, $z^1 = Q + \sum w_i z_i = Q - \sum w_i \cdot \partial Q/\partial w_i$, $\partial Q/\partial w_I = 0$ if $|I| \geq 1$ and formula (8.4) follows."

Alas, the corresponding Lie system (not written here) is not much inspirational. Let us however consider a function $w = w(x_1, \ldots, x_n)$ implicitly defined by an equation $V(x_1, \ldots, x_n, w) = 0$. We may suppose that the transformed function $\mathbf{m}(\lambda)^* w$ satisfies the equation

$$V\left(x_1, \ldots, x_n, \mathbf{m}(\lambda)^* w\right) = \lambda \tag{8.7}$$

without any loss of generality. In infinitesimal terms

$$1 = \frac{\partial(V - \lambda)}{\partial \lambda} = Z(V - \lambda) = -\sum Q_{w_i} V_{x_i} + \left(Q - \sum w_i Q_{w_i}\right) V_w. \tag{8.8}$$

However $w_i = \partial w/\partial x_i = -V_{x_i}/V_w$ may be inserted here, and we have the crucial *Jacobi equation*

$$1 = Q\left(x_1, \ldots, x_n, w, -\frac{V_{x_1}}{V_w}, \ldots, -\frac{V_{x_n}}{V_w}\right) V_w \tag{8.9}$$

(not involving V) which can be uniquely rewritten as the *Hamilton-Jacobi* ($\mathcal{HJ}$) *equation*

$$V_w + \mathcal{H}(x_1, \ldots, x_n, w, p_1, \ldots, p_n) \quad (p_i = V_{x_i}) \tag{8.10}$$

in the "nondegenerate" case $\sum Q_{w_i} V_{x_i} \neq 1$. Let us recall the characteristic curves [22, 23] of the $\mathcal{HJ}$ equation given by the system

$$\frac{dw}{1} = \frac{dx_i}{\mathcal{H}_{p_i}} = -\frac{dp_i}{\mathcal{H}_{x_i}} = \frac{dV}{-\mathcal{H} + \sum p_i \mathcal{H}_{p_i}}. \tag{8.11}$$

The curves may be interpreted as the orbits of the group $\mathbf{m}(\lambda)$. (Hint: look at the well-known classical construction of the solution V of the Cauchy problem [22, 23] in terms of the characteristics. The initial Cauchy data are transferred just along the characteristics, i.e., along the group orbits.) Assume moreover the additional condition $\det(\partial^2 \mathcal{H}/\partial p_i \partial p_j) \neq 0$. We may introduce variational integral (8.3) with the Lagrange function Ł given by the familiar identities

$$\text{Ł} + \mathcal{H} = \sum p_i y_i' \tag{8.12}$$

with interrelations

$$t = w, \quad y_i = x_i, \quad y_i' = \mathcal{H}_{p_i}, \quad p_i = \text{Ł}_{y_i'} \quad (i = 1, \ldots, n) \tag{8.13}$$

between variables t, y_i, y_i' of the space $\mathbf{M}(n, 1)$ and variables x_i, w, w_i of the space $\mathbf{M}(1, n)$. Since (8.11) may be regarded as a Hamiltonian system for the extremals of $\mathcal{VI}$, the metatheorem is clarified. □

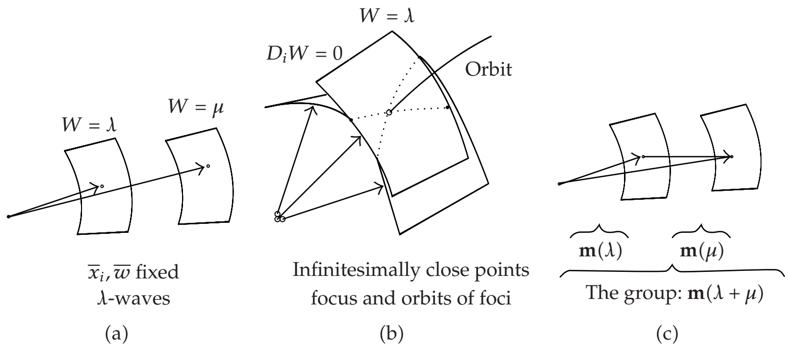

Figure 3

Remark 8.2. Let us recall the Mayer fields of extremals for the $\mathcal{VJ}$ since they provide the true sense of the above construction. The familiar Poincaré-Cartan form

$$\breve{\varphi} = Łdt + \sum Ł_{y_i'}(dy_i - y_i'dt) = -\mathscr{H}dt + \sum p_i dy_i \tag{8.14}$$

is restricted to appropriate subspace $y_i' = g_i(t, y_1, \ldots, y_n)$ $(i = 1, \ldots, n;$ the *slope field*) in order to become a total differential

$$\breve{\varphi}\big|_{y_i' = g_i} = dV(t, y_1, \ldots, y_n) = V_t dt + \sum V_{y_i} dy_i \tag{8.15}$$

of the *action* V. We obtain the requirements $V_t = -\mathscr{H}$, $V_{y_i} = p_i$ identical with (8.10). In geometrical terms: *transformations of a hypersurface $V = 0$ by means of $\mathcal{CT}$ group may be identified with the level sets $V = \lambda$ $(\lambda \in \mathbb{R})$ of the action of a Mayer fields of extremals.*

The last statement is in accordance with (8.11) where

$$dV = \left(-\mathscr{H} + \sum p_i \mathscr{H}_{p_i}\right)dw = \left(-\mathscr{H} + \sum p_i y_i'\right)dt = Łdt, \tag{8.16}$$

use the identifications (8.13) of coordinates. This is the classical definition of the action V in a Mayer field. We have moreover clarified the additive nature of the level sets $V = \lambda$: roughly saying, the composition with $V = \mu$ provides $V = \lambda + \mu$ (see Figure 3(c)) and this is caused by the additivity of the integral $\int Ł\ dt$ calculated along the orbits.

On this occasion, the wave enveloping approach to $\mathcal{CT}$ groups is also worth mentioning.

Lemma 8.3 (see [10–13]). *Let $W(\overline{x}_1, \ldots, \overline{x}_n, \overline{w}, x_1, \ldots, x_n, w)$ be a function of $2n + 2$ variables. Assume that the system $W = D_1W = \cdots = D_nW = 0$ admits a unique solution*

$$\overline{x}_i = F_i(\ldots, x_i', w, w_i', \ldots), \qquad \overline{w} = F^1(\ldots, x_i', w, w_i', \ldots) \tag{8.17}$$

by applying the implicit function theorem and analogously the system $W = \overline{D}_1 W = \cdots = \overline{D}_n W = 0$ *(where* $\overline{D}_i = \partial/\partial\overline{x}_i + \sum \overline{w}_i \partial/\partial\overline{w}$*) admits a certain solution*

$$x_i = \overline{F}_i(\ldots, \overline{x}_{i'}, \overline{w}, \overline{w}_{i'}, \ldots), \qquad w = \overline{F}^1(\ldots, \overline{x}_{i'}, \overline{w}, \overline{w}_{i'}, \ldots). \tag{8.18}$$

Then $\mathbf{m}^* x_i = F_i$, $\mathbf{m}^* w = F^1$ *provides a Lie* $\mathcal{CT}$ *and* $(\mathbf{m}^{-1})^* \overline{x}_i = \overline{F}_i$, $(\mathbf{m}^{-1})^* \overline{w} = \overline{F}^1$ *is the inverse.*

In more generality, if function W in Lemma 8.3 moreover depends on a parameter λ, we obtain a mapping $\mathbf{m}(\lambda)$ which is a certain $\mathcal{CT}$ involving a parameter λ and the inverse $\mathbf{m}(\lambda)^{-1}$. *In favourable case* (see below) *this* $\mathbf{m}(\lambda)$ *may be even a* $\mathcal{CT}$ *group*. The geometrical sense is as follows. Equation $W = 0$ with $\overline{x}_i$, $\overline{w}$ kept fixed represents a wave in the space x_i, w (Figure 3(a)).

