International Journal of Applied Mathematics and Computation
Volume 3(3),pp 181–188, 2011
http://ijamc.psit.in

Modified block method for the direct solution of second order ordinary differential equations

D.O. Awoyemi[1], E. A. Adebile[1], A.O. Adesanya[2], T.A. Anake[2]

[1] *Department of Mathematical Sciences,*
Federal University of Technology, Akure, Nigeria

[2] *Department of Mathematics, Covenant University, Ota, Nigeria*
Email: torlar10@yahoo.com, anaketimothy@yahoo.co.in

Abstract:

The direct solution of general second order ordinary differential equations is considered in this paper. The method is based on the collocation and interpolation of the power series approximate solution to generate a continuous linear multistep method. We modified the existing block method in order to accommodate the general *nth* order ordinary differential equation. The method was found to be efficient when tested on second order ordinary differential equation.

Key words: Collocation; interpolation; approximate solution; continuous linear multistep method; block method

1 Introduction

General second order ordinary differential equations is given as

$$y'' = f(x, y, y') \tag{1.1}$$

Conventionally, (1.1) is solved by reducing it to a system of first order ordinary differential equations, and then any method of solving first order ordinary differential equations can be applied to solve the problem. The setbacks of this approach were reported by Adesanya *et al.* [2] and Awoyemi [5].

The method of collocation and interpolation of the power series approximate solution to generate continuous linear multistep method has been adopted by many scholars among them are Fatunla [9], Awoyemi [4], Awoyemi *et al.* [6], Olabode [14], and Adesanya *et al.* [1], to mention a few. Their approach generates an implicit continuous linear multistep method in which separate predictors are required for its implementation; this method is called the predictor-corrector method. This is the major setback of this method.

In developing numerical methods, the following four basic factors must be taken into consideration;

- the accuracy of the method;

Corresponding author: **T.A. Anake**

- the cost of implementation of the method;
- time taken to develop the method and
- flexibility of the method.

Since the predictor-corrector method has not met these requirements, hence the need to adopt other methods.

In order to address the setbacks of the predictor-corrector method, Fatunla [8], Yahaya [18], Aladeselu [3], Jator [11], Jator and Li [12] and Awoyemi *et al.* [7] independently proposed block method for solving higher order ordinary differential equation. This block method is capable of giving evaluations at different grids points without overlapping as done in the predictor-corrector method. This method does not require the development of separate predictors and more over it is more accurate than the existing methods.

The major set back of this method is that its application to general higher order ordinary differential equation has not been given much attention.

In this paper, we modified the existing block method to accommodate general second order ordinary differential equation and we will generalized it to *nth* order and give condition for the zero stability of the modified block method.

2 Methodology

Firstly, we state a theorem that establishes the existence and uniqueness of the solutions of higher order ordinary differential equations

Theorem 2.1. *Given the general nth order initial value problem*

$$y^{(n)} = f\left(x, y, y', \cdots, y^{(n-1)}\right) \tag{2.1}$$

Let R be the region defined by the inequalities $0 \leq x - x_0 \leq a, |s_k - y_k| < b_k, k = 0, 1, \cdots, n-1$ where $y_k \geq 0$ for $k > 0$. Suppose the function $f(x, s_0, s_1, \cdots, s_{n-1})$ in (2.1) is nonnegative, continuous and non-decreasing in x, and continuous and non-decreasing in s_k for each $k = 0, 1, \cdots, n-1$ in the region R. If in addition $f(x, y_0, y_1, \cdots, y_{n-1}) \neq 0$ in R for $x > x_0$ then, the initial value problem (2.1) has at most one solution in R (see Wend [16] for details)

Theorem 2.2. *Let*

$$u^{(n)} = f\left(x, u, u', \cdots, u^{(n-1)}\right), u^{(k)}(x_0) = c_k \tag{2.2}$$

$k = 0, 1, \cdots, (n-1)$, (u and f are scalars). Let R be the region defined by the inequalities $x_0 \leq x \leq x_0 + a, |s_j - c_j| \leq b, j = 0, 1, \cdots, n-1, (a > 0, b > 0)$. Suppose the function $f(x, s_0, s_1, \cdots, s_{n-1})$ is defined in R and in addition

(a) f is nonnegative and non-decreasing in each of $x, s_0, s_1, \cdots, s_{n-1}$ in R;

(b) $f(x, c_0, c_1, \cdots, c_{n-1}) > 0$ for $x_0 \leq x \leq x_0 + a$ and

(c) $c_k \geq 0, k = 0, 1, 2, \cdots, n-1$,

Then the initial value problem (2.2) has a unique solution in R. (see Wend [17] for details).

