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# Results about the Alexandroff duplicate space

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ABSTRACT

*In this paper, we present some new results about the Alexandroff Duplicate Space.* We prove that if a space $X$ has the property $P$, then its Alexandroff Duplicate space $A(X)$ may not have $P$, where $P$ is one of the following properties: extremally disconnected, weakly extremally disconnected, quasi-normal, pseudocompact. We prove that if $X$ is $\alpha$-normal, epinormal, or has property $wD$, then so is $A(X)$. We prove almost normality is preserved by $A(X)$ under special conditions.

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KEYWORDS: *Alexandroff duplicate; normal; almost normal; mildly normal; quasi-normal; pseudocompact; rroperty $wD$; $\alpha$-normal; epinormal.*

There are various methods of generating a new topological space from a given one. In 1929, Alexandroff introduced his method by constructing the Double Circumference Space [1]. In 1968, R. Engelking generalized this construction to an arbitrary space as follows [6]: Let $X$ be any topological space. Let $X' = X \times \{1\}$. Note that $X \cap X' = \varnothing$. Let $A(X) = X \cup X'$. For simplicity, for an element $x \in X$, we will denote the element $\langle x, 1\rangle$ in $X'$ by $x'$ and for a subset $B \subseteq X$, let $B' = \{x' : x \in B\} = B \times \{1\} \subseteq X'$. For each $x' \in X'$, let $\mathcal{B}(x') = \{\{x'\}\}$. For each $x \in X$, let $\mathcal{B}(x) = \{U \cup (U' \setminus \{x'\}) : U$ is open in $X$ with $x \in U\}$. Then $\mathcal{B} = \{\mathcal{B}(x) : x \in X\} \cup \{\mathcal{B}(x') : x' \in X'\}$ will generate a unique topology on $A(X)$ such that $\mathcal{B}$ is its neighborhood system. $A(X)$ with this topology is called the *Alexandroff Duplicate of* $X$. Now, if $P$ is a topological property and $X$ has $P$, then $A(X)$ may or may not have $P$. Throughout this paper, we denote an ordered pair by $\langle x, y\rangle$, the set of positive integers by $\mathbb{N}$ and the set of real numbers by $\mathbb{R}$. For a subset $A$ of a space $X$,

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int$A$ and $\overline{A}$ denote the interior and the closure of $A$, respectively. An ordinal $\gamma$ is the set of all ordinal $\alpha$ such that $\alpha < \gamma$. The first infinite ordinal is $\omega$ and the first uncountable ordinal is $\omega_1$.

A topological space $X$ is called *$\alpha$-normal* [3] if for any two disjoint closed subsets $A$ and $B$ of $X$, there exist two open disjoint subsets $U$ and $V$ of $X$ such that $A \cap U$ dense in $A$ and $B \cap V$ dense in $B$.

**Theorem 0.1.** *If $X$ is $\alpha$-normal, then so is its Alexandroff Duplicate $A(X)$.*

*Proof.* Let $E$ and $F$ be any two disjoint closed sets in $A(X)$. Write $E = E_1 \cup E_2$, where $E_1 = E \cap X$, $E_2 = E \cap X'$ and $F = F_1 \cup F_2$, where $F_1 = F \cap X$, $F_2 = F \cap X'$. So, we have $E_1$ and $F_1$ are two disjoint closed sets in $X$. By $\alpha$-normality of $X$, there exist two disjoint open sets $U$ and $V$ of $X$ such that $E_1 \cap U$ is dense in $E_1$ and $F_1 \cap V$ is dense in $F_1$. Let $W_1 = (U \cup U' \cup E_2) \setminus F$ and $W_2 = (V \cup V' \cup F_2) \setminus E$. Then $W_1$ and $W_2$ are disjoint open sets in $A(X)$. Now, we prove $W_1 \cap E$ is dense in $E$. Note that $W_1 \cap E = (W_1 \cap E_1) \cup (W_1 \cap E_2) = (U \cap E_1) \cup E_2$, hence $\overline{W_1 \cap E} = \overline{(U \cap E_1) \cup E_2} = \overline{(U \cap E_1)} \cup \overline{E_2} \supset E_1 \cup \overline{E_2} \supset E$. Therefore, $W_1 \cap E$ is dense in $E$. Similarly, $W_2 \cap F$ is dense in $F$. Therefore, $A(X)$ is $\alpha$-normal. □

A space $X$ is called *extremally disconnected* [5] if it is $T_1$ and the closure of any open set is open. Extremally disconnectedness is not preserved by the Alexandroff Duplicate space and here is a counterexample.