The total system $W = D_1 W = \cdots = D_n W = 0$ provides the intersection (envelope) of infinitely close waves (Figure 3(b)) with the resulting transform, the focus point $\mathbf{m}$ (or $\mathbf{m}(\lambda)$ if the parameter λ is present). The reverse waves with the role of variables interchanged gives the inversion. Then the group property holds true if the waves can be composed (Figure 3(c)) within the parameters λ, μ, but this *need not* be in general the case.

Let us eventually deal with the condition ensuring the group composition property. Without loss of generality, we may consider the λ-depending wave

$$W(\overline{x}_1, \ldots, \overline{x}_n, \overline{w}, x_1, \ldots, x_n, w) - \lambda = 0. \tag{8.19}$$

If $\overline{x}_i$, $\overline{w}$ are kept fixed, the previous results may be applied. We obtain a group if and only if the $\mathcal{HJ}$ equation (8.10) holds true, therefore

$$W_w + \mathcal{H}(x_1, \ldots, x_n, w, W_{x_1}, \ldots, W_{x_n}) = 0. \tag{8.20}$$

The existence of such function $\mathcal{H}$ means that functions $W_w, W_{x_1}, \ldots, W_{x_n}$ *of dashed variables* are functionally dependent whence

$$\det \begin{pmatrix} W_{w\overline{w}} & W_{w\overline{x}_{i'}} \\ W_{x_i\overline{w}} & W_{x_i\overline{x}_{i'}} \end{pmatrix} = 0, \qquad \det\left(W_{x_i\overline{x}_{i'}}\right) \neq 0. \tag{8.21}$$

The symmetry $\overline{x}_i, \overline{w} \leftrightarrow x_i, w$ is not surprising here since the change $\lambda \leftrightarrow -\lambda$ provides the inverse mapping: equations

$$W(\ldots, \overline{x}_i, \overline{w}, \ldots, x_i, w) = \lambda, \qquad W(\ldots, x_i, w, \ldots, \overline{x}_i, \overline{w}) = -\lambda \tag{8.22}$$

are equivalent. In particular, it follows that

$$W(\ldots, \overline{x}_i, \overline{w}, \ldots, x_i, w) = -W(\ldots, x_i, w, \ldots, \overline{x}_i, \overline{w}), \qquad W(\ldots, x_i, w, \ldots, x_i, w) = 0 \tag{8.23}$$

and the wave $W - \lambda = 0$ corresponds to the Mayer central field of extremals.

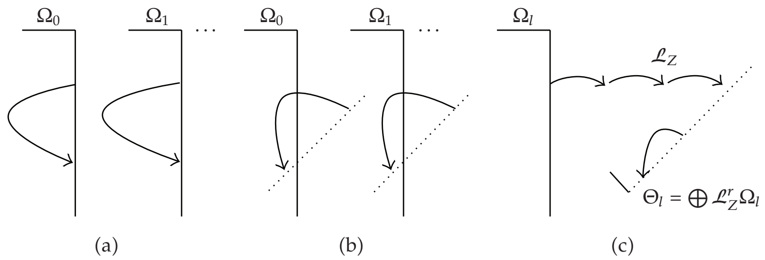

Figure 4

Summary 4. Conditions (8.21) ensure the existence of $\mathcal{HJ}$ equation (8.20) for the λ-wave (8.19) and therefore the group composition property of waves (8.19) in the nondegenerate case $\det(\partial^2 \mathcal{H}/\partial p_i \partial p_j) \neq 0$.

Remark 8.4. A reasonable theory of Mayer fields of extremals and Hamilton-Jacobi equations can be developed also for the constrained variational integrals (the Lagrange problem) within the framework of jet spaces, that is, without the additional Lagrange multipliers [9, Chapter 3]. It follows that there do exist certain groups of generalized Lie's contact transformations with differential constraints.

9. On the Order-Destroying Groups in Jet Space

We recall that in the order-preserving case, the filtration

$$\Omega(m,n)_* : \Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega(m,n) = \cup \Omega_l \tag{9.1}$$

of module $\Omega(m,n)$ is preserved (Figure 4(a)). It follows that certain invariant submodules $\Omega_l \subset \Omega(m,n)$ are a priori prescribed which essentially restricts the store of the symmetries (the Lie-Bäcklund theorem). The order-destroying groups also preserve certain submodules of $\Omega(m,n)$ due to approximation results, however, they are not known in advance (Figure 4(b)) and appear after certain saturation (Figure 4(c)) described in technical theorem 5.1.

The saturation is in general a toilsome procedure. It may be simplified by applying two simple principles.

Lemma 9.1 (going-up lemma). *Let a group of morphisms* $\mathbf{m}(\lambda)$ *preserve a submodule* $\Theta \subset \Omega(m,n)$. *Then also the submodule*

$$\Theta + \sum \mathcal{L}_{D_i} \Theta \subset \Omega(m,n) \tag{9.2}$$

is preserved.

Proof. We suppose $\mathcal{L}_Z \Theta \subset \Theta$. Then

$$\mathcal{L}_Z\left(\Theta + \sum \mathcal{L}_{D_i}\Theta\right) = \mathcal{L}_Z\Theta + \left(\mathcal{L}_{D_i}\mathcal{L}_Z\Theta - \sum D_i z_{i'} \mathcal{L}_{D_{i'}}\Theta\right) \subset \Theta + \sum \mathcal{L}_{D_i}\Theta \tag{9.3}$$

by using the commutative rule (5.17). □

Lemma 9.2 (going-down lemma). *Let the group of morphisms* $\mathbf{m}(\lambda)$ *preserve a submodule* $\Theta \subset \Omega(m,n)$. *Let* $\Theta' \subset \Theta$ *be the submodule of all* $\omega \in \Theta$ *satisfying* $\mathcal{L}_{D_i}\omega \in \Theta$ $(i = 1,\ldots,n)$. *Then* Θ' *is preserved, too.*

Proof. Assume $\omega \in \Theta'$ hence $\mathcal{L}_{D_i}\omega \in \Theta$. Then $\mathcal{L}_{D_i}\mathcal{L}_Z\omega = \mathcal{L}_Z\mathcal{L}_{D_i}\omega + \mathcal{L}_{\sum D_i z_{i'} \cdot D_{i'}}\omega \in \Theta$ hence $\mathcal{L}_Z\omega \in \Theta'$ and Θ' is preserved. □

We are passing to illustrative examples.