2.1 Specification of the Method

We consider an approximate solution in the power series of the form

$$y(x) = \sum_{j=0}^{2k-1} a_j x_j(x) \tag{2.3}$$

Where k=3.
The second derivative of (2.3) is given as

$$y'' = \sum_{j=2}^{2k-1} j(j-1)a_j x_{j-2}(x) \tag{2.4}$$

Substituting (2.4) into (1.1) gives

$$y'' = \sum_{j=2}^{2k-1} j(j-1)a_j x_{j-2}(x) = f(x, y, y') \tag{2.5}$$

Evaluating (2.5) at $x_{n+j}, j = 0(1)k$ and (2.3) at $x_{n+j}, j = 1(1)2$ gives a system of equations of the form

$$UA = B \tag{2.6}$$

Where

$$U = \begin{bmatrix} 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+2}^3 \end{bmatrix}, A = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}, B = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix}$$

Solving (2.6) for a_j's, $j = 0(1)5$ using Gaussian elimination method and substituting in (2.3) gives a continuous linear multistep method of the form

$$y(x) = \sum_{j=1}^{k-1} \phi_j y_{n+j}(x) + \sum_{j=0}^{k} \beta_j f_{n+j}(x) \tag{2.7}$$

Where

$$\begin{aligned} \phi_1(t) &= -t \\ \phi_2(t) &= 1 + t \\ \beta_0(t) &= \frac{h^2}{360}(-3t^5 + 10t^3 - 7t) \\ \beta_1(t) &= \frac{h^2}{120}(3t^5 + 5t^4 - 20t^3 + 22t) \\ \beta_2(t) &= \frac{h^2}{120}(-3t^5 - 10t^4 + 10t^3 + 60t^2 + 43t) \\ \beta_3(t) &= \frac{h^2}{360}(3t^5 + 15t^4 + 20t^3 - 8t) \end{aligned} \tag{2.8}$$

Where $t = \dfrac{x - x_{n+2}}{h}$. Evaluating (2.8) at $t = -2$ and $t = -1$ and substituting into (2.7) gives

$$12y_{n+2} - 24y_{n+1} + 12y_n = h^2 \left(f_{n+2} + 10f_{n+1} + f_n\right) \tag{2.9}$$

$$12y_{n+3} - 24y_{n+2} + 12y_n = h^2\left(f_{n+3} + 10f_{n+2} + f_{n+1}\right) \tag{2.10}$$

Evaluating the first derivative of (2.8) at $t = -2, -1, 0$ and 1 and substituting into (2.7) gives

$$\begin{aligned}
360hy'_n - 360y_{n+2} + 360y_{n+1} &= h^2(-8f_{n+3} + 9f_{n+2} - 414f_{n+1} - 127f_n) && (2.11)\\
360hy'_{n+1} - 360y_{n+2} + 360y_{n+1} &= h^2(7f_{n+3} - 66f_{n+2} - 129f_{n+1} + 8f_n) && (2.12)\\
360hy'_{n+2} - 360y_{n+2} + 360y_{n+1} &= h^2(-8f_{n+3} + 129f_{n+2} + 66f_{n+1} - 7f_n) && (2.13)\\
360hy'_{n+3} - 360y_{n+2} + 360y_{n+1} &= h^2(127f_{n+3} + 414f_{n+2} - 7f_{n+1} + 8f_n) && (2.14)
\end{aligned}$$

3 Modified Block Method

In order to evaluate the unknown parameters independently, we need to modify the existing method proposed by Fatunla [10]. We now give the modified block method in the form

$$A^0 h^\lambda Y_m^{(n)} = h^\lambda \sum_{i=0}^{k} A^{(i)} Y_{m-i}^{(n)} + h^\mu \sum_{i=1}^{k} B^{(i)} F_{m-i} \tag{3.1}$$

Where n is the power of the derivative, μ is the order of the differential equation. A^0 and $A^{(i)}$ are $R \times R$ identity matrices and λ is the power of h relative to the derivative of the differential equation. Also,

$$\begin{aligned}
&h^\lambda Y_m^{(n)} = \left[y_{n+1}, y_{n+2}, \cdots, hy'_{n+1}, \cdots, h^2y''_{n+1}, h^2y''_{n+2}, \cdots, h^n y^n_{n+m}\right]^T\\
&h^\lambda Y_{m-i}^{(n)} = \left[y_{n-1}, y_{n-2}, \cdots, y_n, hy'_{n-1}, \cdots, hy'_n, h^2y''_{n-1}, h^2y''_{n-2}, \cdots, h^2y''_n \cdots, h^m y^m_n\right]^T\\
&F_{m-i} = [f_{n-1}, f_{n-2}, \cdots, f_m, f_{n+1}, f_{n+2}, \cdots, f_m]^T
\end{aligned}$$