**Example 0.2.** Consider the Stone-Čech compactification space $\beta\omega$ which is compact Hausdorff, hence Tychonoff. It is well-known that $\beta\omega$ is extremally disconnected. Clearly $\omega$ is open in $A(\beta\omega)$ and $\overline{\omega}^{A(\beta\omega)} = \beta\omega$ which is not open in $A(\beta\omega)$.

A space $X$ is called *weakly extremally disconnected* [8] if the closure of any open set is open. Weakly extremally disconnected is not preserved and the above example is a counterexample. The following question is interesting and still open: *"Does there exist a Tychonoff non-discrete space $X$ such that $A(X)$ is extremally disconnected?"*.

A subset $B$ of a space $X$ is called a *closed domain* [5] if $B = \overline{\text{int}B}$. A finite intersection of closed domains is called *$\pi$-closed* [9]. A topological space $X$ is called *mildly normal* [7] if for any two disjoint closed domains $A$ and $B$ of $X$, there exist two open disjoint subsets $U$ and $V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$. A topological space $X$ is called *quasi-normal* [9] if for any two disjoint $\pi$-closed subsets $A$ and $B$ of $X$, there exist two open disjoint subsets $U$ and $V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$. It is clear from the definitions that every quasi-normal space is mildly normal. Mild normality is not preserved by the Alexandroff Duplicate space [7]. Quasi-normality is not preserved by the Alexandroff Duplicate space and here is a counterexample. We denote the set of all limit points of a set $B$ by $B^d$ and call it *the derived set of $B$*.

**Example 0.3.** Consider $\mathbb{R}^{\omega_1}$, which is Tychonoff separable non-normal space ([5], 2.3.15). It is well-known that every closed domain in $\mathbb{R}^{\omega_1}$ depends on a countable set [12]. It follows that every $\pi$-closed set in $\mathbb{R}^{\omega_1}$ depends on a countable set. Now, if $A$ and $B$ are disjoint $\pi$-closed sets in $\mathbb{R}^{\omega_1}$, then there is a countable $S \subset \omega_1$ such that $A = \pi_S(A) \times \mathbb{R}^{\omega_1 \setminus S}$ and $B = \pi_S(B) \times \mathbb{R}^{\omega_1 \setminus S}$, where $\pi_S$ is the projection function $\pi_S : \mathbb{R}^{\omega_1} \longrightarrow \mathbb{R}^S$. It follows that $\pi_S(A)$ and $\pi_S(B)$ are disjoint closed sets in $\mathbb{R}^S$. Since $S$ is countable, $\mathbb{R}^S$ is metrizable. So, there exist two open disjoint sets $U_1, V_1 \subset \mathbb{R}^S$ such that $\pi_S(A) \subseteq U_1$ and $\pi_S(B) \subseteq V_1$. Then $A \subseteq U = U_1 \times \mathbb{R}^{\omega_1 \setminus S}$ and $B \subseteq V = V_1 \times \mathbb{R}^{\omega_1 \setminus S}$ where $U$ and $V$ are open in $\mathbb{R}^{\omega_1}$ and disjoint. Therefore, $\mathbb{R}^{\omega_1}$ is quasi-normal.

We show that the Alexandroff Duplicate space $A(\mathbb{R}^{\omega_1})$ is not quasi-normal by showing that it is not mildly normal. Let

$$E = \{\langle n_\xi : \xi < \omega_1\rangle \in \mathbb{N}^{\omega_1} : \forall m \in \mathbb{N} \setminus \{1\}(|\{\xi < \omega_1 : n_\xi = m\}| \leq 1)\}.$$

$$F = \{\langle n_\xi : \xi < \omega_1\rangle \in \mathbb{N}^{\omega_1} : \forall m \in \mathbb{N} \setminus \{2\}(|\{\xi < \omega_1 : n_\xi = m\}| \leq 1)\}.$$

$E$ and $F$ are disjoint closed subsets in $\mathbb{N}^{\omega_1}$, hence closed in $\mathbb{R}^{\omega_1}$. They cannot be separated by disjoint open sets, see [14], and they are perfect, i.e., $E = E^d$ and $F = F^d$. By a theorem from [7] which says: *"If $A$ and $B$ are disjoint subsets of a space $X$ such that $A^d$ and $B^d$ cannot be separated, then $A(X)$ is not mildly normal."*, we conclude that $A(\mathbb{R}^{\omega_1})$ is not mildly normal.