Example 9.3. Let us consider the vector field (the variation of jet structure)

$$Z = \sum z_I^j \frac{\partial}{\partial w_I^j} \quad \left(z_I^j = D_I z^j,\ D_I = D_{i_1} \cdots D_{i_n}\right), \tag{9.4}$$

see (5.6) and (5.10) for the particular case $z_i = 0$. Then $Z^r x_i = 0$ $(i = 1,\ldots,n)$ and the *sufficient* requirement $Z^2 w^j = 0$ $(j = 1,\ldots,m)$ ensures $Z \in \mathbb{G}$, see (ι) of Lemma 5.4. We will deal with the linear case where

$$z^j = \sum a_{i'}^{jj'} w_{i'}^{j'} \quad \left(a_{i'}^{jj'} \in \mathbb{R}\right) \tag{9.5}$$

is supposed. Then

$$Z^2 w^j = Z z^j = \sum a_{i'}^{jj'} z_{i'}^{j'} = \sum a_{i'}^{jj'} a_{i''}^{j'j''} w_{i'i''}^{j''} = 0 \tag{9.6}$$

identically if and only if

$$\sum_{j'} \left(a_{i'}^{jj'} a_{i''}^{j'j''} + a_{i''}^{jj'} a_{i'}^{j'j''} \right) = 0 \quad \left(i', i'' = 1,\ldots,n;\ j, j', j'' = 1,\ldots,m\right). \tag{9.7}$$

This may be expressed in terms of matrix equations

$$A_i A_{i'} = 0 \quad \left(i, i' = 1,\ldots,n;\ A_i = \left(a_i^{jj'}\right)\right) \tag{9.8}$$

or, in either of more geometrical transcriptions

$$A^2 = 0, \qquad \operatorname{Im} A \subset \operatorname{Ker} A \quad \left(A = \sum \lambda_i A_i,\ \lambda_i \in \mathbb{R}\right), \tag{9.9}$$

where A is regarded as (a matrix of an) operator acting in m-dimensional linear space and depending on parameters $\lambda_1,\ldots,\lambda_n$. We do not know explicit solutions A in full generality, however, solutions A such that Ker A does not depend on the parameters $\lambda_1,\ldots,\lambda_n$ can be easily found (and need not be stated here). The same approach can be applied to the more general *sufficient* requirement $Z^r w^j = 0$ $(j = 1,\ldots,m$; fixed $r)$ ensuring $Z \in \mathbb{G}$. If $r \geq n$, the requirement is *equivalent* to the inclusion $Z \in \mathbb{G}$.

Example 9.4. Let us consider vector field (5.6) where $z^1 = \cdots = z^m = 0$. In more detail, we take

$$Z = \sum z_i \frac{\partial}{\partial x_i} + \sum z_i^j \frac{\partial}{\partial w_i^j} + \cdots \quad \left(z_i^j = -\sum w_{i'}^j D_i z_{i'}\right). \tag{9.10}$$

Then $Z^r w^j = 0$ and we have to deal with functions $Z^r x_i$ in order to ensure the inclusion $Z \in \mathbb{G}$. This is a difficult task. Let us therefore suppose

$$z_1 = z\left(\ldots, x_{i'}, w^{j'}, w_1^{j'}, \ldots\right), \qquad z_k = c_k \in \mathbb{R} \quad (k = 2, \ldots, n). \tag{9.11}$$

Then $Zx_k = 0$ $(k = 2, \ldots, n)$ and

$$Z^2 x_1 = Zz = \sum \frac{\partial z}{\partial x_i} z_i + \sum \frac{\partial z}{\partial w_1^j} z_1^j, \tag{9.12}$$

where

$$z_1^j = -w_1^j D_1 z = -w_1^j \left(\frac{\partial z}{\partial x_1} + \sum \frac{\partial z}{\partial w^{j'}} w_1^{j'} + \sum \frac{\partial z}{\partial w_1^{j'}} w_{11}^{j'} \right). \tag{9.13}$$

The second-order summand

$$Z^2 x_1 = \cdots + \sum \frac{\partial z}{\partial w_1^j} z_1^j = \cdots - \sum \frac{\partial z}{\partial w_1^j} w_1^j \frac{\partial z}{\partial w_1^{j'}} w_{11}^{j'} \tag{9.14}$$

identically vanishes for the choice

$$z = f\left(\ldots, x_{i'}, w^{j'}, u^l, \ldots\right) \quad \left(u^l = \frac{w_1^l}{w_1^1};\ l = 2, \ldots, m\right) \tag{9.15}$$

as follows by direct verification. Quite analogously

$$Zu^l = Z\frac{w_1^l}{w_1^1} = z_1^l \frac{1}{w_1^1} - z_1^1 \frac{w_1^l}{\left(w_1^1\right)^2} = \left(-w_1^l \frac{1}{w_1^1} + w_1^1 \frac{w_1^l}{\left(w_1^1\right)^2} \right) D_1 z = 0. \tag{9.16}$$

It follows that all functions $Z^r x_i$, $Z^r w^j$ can be expressed in terms of the *finite family* of functions x_i $(i = 1, \ldots, n)$, w^j $(j = 1, \ldots, m)$, u^l $(l = 2, \ldots, m)$ and therefore $Z \in \mathbb{G}$.

Remark 9.5. On this occasion, let us briefly mention the groups generated by vector fields Z of the above examples. The Lie system of the vector field (9.4) and (9.5) reads

$$\frac{\mathrm{d}G_i}{\mathrm{d}\lambda} = 0, \qquad \frac{\mathrm{d}G^j}{\mathrm{d}\lambda} = \sum a_{i'}^{jj'} G_{i'}^{j'} \quad (i = 1, \ldots, n;\ j = 1, \ldots, m), \tag{9.17}$$

where we omit the prolongations. It is resolved by

$$G_i = x_i, \quad G^j = w^j + \lambda \sum a^{jj'}_{i'} w^{j'}_{i'} \quad (i = 1, \ldots, n;\ j = 1, \ldots, m) \tag{9.18}$$

as follows either by direct verification or, alternatively, from the property $Z^2 x_i = Z z_i = 0$ $(i = 1, \ldots, n)$ which implies

$$\frac{\mathrm{d} \sum a^{jj'}_{i'} G^{j'}_{i'}}{\mathrm{d}\lambda} = 0, \qquad \sum a^{jj'}_{i'} G^{j'}_{i'} = \sum a^{jj'}_{i'} G^{j'}_{i'} \Big|_{\lambda=0} = \sum a^{jj'}_{i'} w^{j'}_{i'}. \tag{9.19}$$

Quite analogously, the Lie system of the vector field (9.10), (9.11), (9.15) reads

$$\frac{\mathrm{d}G_1}{\mathrm{d}\lambda} = f\left(\ldots, G_{i'}, G^{j'}, \frac{G^{l'}_1}{G^1_1}, \ldots\right), \quad \frac{\mathrm{d}G_k}{\mathrm{d}\lambda} = c_k, \quad \frac{\mathrm{d}G^j}{\mathrm{d}\lambda} = 0 \quad (k = 2, \ldots, n;\ j = 1, \ldots, m) \tag{9.20}$$

and may be completed with the equations

$$\frac{\mathrm{d}\left(G^l_1 / G^1_1\right)}{\mathrm{d}\lambda} = 0 \quad (l = 2, \ldots, m) \tag{9.21}$$

following from (9.16). This provides a classical self-contained system of ordinary differential equations where the common existence theorems can be applied.

The above Lie systems admit many nontrivial *first integrals* $F \in \mathcal{F}$, that is, functions F that are constant on the orbits of the group. Conditions $F = 0$ may be interpreted as differential equations in the total jet space, and the above transformation groups turn into the *external generalized symmetries* of such differential equations, see Section 11 below.