Evaluating (2.9)- (2.14) for $h^\lambda y^\lambda_{n+m}, m = 0(1)k$ gives

$$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.2}$$

$$A^i = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 & 0 & 2\\ 0 & 0 & 1 & 0 & 0 & 3\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.3}$$

$$B^i = \begin{bmatrix} \frac{97}{360} & \frac{19}{60} & \frac{-13}{360} & \frac{1}{45}\\ \frac{28}{45} & \frac{22}{15} & \frac{-2}{15} & \frac{2}{45}\\ \frac{39}{40} & \frac{27}{10} & \frac{27}{40} & \frac{3}{20}\\ \frac{3}{8} & \frac{19}{24} & \frac{-5}{24} & \frac{1}{24}\\ \frac{1}{3} & \frac{4}{3} & \frac{1}{3} & 0\\ \frac{3}{8} & \frac{9}{8} & \frac{9}{8} & \frac{3}{8} \end{bmatrix} \tag{3.4}$$

Substituting (3.2) - (3.4) in (3.1) gives

$$\begin{aligned}
y_{n+1} &= y_n + hy'_n + \frac{h^2}{360}\left[97f_n + 114f_{n+1} - 39f_{n+2} + 8f_{n+3}\right] && (3.5)\\
y_{n+2} &= y_n + 2hy'_n + \frac{h^2}{45}\left[28f_n + 66f_{n+1} - 6f_{n+2} + 2f_{n+3}\right] && (3.6)\\
y_{n+3} &= y_n + 3hy'_n + \frac{h^2}{40}\left[39f_n + 108f_{n+1} + 27f_{n+2} + 6f_{n+3}\right] && (3.7)\\
y'_{n+1} &= y'_n + \frac{h}{24}\left[9f_n + 19f_{n+1} - 5f_{n+2} + f_{n+3}\right] && (3.8)\\
y'_{n+2} &= y'_n + \frac{h}{3}\left[f_n + 4f_{n+1} + f_{n+2}\right] && (3.9)\\
y'_{n+3} &= y'_n + \frac{h}{8}\left[3f_n + 9f_{n+1} + 9f_{n+2} + 3f_{n+3}\right] && (3.10)
\end{aligned}$$

4 Analysis Of The Method

We verify the basic properties of the block which include the order, zero stability and the convergence of the method

4.1 Order of the block

We adopted the method proposed by Fatunla [10], and Lambert [15] to obtain the order of our methods (2.9)-(2.14) as $(4,4,4,4,4,4,4)^T$ and error constants as $\left(\frac{-1}{240}, \frac{-1}{240}, \frac{27}{4}, \frac{-11}{24}, \frac{11}{24}, \frac{-27}{4}\right)^T$.

4.2 Zero Stability of the Block

To obtain the zero stability of the block method we consider the following conditions:

(i) The block (3.1) is said to be stable if as $h \to 0$ the roots $r_j, j = 1(1)k$ of the first characteristic polynomial $\rho(R) = 0$, that is $\rho(R) = det\left[\sum A^{(i)} R^{k-1}\right] = 0$, satisfy $|R| \leq 1$ and for those roots with $|R| \leq 1$, must have multiplicity equal to unity. (See Fatunla [8]) for details).

(ii) If (3.1) be an $R \times R$ matrix then, it is zero stable if as $h^\mu \to 0, \left|RA^0 - A^i\right| = R^{r-\mu}(R-1) = 0$. For those root with $|R_j| \leq 1$, the multiplicity must not exceed the order of the differential equation.

For our method,

$$\lambda A^0 - A^i = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & -1 \\ 0 & \lambda & 0 & 0 & 0 & -2 \\ 0 & 0 & \lambda - 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & \lambda & 0 & -1 \\ 0 & 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & 0 & \lambda - 1 \end{bmatrix} = 0 \qquad (4.1)$$

As $h \to 0$ in (3.1), ((4.1) reduces to

$$\begin{bmatrix} \lambda & 0 & -1 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda - 1 \end{bmatrix} = 0$$

the determinat of which yields the values $\lambda = 0, 0, 1$. Now, taking the determinant of (4.1) we have

$$\lambda^4(\lambda - 1)^2 = 0$$

Since all the two conditions above are satisfied, we conclude that the block method converges. Notice that in the limit as $h \to 0$ and after normalization, (3.1) reduces to the block form proposed by Fatunla [8]. Similarly, as $hy_{n+i} \to 0, i = 1(1)k$ (3.1) reduces to the form proposed by Jator [12].