A space $X$ is *pseudocompact* [5] if $X$ is Tychonoff and any continuous real-valued function defined on $X$ is bounded. Equivalently, a Tychonoff space $X$ is pseudocompact if and only if any locally finite family consisting of non-empty open subsets is finite [5]. We will conclude that pseudocompactness is not preserved by the Alexandroff Duplicate space by the following theorem.

**Theorem 0.4.** *Let $X$ be a Tychonoff space. The Alexandroff Duplicate $A(X)$ is pseudocompact if and only if $X$ is countably compact.*

*Proof.* If $X$ is not countably compact, then there exists a countably infinite closed discrete subset $D$ of $X$. Then $D \times \{1\} = D'$ is closed and open set in $A(X)$ which is also countably infinite, hence $A(X)$ is not pseudocompact. Now, assume that $X$ is countably compact. Since for a set $B \subseteq X$ and a point $a \in X$, we have that $a$ is a limit point for $B'$ if and only if $a$ is a limit point for $B$, we do have that $A(X)$ is also countably compact and hence pseudocompact. □

So, any pseudocompact space which is not countably compact will be an example of a pseudocompact space whose Alexandroff Duplicate space $A(X)$ is not pseudocompact. A Mrówka space $\Psi(\mathcal{A})$ , where $\mathcal{A} \subset [\omega]^\omega$ is maximal, is such a space, see [4] and [10].

Arhangel'skii introduced the notions of epinormality and $C$-normality in 2012, when he was visiting the department of mathematics, King Abdulaziz University, Jeddah, Saudi Arabia. A topological space $X$ is called *$C$-normal* [2] if there exist a normal space $Y$ and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|C} : C \longrightarrow f(C)$ is a homeomorphism for each compact subspace $C \subseteq X$. It was proved in [2] that if $X$ is $C$-normal, then so is its Alexandroff Duplicate. A topological space $(X, \tau)$ is called *epinormal* [2] if there is a coarser topology $\tau'$ on $X$ such that $(X, \tau')$ is $T_4$.

**Theorem 0.5.** *If $X$ is epinormal, then so is its Alexandroff Duplicate $A(X)$.*

*Proof.* Let $X$ be any space which is epinormal, let $\tau$ be a topology on $X$, since $X$ is epinormal, then there is a coarser topology $\tau^*$ on $X$ such that $(X, \tau^*)$ is $T_4$. Since $T_4$ is preserved by the Alexandroff Duplicate space, then $A(X, \tau^*)$ is also $T_4$ and it is coarser than $A(X, \tau)$ by the topology of the Alexandroff Duplicate. Hence, $A(X)$ is epinormal. $\square$

A space X is said to satisfy *Property wD* [11] if for every infinite closed discrete subspace $C$ of $X$, there exists a discrete family $\{U_n : n \in \omega\}$ of open subsets of $X$ such that each $U_n$ meets $C$ at exactly one point.

**Theorem 0.6.** *If $X$ satisfies property $wD$, then so does its Alexandroff Duplicate $A(X)$.*

*Proof.* Let $X$ be any space which satisfies property $wD$ and consider its Alexandroff Duplicate $A(X)$. To show that $A(X)$ has Property $wD$, let $C \subseteq A(X)$ be any infinite closed discrete subspace of $A(X)$. Write $C = (C \cap X) \cup (C \cap X')$. For each $x \in C \cap X$, fix an open set $U_x$ in $X$ such that $V_x = U_x \cup (U'_x \setminus \{x'\})$ open in $A(X)$ and