10. Towards the Main Algorithm

We briefly recall the algorithm [10–13] for determination of all individual automorphisms $\mathbf{m}$ of the jet space $\mathbf{M}(m, n)$ in order to compare it with the subsequent calculation of vector field $Z \in \mathbb{G}$.

Morphisms $\mathbf{m}$ of the jet structure were defined by the property $\mathbf{m}^*\Omega(m, n) \subset \Omega(m, n)$. The inverse $\mathbf{m}^{-1}$ exists if and only if

$$\Omega_0 \subset \mathbf{m}^*\Omega(m, n), \quad \text{equivalently } \Omega_0 \subset \mathbf{m}^*\Omega_l \quad (l = l(\mathbf{m})) \tag{10.1}$$

for appropriate term $\Omega_{l(\mathbf{m})}$ of filtration (9.1). However

$$\mathbf{m}^*\Omega_{l+1} = \mathbf{m}^*\Omega_l + \sum \mathcal{L}_{D_i} \mathbf{m}^*\Omega_l \tag{10.2}$$

and it follows that criterion (10.1) can be verified by repeated use of operators $\mathcal{L}_{D_i}$. In more detail, we start with equations

$$\mathbf{m}^*\omega^j = \sum a^{jj'}_{I'}\omega^{j'}_{I'} \quad \left(= \mathrm{d}\mathbf{m}^*w^j - \sum \mathbf{m}^*w^j_i \mathrm{d}\mathbf{m}^*x_i\right) \tag{10.3}$$

with uncertain coefficients. Formulae (10.3) determine the module $\mathbf{m}^*\Omega_0$. Then we search for lower-order contact forms, especially forms from Ω_0, lying in $\mathbf{m}^*\Omega_l$ with the use of (10.2). Such forms are ensured if certain *linear relations among coefficients exist*. The calculation is finished on a certain level $l = l(\mathbf{m})$ and this is the *algebraic part* of the algorithm. With this favourable choice of coefficients $a^{jj'}_{I'}$, functions $\mathbf{m}^*x_i$, $\mathbf{m}^*w^j$ (and therefore the invertible morphism $\mathbf{m}$) can be determined by inspection of the bracket in (10.3). This is the *analytic part* of algorithm.

Let us turn to the infinitesimal theory. Then the main technical tool is the rule (5.17) in the following transcription:

$$\mathcal{L}_Z\mathcal{L}_{D_i} = \mathcal{L}_{D_i}\mathcal{L}_Z - \sum D_i z_{i'}\mathcal{L}_{D_{i'}} \tag{10.4}$$

or, when applied to basical forms

$$\mathcal{L}_Z\omega^j_{Ii} = \mathcal{L}_{D_i}\mathcal{L}_Z\omega^j_I - \sum D_i z_{i'}\omega^j_{Ii'}. \tag{10.5}$$

We are interested in vector fields $Z \in \mathbb{G}$. They satisfy the recurrence (5.10) together with requirements

$$\dim \oplus \mathcal{L}^r_Z\Omega_0 < \infty, \quad \text{equivalently } \mathcal{L}^r_Z\Omega_0 \subset \Omega_{l(Z)} \quad (r = 0, 1, \ldots) \tag{10.6}$$

for appropriate $l(Z) \in \mathbb{N}$. Due to the recurrence (10.5) these requirements can be effectively investigated. In more detail, we start with equations

$$\mathcal{L}_Z\omega^j = \sum a^{jj'}_{I'}\omega^{j'}_{I'} \quad \left(= \mathrm{d}z^j - \sum z^j_i \mathrm{d}x_i - \sum w^j_i \mathrm{d}z_i\right). \tag{10.7}$$

Formulae (10.7) determine module $\mathcal{L}_Z\Omega_0$. Then, choosing $l(Z) \in \mathbb{N}$, operator $\mathcal{L}_Z$ is to be repeatedly applied and requirements (10.6) provide certain *polynomial relations for the coefficients* by using (10.5). This is the *algebraical part* of the algorithm. With such coefficients $a^{jj'}_{I'}$ available, functions $z_i = \mathcal{L}_Z x_i$, $z^j = \mathcal{L}_Z w^j$ (and therefore the vector field $Z \in \mathbb{G}$) can be determined by inspection of the bracket in (10.7) or, alternatively, with the use of formulae (5.12) for the particular case $I = \phi$ empty

$$\mathcal{L}_Z\omega^j = \sum\left(\frac{\partial z^j}{\partial w^{j'}_{I'}} - \sum w^j_i \frac{\partial z_i}{\partial w^{j'}_{I'}}\right)\omega^{j'}_{I'}. \tag{10.8}$$

This is the *analytic part* of the algorithm.

Altogether taken, the algorithm is not easy and the conviction [7, page 121] that the "exhaustive description of integrable C-fields (fields $Z \in \mathbb{Z}$ in our notation) is given in [16]" is disputable. We can state only one optimistic result at this place.

Theorem 10.1. *The jet spaces* $\mathbf{M}(1, n)$ *do not admit any true generalized infinitesimal symmetries* $Z \in \mathbb{G}$.

Proof. We suppose $m = 1$ and then (10.7) reads

$$\mathcal{L}_Z \omega^1 = \sum a^{11}_{I'} \omega^1_{I'} = \cdots + a^{11}_{I''} \omega^1_{I''} \quad \left(a^{11}_{I''} \neq 0 \right), \tag{10.9}$$

where we state a summand of maximal order. Assuming $I'' = \phi$, the Lie-Bäcklund theorem can be applied and we do not have the true generalized symmetry Z. Assuming $I'' \neq \phi$, then

$$\mathcal{L}^r_Z \omega^1 = \cdots + a^{11}_{I''} \omega^1_{I'' \cdots I''} \quad (r \text{ terms } I'') \tag{10.10}$$

by using rule (10.5) where the last summand may be omitted. It follows that (10.6) is not satisfied hence $Z \notin \mathbb{G}$. □

Example 10.2. We discuss the simplest possible but still a nontrivial particular example. Assume $m = 2$, $n = 1$ and $l(Z) = 1$. Let us abbreviate

$$x = x_1, \quad D = D_1, \quad Z = z \frac{\partial}{\partial x} + \sum z^j_I \frac{\partial}{\partial w^j_I} \quad (j = 1, 2;\ I = 1 \cdots 1). \tag{10.11}$$

Then, due to $l(Z) = 1$, requirement (10.6) reads

$$\mathcal{L}^r_Z \Omega_0 \subset \Omega_1 \quad (r = 0, 1, \ldots). \tag{10.12}$$

In particular (if $r = 1$) we have (10.7) written here in the simplified notation

$$\mathcal{L}_Z \omega^j = a^{j1} \omega^1 + a^{j2} \omega^2 + b^{j1} \omega^1_1 + b^{j2} \omega^2_1 \quad (j = 1, 2). \tag{10.13}$$

The next requirement ($r = 2$) *implies the* (only seemingly) *stronger inclusion*

$$\mathcal{L}^2_Z \Omega_0 \subset \mathcal{L}_Z \Omega_0 + \Omega_0 \tag{10.14}$$

which already ensures (10.12) for all r and therefore $Z \in \mathbb{G}$ (easy). *We suppose* (10.14) *from now on*.

"Hint for proof of (10.14): assuming (10.12) and moreover the equality

$$\mathcal{L}^2_Z \Omega_0 + \mathcal{L}_Z \Omega_0 + \Omega_0 = \Omega_1, \tag{10.15}$$

it follows that

$$\mathcal{L}_Z\Omega_1 \subset \mathcal{L}_Z^3\Omega_0 + \mathcal{L}_Z^2\Omega_0 + \mathcal{L}_Z\Omega_0 \subset \Omega_1 \tag{10.16}$$

and Lie-Bäcklund theorem can be applied whence $\mathcal{L}_Z\Omega_0 \subset \Omega_0$, $l(Z) = 0$ which we exclude. It follows that necessarily

$$\dim\left(\mathcal{L}_Z^2\Omega_0 + \mathcal{L}_Z\Omega_0 + \Omega_0\right) < \dim\Omega_1 = 4. \tag{10.17}$$

On the other hand $\dim(\mathcal{L}_Z\Omega_0 + \Omega_0) \geq 3$ and the inclusion (10.14) follows."

After this preparation, we are passing to the proper algebra. Clearly

$$\mathcal{L}_Z^2\omega^j = \cdots + b^{j1}\mathcal{L}_Z\omega_1^1 + b^{j2}\mathcal{L}_Z\omega_1^2 = \cdots + b^{j1}\left(b^{11}\omega_{11}^1 + b^{12}\omega_{11}^2\right) + b^{j2}\left(b^{21}\omega_{11}^1 + b^{22}\omega_{11}^2\right) \tag{10.18}$$

by using the commutative rule (10.5). Due to "weaker" inclusion (10.12) with $r = 2$, we obtain identities

$$b^{j1}b^{11} + b^{j2}b^{21} = 0, \quad b^{j1}b^{12} + b^{j2}b^{22} = 0 \quad \left(j = 1, 2\right). \tag{10.19}$$

Omitting the trivial solution, they are satisfied if either

$$b^{11} + b^{22} = 0, \qquad b^{12} = cb^{11}, \qquad b^{11} + cb^{21} = 0 \tag{10.20}$$

for appropriate factor c (where $b^{11} \neq 0$ and either $b^{12} \neq 0$ or $b^{21} \neq 0$ is supposed) or

$$b^{11} = b^{22} = 0, \quad \text{either } b^{12} = 0 \quad \text{or } b^{21} = 0. \tag{10.21}$$

We deal only with the (more interesting) identities (10.20) here. Then

$$\begin{aligned} \mathcal{L}_Z\omega^1 &= a^{11}\omega^1 + a^{12}\omega^2 - cb\left(\omega_1^1 + c\omega_1^2\right), \\ \mathcal{L}_Z\omega^2 &= a^{21}\omega^1 + a^{22}\omega^2 + b\left(\omega_1^1 + c\omega_1^2\right) \end{aligned} \tag{10.22}$$

(abbreviation $b = b^{21}$) by inserting (10.20) into (10.13). It follows that

$$\mathcal{L}_Z\left(\omega^1 + c\omega^2\right) = a^1\omega^1 + a^2\omega^2 \quad \left(a^1 = a^{11} + ca^{21}, a^2 = a^{12} + ca^{22} + Zc\right). \tag{10.23}$$

It may be seen by direct calculation of $\mathcal{L}_Z^2\omega^2$ that the "stronger" inclusion (10.14) is equivalent to the identity $ca^1 = a^2$, that is,

$$\mathcal{L}_Z\left(\omega^1 + c\omega^2\right) = a\left(\omega^1 + c\omega^2\right) \tag{10.24}$$

(abbreviation $a = a^1$). Alternatively, (10.24) can be proved by using Lemma 9.2.

"Hint: denoting $\Theta = \mathcal{L}_Z\Omega_0 + \Omega_0$, (10.14) implies $\mathcal{L}_Z\Theta \subset \Theta$. Moreover $\mathcal{L}_D(\omega^1 + c\omega^2) \in \Theta$ by using (10.22). Lemma 9.2 can be applied: $\omega^1 + c\omega^2 \in \Theta'$ and Θ' involves just all multiples of form $\omega^1 + c\omega^2$. Therefore $\mathcal{L}_Z(\omega^1 + c\omega^2) \in \Theta'$ is a multiple of $\omega^1 + c\omega^2$."

The algebraical part is concluded. We have congruences

$$\mathcal{L}_Z\omega^1 \cong -cb\left(\omega_1^1 + c\omega_1^2\right), \quad \mathcal{L}_Z\omega^2 \cong b\left(\omega_1^1 + c\omega_1^2\right) \quad (\mathrm{mod}\ \Omega_0) \tag{10.25}$$

and equality

$$\mathcal{L}_Z\omega^1 + c\mathcal{L}_Z\omega^2 + Zc\omega^2 = a\left(\omega^1 + c\omega^2\right). \tag{10.26}$$

If Z is a variation then these three conditions together ensure the "stronger inclusion" (10.14) hence $Z \in \mathbb{G}$.

We turn to analysis. Abbreviating

$$Z_{I'}^{jj'} = \frac{\partial z^j}{\partial w_{I'}^{j'}} - w_1^j \frac{\partial z}{\partial w_{I'}^{j'}} \quad (j, j' = 1, 2;\ I' = 1 \cdots 1) \tag{10.27}$$

and employing (10.8), the above conditions (10.25) and (10.26) read

$$\begin{gathered}\sum Z_{I'}^{1j'}\omega_{I'}^{j'} = -cb\left(\omega_1^1 + c\omega_1^2\right), \quad \sum Z_{I'}^{2j'}\omega_{I'}^{j'} = b\left(\omega_1^1 + c\omega_1^2\right) \quad \left(\left|I'\right| \geq 1\right), \\ \sum\left(Z_{I'}^{1j'} + cZ_{I'}^{2j'}\right)\omega_{I'}^{j'} + Zc\omega^2 = a\left(\omega^1 + c\omega^2\right).\end{gathered} \tag{10.28}$$

We compare coefficients of forms ω_I^j on the level $s = |I'|$

$$s = 0\colon\ Z^{11} + cZ^{21} = a, \qquad Z^{12} + cZ^{22} + Zc = ac, \tag{10.29}$$

$$s = 1\colon\ Z_1^{11} = -cb, \qquad Z_1^{12} = -(c)^2 b, \qquad Z_1^{21} = b, \qquad Z_1^{22} = bc, \qquad Z_1^{1j} + cZ_1^{2j} = 0, \tag{10.30}$$

$$s \geq 2\colon\ Z_{I'}^{jj'} = 0, \qquad Z_{I'}^{1j'} + cZ_{I'}^{2j'} = 0. \tag{10.31}$$

We will successively delete the coefficients a, b, c in order to obtain interrelations only for variables $Z_{I'}^{jj'}$. Clearly

$$\begin{gathered}s = 0\colon\ Z^{12} + cZ^{22} + Zc = \left(Z^{11} + cZ^{21}\right)c, \\ s = 1\colon\ Z_1^{11} + Z_1^{22} = 0, \qquad Z_1^{11}Z_1^{22} = Z_1^{12}Z_1^{21},\end{gathered} \tag{10.32}$$

and we moreover have three compatible equations

$$c = -\frac{Z_1^{11}}{Z_1^{21}} = -\frac{Z_1^{12}}{Z_1^{22}}, \qquad (c)^2 = -\frac{Z_1^{12}}{Z_1^{21}} \tag{10.33}$$

for the coefficient c. To cope with levels $s \geq 2$, we introduce functions

$$Q^j = \omega^j(Z) = z^j - w_1^j z \quad (j = 1, 2). \tag{10.34}$$

Then substitution into (10.27) with the help of (10.31) gives

$$\frac{\partial Q^j}{\partial w_{I'}^{j'}} = 0 \quad \left(j, j' = 1, 2;\ \left|I'\right| \geq 2\right). \tag{10.35}$$