$$(*) \qquad V_x \cap C = \{x\}$$

Case 1: $C \cap X$ is finite. This implies that $C \cap X'$ is infinite. Let $\{x'_n : n \in \omega\} \subseteq C \cap X'$ such that $x'_i \neq x'_j$, for all $i, j \in \omega$ with $i \neq j$. Now, consider the family $\{\{x'_n\} : n \in \omega\}$, then it consists of open sets and each $\{x'_n\}$ meets $C$ at exactly one point. Now, we will show that $\{\{x'_n\} : n \in \omega\}$ is a discrete family in $A(X)$. It is obvious that $\{x'_n : n \in \omega\}$ is discrete and it is closed because if $x \in A(X) \setminus C$, then there is open set $U_x$ containing $x$ such that $U_x \cap C = \varnothing$, and if $x \in C \cap X$ hence, by (*) there is an open set $V_x$ in $A(X)$ containing $x$ such that $V_x \cap C = \{x\}$. Thus, $\{\{x'_n\} : n \in \omega\}$ is discrete family. Therefore, in this case, $A(X)$ satisfies property $wD$.

Case 2: $C \cap X$ is infinite. Then $C \cap X$ is an infinite closed discrete subspace of $X$. Since $X$ satisfies the property $wD$, then there exists a discrete family $\{V_n : n \in \omega\}$ of open subsets of $X$ such that each $V_n$ meets $C \cap X$ at exactly one point $\{x_n\}$. Hence, $\{V_n \cup (V'_n \setminus \{x'_n\}) : n \in \omega\}$ is discrete in $A(X)$ and then meets $C$ in exactly one point $\{x_n\}$. Therefore, also in this case, $A(X)$ satisfies the property $wD$. $\square$

A space $X$ is called *almost normal* [8] if for any two disjoint closed subsets $A$ and $B$ of $X$ one of which is a closed domain there exist two disjoint open sets $U$ and $V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$.

 

**Proposition 0.7.** *Let $X$ be a topological space. The following are equivalent:*

(1) *$X$ is the only non-empty closed domain in $X$.*
(2) *Each non-empty open subset is dense in $X$.*
(3) *The interior of each non-empty proper closed subset of $X$ is empty.*

*Proof.* (1) $\Longrightarrow$ (2) Let $U$ be any non-empty open set. Suppose $U$ is not dense in $X$, then $\overline{U}$ is a non-empty proper closed domain in $X$, which contradicts the hypothesis, hence $U$ is dense in $X$.
(2) $\Longrightarrow$ (3) Suppose that $E$ is a non-empty proper closed subset of $X$ such that $\text{int}E \neq \varnothing$, then $X = \overline{\text{int}E} \subseteq \overline{E} = E$ which is a contradiction.
(3) $\Longrightarrow$ (1) Suppose that there exists a closed domain $B$ such that $\varnothing \neq B \neq X$, then $\varnothing \neq \overline{\text{int}B} \neq X$ , which contradicts the hypothesis, thus $X$ is the only non-empty closed domain in $X$. □

It is clear that any space that satisfies the conditions of Proposition 7 will be almost normal.

**Corollary 0.8.** *If $X$ satisfies the conditions of Proposition 7, then its Alexandroff Duplicate $A(X)$ is almost normal.*

*Proof.* Let $E$ and $F$ be any non-empty disjoint closed subsets of $A(X)$ such that $E$ is a closed domain. Let $W$ be an open set in $A(X)$ such that $\overline{W} = E$. If $W \cap X \neq \varnothing$, then $W \cap X$ is dense in $X$ by Proposition 7, so $X \subset \overline{W} = E$. It follows that $F \subset X'$, so $E$ is closed and open, hence there are two disjoint open sets $U = A(X) \setminus F$ and $V = F$ in $A(X)$ containing $E$ and $F$ respectively.

If $W \cap X = \varnothing$, then $E \subset X'$, so $E$ is closed and open, thus there are two disjoint open sets $U = E$ and $V = A(X) \setminus E$ in $A(X)$ containing $E$ and $F$ respectively. □

**Corollary 0.9.** *The Alexandroff Duplicate of the countable complement space [13], the finite complement space [13], and the particular point space [13] are all almost normal.*

The following problems are still open:

(1) If $X$ is almost normal, is then its Alexandroff Duplicate $A(X)$ almost normal?
(2) If $X$ is $\beta$-normal [3], is then its Alexandroff Duplicate $A(X)$ $\beta$-normal?

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