It follows moreover easily that

$$Z_1^{jj'} = \frac{\partial Q^j}{\partial w_1^{j'}} \quad \left(j \neq j'\right), \qquad Z_1^{jj} = z + \frac{\partial Q^j}{\partial w_1^j}, \qquad Z^{jj'} = \frac{\partial Q^j}{\partial w^{j'}} \tag{10.36}$$

and we have *the final differential equations*

$$s = 0\colon \frac{\partial Q^1}{\partial w^2} + c\frac{\partial Q^2}{\partial w^2} + Zc = \left(\frac{\partial Q^1}{\partial w^1} + c\frac{\partial Q^2}{\partial w^1}\right)c, \tag{10.37}$$

$$s = 1\colon 2z + \frac{\partial Q^1}{\partial w_1^1} + \frac{\partial Q^2}{\partial w_1^2} = 0, \quad \left(z + \frac{\partial Q^1}{\partial w_1^1}\right)\left(z + \frac{\partial Q^2}{\partial w_1^2}\right) = \frac{\partial Q^1}{\partial w_1^2}\frac{\partial Q^2}{\partial w_1^1} \tag{10.38}$$

for the unknown functions

$$z = z\left(x, w^1, w^2, w_1^1, w_1^2\right), \qquad Q^j = Q^j\left(x, w^1, w^2, w_1^1, w_1^2\right). \tag{10.39}$$

The coefficient c is determined by (10.33) *and* (10.36) *in terms of functions Q^j. This concludes the analytic part of the algorithm since trivially $z^j = w_1^j z + Q^j$ and the vector field Z is determined.*

The system is compatible: particular solutions with functions Q^j quadratic in jet variables and $c = \text{const.}$ can be found as follows. Assume

$$Q^j = A^j\left(w_1^1\right)^2 + 2B^j w_1^1 w_1^2 + C^j\left(w_1^2\right)^2 \quad (j = 1, 2) \tag{10.40}$$

with constant coefficients $A^j, B^j, C^j \in \mathbb{R}$. We also suppose $c \in \mathbb{R}$ and then (10.37) is trivially satisfied.

On the other hand, (10.33) provide the requirements

$$z+\frac{\partial Q^1}{\partial w_1^1}+c\frac{\partial Q^2}{\partial w_1^1}=\frac{\partial Q^1}{\partial w_1^2}+c\left(z+\frac{\partial Q^2}{\partial w_1^2}\right)=\frac{\partial Q^1}{\partial w_1^2}+(c)^2\frac{\partial Q^2}{\partial w_1^1}=0 \tag{10.41}$$

by using (10.36). If we put

$$z=-\frac{\partial Q^1}{\partial w_1^1}-\frac{\partial Q^2}{\partial w_1^2}=-\left(A^1+B^1\right)w_1^1-\left(B^1+C^2\right)w_1^2, \tag{10.42}$$

then (10.38) is satisfied (a clumsy direct verification).

The above requirements turn to a system of six homogeneous linear equations (not written here) for the six constants A^j, B^j, C^j $(j=1,2)$ with determinant $\Delta=c^2(c^2-8)$ if the values z, Q^1, Q^2 are inserted and the coefficients of w_1^1 and w_1^2 are compared. The roots $c=0$ and $c=\pm 2\sqrt{2}$ of the equation $\Delta=0$ provide rather nontrivial infinitesimal transformation Z, however, we can state only the simplest result for the trivial root $c=0$ for obvious reason. It reads

$$Q^1=A^1\left(w_1^1\right)^2,\quad Q^2=A^2\left(w_1^1\right)^2,\quad z=-A^1w_1^1,\quad z^1=0,\quad z^2=w_1^1\left(A^2w_1^1+A^1w_1^2\right), \tag{10.43}$$

where A^1, A^2 are arbitrary constants.

Remark 10.3. It follows that investigation of vector fields $Z\in\mathbb{G}$ cannot be regarded for easy task and some new powerful methods are necessary, for example, better use of differential forms (involutive systems) with pseudogroup symmetries of the problem (moving frames).

11. A Few Notes on the Symmetries of Differential Equations

The *external theory* deals with (systems of) differential equations ($\mathfrak{D}\mathcal{E}$) that are firmly localized in the jet spaces. This is the common approach and it runs as follows. A given finite system of $\mathfrak{D}\mathcal{E}$ is infinitely prolonged in order to ensure the compatibility. In general, this prolongation is a toilsome and delicate task, in particular the "singular solutions" are tacitly passed over. The prolongation procedure is expressed in terms of jet variables and as a result a *fixed subspace* of the (infinite-order) jet space appears which represents the $\mathfrak{D}\mathcal{E}$ under consideration. Then the *external symmetries* [2, 3, 6, 7] are such symmetries of the ambient jet space which preserve the subspace. In this sense we may speak of *classical symmetries* (point and contact transformations) and *higher-order symmetries* (which destroy the order of derivatives).

The *internal theory* of $\mathfrak{D}\mathcal{E}$ is irrelevant to the jet localization, in particular to the choice of the hierarchy of independent and dependent variables. This point of view is due to E. Cartan and actually the congenial term "diffiety" was introduced in [6, 7]. Alas, these diffieties were defined as objects *locally identical with appropriate external* $\mathfrak{D}\mathcal{E}$ restricted to the corresponding subspace of the ambient total jet space. This can hardly be regarded as a coordinate-free (or jet theory-free) approach since the model objects (*external* $\mathfrak{D}\mathcal{E}$) and the intertwining mappings (*higher-order symmetries*) essentially need the use of the above hard jet theory mechanisms and concepts.

In reality, the final result of prolongation, the infinitely prolonged $\mathfrak{DE}$, can be alternatively characterized by three simple axioms as follows [8, 9, 24–27].

Let $\mathbf{M}$ be a space *modelled on* $\mathbb{R}^\infty$ (local coordinates $h^1, h^2, \dots$ as in Sections 1 and 2 above). Denote by $\mathcal{F}(\mathbf{M})$ the *structural module* of all smooth functions f on $\mathbf{M}$ (locally depending on a finite number $m(f)$ of coordinates). Let $\Phi(\mathbf{M})$, $\mathcal{T}(\mathbf{M})$ be the $\mathcal{F}(\mathbf{M})$-modules of all differential 1-forms and vector fields on $\mathbf{M}$, respectively. For every submodule $\Omega \subset \Phi(\mathbf{M})$, we have the "orthogonal" submodule $\Omega^\perp = \mathcal{H} \subset \mathcal{T}(\mathbf{M})$ of all $X \in \mathcal{H}$ such that $\Omega(X) = 0$.

Then an $\mathcal{F}(\mathbf{M})$-submodule $\Omega \subset \Phi(\mathbf{M})$ is called a *diffiety* if the following three requirements are locally satisfied.

($\mathcal{A}$) Ω is of codimension $n < \infty$, equivalent $\mathcal{H}$ is of dimension $n < \infty$.
Here n is the *number of independent variables*. The independent variables provide the complementary module to Ω in $\Phi(\mathbf{M})$ which is not prescribed in advance.

($\mathcal{B}$) $d\Omega \cong 0 \pmod{\Omega}$, equivalent $\mathcal{L}_{\mathcal{H}}\Omega \subset \Omega$, equivalently: $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$.
This *Frobenius condition* ensures the classical *passivity requirement*: we deal with the compatible infinite prolongation of differential equations.

($\mathcal{C}$) There exists filtration $\Omega_* : \Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega = \cup\Omega_l$ by finite-dimensional submodules $\Omega_l \subset \Omega$ such that

$$\mathcal{L}_{\mathcal{H}}\Omega_l \subset \Omega_{l+1} \quad (\text{all } l), \quad \Omega_{l+1} = \Omega_l + \mathcal{L}_{\mathcal{H}}\Omega_l \quad (l \text{ large enough}). \tag{11.1}$$

This condition may be expressed in terms of a $\odot\mathcal{H}$-polynomial algebra on the graded module $\oplus\, \Omega_l/\Omega_{l-1}$ (the *Noetherian property*) and ensures the *finite number of dependent variables*. Filtration Ω_* may be capriciously modified. In particular, various localizations of Ω in jet spaces $\Omega(m, n)$ can be easily obtained.

The *internal symmetries* naturally appear. For instance, a vector field $Z \in \mathcal{T}(\mathbf{M})$ is called a (*universal) variation* of diffiety Ω if $\mathcal{L}_Z\Omega \subset \Omega$ and *infinitesimal symmetry* if moreover Z generates a local group, that is, if and only if $Z \in \mathbb{G}$.

Theorem 11.1 (technical theorem). *Let Z be a variation of diffiety Ω. Then $Z \in \mathbb{G}$ if and only if there is a finite-dimensional $\mathcal{F}(M)$-submodule $\Theta \subset \Omega$ such that*

$$\oplus\mathcal{L}^r_{\mathcal{H}}\Theta = \Omega, \qquad \dim \oplus\mathcal{L}^r_Z\Theta < \infty. \tag{11.2}$$

This is exactly counterpart to Theorem 5.6: submodule $\Theta \subset \Omega$ stands here for the previous submodule $\Omega_0 \subset \Omega(m, n)$. We postpone the proof of Theorem 11.1 together with applications to some convenient occasion.

Remark 11.2. There may exist conical symmetries Z of a diffiety Ω, however, they are all lying in $\mathcal{H}$ and generate just the *Cauchy characteristics* of the diffiety [9, page 155].

We conclude with two examples of internal theory of underdetermined ordinary differential equations. The reasonings to follow can be carried over quite general diffieties without any change.

Example 11.3. Let us deal with the *Monge equation*

$$\frac{dx}{dt} = f\left(t, x, y, \frac{dy}{dt}\right). \tag{11.3}$$

The prolongation can be represented as the Pfaffian system

$$\mathrm{d}x - f(t, x, y, y')\mathrm{d}t = 0, \quad \mathrm{d}y - y'\mathrm{d}t = 0, \quad \mathrm{d}y' - y''\mathrm{d}t = 0, \ldots. \tag{11.4}$$

Within the framework of diffieties, we introduce space $\mathbf{M}$ with coordinates

$$t, x_0, y_0, y_1, y_2, \ldots \tag{11.5}$$

and submodule $\Omega \subset \Phi(\mathbf{M})$ with generators

$$\mathrm{d}x_0 - f\mathrm{d}t, \quad (\omega_r =)\mathrm{d}y_r - y_{r+1}\mathrm{d}t \quad (r = 0, 1, \ldots;\ f = f(t, x_0, y_0, y_1)). \tag{11.6}$$

Clearly $\mathscr{H} = \Omega^{\perp} \subset \mathcal{T}(\mathbf{M})$ is one-dimensional subspace including the vector field

$$D = \frac{\partial}{\partial t} + f\frac{\partial}{\partial x_0} + \sum y_{r+1}\frac{\partial}{\partial y_r}. \tag{11.7}$$

One can easily find that we have a diffiety. ($\mathcal{A}$ and $\mathcal{B}$ are trivially satisfied. The *common order preserving filtrations* where Ω_l involves $\mathrm{d}x_0 - f\mathrm{d}t$ and ω_r with $r \le l$ is enough for $\mathcal{C}$.)

We introduce a new (*standard* [9]) *filtration* $\overline{\Omega}_*$ where the submodule $\overline{\Omega}_l \subset \Omega$ is generated by the forms

$$\vartheta_0 = \mathrm{d}x_0 - f\mathrm{d}t - \frac{\partial f}{\partial y_1}\omega_0, \omega_r \quad (r \le l - 1). \tag{11.8}$$

This is indeed a filtration since

$$\begin{aligned}\mathcal{L}_D\vartheta_0 &= \mathrm{d}f - Df\mathrm{d}t - D\frac{\partial f}{\partial y_1}\cdot\omega_0 - \frac{\partial f}{\partial y_1}\omega_1 = \frac{\partial f}{\partial x_0}(\mathrm{d}x_0 - f\mathrm{d}t) + \left(\frac{\partial f}{\partial y_0} - D\frac{\partial f}{\partial y_1}\right)\omega_0 \\ &= \frac{\partial f}{\partial x_0}\vartheta_0 + A\omega_0 \quad \left(A = \frac{\partial f}{\partial y_0} + \frac{\partial f}{\partial x_0}\frac{\partial f}{\partial y_1} - D\frac{\partial f}{\partial y_1}\right)\end{aligned} \tag{11.9}$$

and (trivially) $\mathcal{L}_D\omega_r = \omega_{r+1}$. Assuming $A \ne 0$ from now on (this is satisfied if $f_{y_1y_1} \ne 0$) *every module $\overline{\Omega}_l$ is generated by the forms $\vartheta_r = \mathcal{L}_D^r\vartheta_0$ $(r \le l)$.*

The forms ϑ_r satisfy the recurrence $\mathcal{L}_D\vartheta_r = \vartheta_{r+1}$. Then the formula

$$\vartheta_{r+1} = \mathcal{L}_D\vartheta_r = D\rfloor\mathrm{d}\vartheta_r + \mathrm{d}\vartheta_r(D) = D\rfloor\mathrm{d}\vartheta_r \tag{11.10}$$

implies the congruence $\mathrm{d}\vartheta_r \cong \mathrm{d}t \wedge \vartheta_{r+1}$ (mod $\Omega \wedge \Omega$). Let

$$Z = z\frac{\partial}{\partial t} + z^0\frac{\partial}{\partial x_0} + \sum z_r \frac{\partial}{\partial y_r} \tag{11.11}$$

be a variation of Ω in the common sense $\mathcal{L}_Z\Omega \subset \Omega$. This inclusion is equivalent to the congruence

$$\mathcal{L}_Z\vartheta_r = Z\rfloor \mathrm{d}\vartheta_r + \mathrm{d}\vartheta_r(Z) \cong -\vartheta_{r+1}(Z)\mathrm{d}t + D\vartheta_r(Z)\mathrm{d}t = 0 \,(\mathrm{mod}\ \Omega) \tag{11.12}$$

whence to the recurrence

$$\vartheta_{r+1}(Z) = D\vartheta_r(Z) \tag{11.13}$$

quite analogous to the recurrence (5.10), see Remark 5.3. It follows that the functions

$$z = Zt = \mathrm{d}t(Z), \qquad g = \vartheta_0(Z) \tag{11.14}$$

can be quite arbitrarily chosen. Then functions $\vartheta_r(Z) = D^r g$ are determined and we obtain *quite explicit formulae for the variation Z*. In more detail

$$\begin{aligned} g = \vartheta_0(Z) &= \left(\mathrm{d}x_0 - f\mathrm{d}t - \frac{\partial f}{\partial y_1}\omega_0\right)(Z) = z^0 - fz - \frac{\partial f}{\partial y_1}(z_0 - y_1 z), \\ Dg = \vartheta_1(Z)&\left(\frac{\partial f}{\partial x_0}\vartheta_0 + A\omega_0\right)(Z) = \frac{\partial f}{\partial x_0}g + A(z_0 - y_1 z) \end{aligned} \tag{11.15}$$

and these equations determine coefficients z^0 and z_0 in terms of functions z and g. Coefficients z_r $(r \geq 1)$ follow by prolongation (not stated here). If moreover

$$\dim \{\mathcal{L}_Z^r\vartheta_0\}_{r\in\mathbb{N}} < \infty \tag{11.16}$$

we have infinitesimal symmetry $Z \in \mathbb{G}$, see Theorem 11.1.

Example 11.4. Let us deal with the *Hilbert-Cartan equation* [3]

$$\frac{dy}{dt} = \left(\frac{d^2x}{dt^2}\right)^2. \tag{11.17}$$

Passing to the diffiety, we introduce space $\mathbf{M}$ with coordinates

$$t, x_0, x_1, y_0, y_1, y_2, \ldots \tag{11.18}$$

and submodule $\Omega \subset \Phi(\mathbf{M})$ generated by forms

$$\mathrm{d}x_0 - x_1\mathrm{d}t, \quad \mathrm{d}x_1 - \sqrt{y_1}\mathrm{d}t, \quad (\omega_r =)\mathrm{d}y_r - y_{r+1}\mathrm{d}t \quad (r = 0, 1, \ldots). \tag{11.19}$$

The submodule $\mathcal{H} = \Omega^{\perp} \subset \mathfrak{T}(\mathbf{M})$ is generated by the vector field

$$D = \frac{\partial}{\partial t} + x_1\frac{\partial}{\partial x_0} + \sqrt{y_1}\frac{\partial}{\partial x_1} + \sum y_{r+1}\frac{\partial}{\partial y_r}. \tag{11.20}$$

We introduce the form

$$\vartheta_0 = \mathrm{d}x_0 - x_1\mathrm{d}t + B\left\{\mathrm{d}x_1 - \sqrt{y_1}\mathrm{d}t - \frac{1}{2\sqrt{y_1}}\omega_0\right\} \quad \left(B = \frac{1/\sqrt{y_1}}{D(1/\sqrt{y_1})}\right) \tag{11.21}$$

and moreover the forms

$$\begin{aligned}
&\vartheta_1 = \mathcal{L}_D\vartheta_0 = (1 + DB)\{\cdots\},\\
&\vartheta_2 = \mathcal{L}_D\vartheta_1 = D^2B\{\cdots\} - C\omega_0 \quad \left(C = (1 + DB)D\frac{1}{2\sqrt{y_1}}\right),\\
&\vartheta_3 = \cdots + C\omega_1,\\
&\vartheta_4 = \cdots + C\omega_2,\\
&\quad\vdots
\end{aligned} \tag{11.22}$$

Assuming $C \neq 0$, we have a standard filtration $\overline{\Omega}_*$ where the submodules $\overline{\Omega}_l \subset \Omega$ are generated by forms ϑ_r $(r \leq l)$. Explicit formulae for variations

$$Z = z\frac{\partial}{\partial t} + z^0\frac{\partial}{\partial x_0} + z^1\frac{\partial}{\partial x_1} + \sum z_r\frac{\partial}{\partial y_r} \tag{11.23}$$

can be obtained analogously as in Example 11.3 (and are omitted here). Functions z and $g = \vartheta_0(Z)$ can be arbitrarily chosen. Condition (11.16) ensures $Z \in \mathbb{G}$.

Appendix

For the convenience of reader, we survey some results [9, 18, 19] on the modules Adj. Our reasonings are carried out in the space $\mathbb{R}^n$ and will be true *locally* near *generic points*.

Let Θ be a given module of 1-forms and $A(\Theta)$ the module of all vector fields X such that $\mathcal{L}_{fX}\Theta \subset \Theta$ for all functions f, see [9]. Clearly

$$\mathcal{L}_{[X,Z]}\Theta = (\mathcal{L}_X\mathcal{L}_Z - \mathcal{L}_Z\mathcal{L}_X)\Theta \subset \Theta \quad (X, Z \in A(\Theta)) \tag{A.1}$$

and it follows that identity

$$f[X,Y]=[X,Z]+Xf\cdot Y\quad \left(X,Y\in A(\Theta);\ Z=fY\right) \tag{A.2}$$

implies $\mathcal{L}_{f[X,Y]}\Theta\subset\Theta$ whence $[A(\Theta),A(\Theta)]\subset A(\Theta)$.

Let Θ be of a finite dimension I. The Frobenius theorem can be applied, and it follows that module $\mathrm{Adj}\,\Theta = A(\Theta)^{\perp}$ (of all forms φ satisfying $\varphi(A(\Theta))=0$) has a certain basis $\mathrm{d}f^1,\dots,\mathrm{d}f^K$ $(K\geq I)$.

On the other hand, identity

$$\mathcal{L}_{fX}\vartheta = \quad fX\rfloor \mathrm{d}\vartheta+\mathrm{d}\left(f\vartheta(X)\right)=f\mathcal{L}_X\vartheta+\vartheta(X)\vartheta \tag{A.3}$$

implies that $X\in A(\Theta)$ if and only if

$$\vartheta(X)=0,\ X\rfloor\mathrm{d}\vartheta\in\Theta\quad(\vartheta\in\Theta) \tag{A.4}$$

which is the classical definition, see [2]. In particular $\Theta\subset\mathrm{Adj}\,\Theta$ so we may suppose the generators

$$\vartheta^i=\mathrm{d}f^i+g^i_{I+1}\mathrm{d}f^{I+1}+\cdots+g^i_K\mathrm{d}f^K\in\Theta\quad(i=1,\dots,I) \tag{A.5}$$

of module Θ. Recall that $Xf^k=0\ \ (k=1,\dots,K;\ X\in A(\Theta))$ whence

$$\mathcal{L}_X\vartheta^i=Xg^i_{I+1}\mathrm{d}f^{I+1}+\cdots+Xg^i_K\mathrm{d}f^K\in\Theta \tag{A.6}$$

and this implies $Xg^i_{I+1}=\cdots=Xg^i_K=0$. It follows that

$$\mathrm{d}g^i_{I+1},\dots,\mathrm{d}g^i_K\in\mathrm{Adj}\,\Theta\quad(i=1,\dots,I) \tag{A.7}$$

and therefore all coefficients g^i_k depend only on variables $f^1,\dots,f^K$.

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