5 Numerical Examples

In this section, we test our method with general second order initial value problems which have been solved using methods proposed by other scholars. Our results are then compared with those obtained from these existing methods.

Problem 1
We consider the non linear initial value problem (I.V.P) which was solved by Awoyemi [5]) and Awoyemi and Kayode[6] for step length $h = 0.003125$

$$y'' - x(y')^2 = 0, y(0) = 1, y'(0) = \frac{1}{2}$$

Exact Solution:

$$y(x) = 1 + \frac{1}{2} ln\left(\frac{2+x}{2-x}\right)$$

The results at selected grid point are shown in table 1. Despite the higher order of the existing result, our method compare favorably with them. This problem was also solved by Yahaya and Badmus [19] for $h = 1/30$. Though the details was not shown, it was discovered that our method gives a better result.

Table 1: Comparison of new result with results from existing methods

x	Exact result $y(x)$	New result $y_n(x)$	Error in New result	Error in [6]	Error in [5]
0.1	1.050041729278	1.050041729236	6.5501D-11	6.6391D-13	2.6070D-13
0.2	1.100335347731	1.100335347378	5.4803D-10	2.0012D-09	1.9816D-09
0.3	1.151140404394	1.151140434699	1.9256D-09	1.7201D-09	6.5074D-09
0.4	1.202732554054	1.202732550972	4.8029D-09	5.8946D-09	1.5592D-08
0.5	1.256746590600	1.256746584072	1.0006D-08	1.4435D-08	3.1504D-08
0.6	1.309519604203	1.309519592193	1.8727D-08	4.1864D-08	5.6374D-08
0.7	1.348449765089	1.348449747233	3.2746D-08	5.3110D-08	9.6164D-08
0.8	1.423648930193	1.423648930194	5.3969D-08	9.1317D-08	1.5686D-07
0.9	1.484700278595	1.484700278594	8.8004D-08	1.4924D-07	2.4869D-07
1.0	1.549306144334	1.549306144334	1.4353D-07	2.3719D-07	3.8798D-07

Problem 2
We consider a highly oscillatory test problem $y'' + \lambda^2 y = 0$, we take $\lambda = 2$ with initial condition $y(0) = 1, y'(0) = 2$
Exact solution: $y(x) = cos2x + sin2x$
Our result was compared with Okunuga [19] who solved the same problem in table 2.

6 Conclusion

We have proposed a direct method for solving second order ordinary differential equation using modified block method. This method does not require developing separate predictors to implement the method. Despite the high error constant exhibited by the schemes, it gives better accuracy with low cost of implementation within a very short time when compared with the existing methods.

Table 2: Comparison of new result with results from Okunuga [19]

x	Exact result $y(x)$	New result $y_n(x)$	Error in New result	Error in [19]
0.01	1.01979867335991	1.019798673333	2.6577D-11	-
0.02	1.03918944084761	1.03918944	8.4761D-10	2.65-06
0.03	1.05816454641465	1.05816454	6.4146D-09	3.98D-06
0.04	1.07671640027179	1.076716393565	6.7071D-09	5.30D-06
0.05	1.09483758192485	1.094837480388	7.1209D-09	6.62D-06
0.06	1.11252084314279	1.112520835490	7.6530D-09	7.94D-06
0.07	1.12975911085687	1.129759102557	8.3601D-09	9.25D-06
0.08	1.14654548998987	1.146545480931	9.0592D-09	1.06D-05
0.09	1.16287326621395	1.162873256288	9.9268D-09	1.19D-05
0.10	1.1787359086363	1.178735897737	1.0899D-08	1.32D-05

References

[1] Adesanya, A. O., Anake, T. A., Udoh, M. O. (2008): Improved continuous method for direct solution of general second order ordinary differential equation. Journal of the Nigerian Association of Mathematical Physics, 13, 59-62

[2] Adesanya, A. O., Anake, T. A., Bishop S. A., Osilagun, J. A (2009): Two steps block method for the solution of general second order initial value problems of ordinary differential equation. Asset 8(1), 59-68

[3] Aladeselu, V. A (2007): Improved family of block method for special second order initial value problems journal of Nigerian Association of mathematical physics, 11, 153-158

[4] Awoyemi, D. O. (1991): A class of continuous linear multistep method for general second order initial value problem in ordinary differential equation. Intern. J. Comp. Math. 72, 29-37

[5] Awoyemi, D. O (2001): A new Sixth-order algorithm for general secondary order ordinary differential equation. Intern. J. Comp. Math. 77, 117-124

[6] Awoyemi, D. O, Kayode, S. J (2005): A maximal order collocation method for direct solution of initial value problems of general second order ordinary differential equation. Proceedings of the conference organized by the National mathematical centre, Abuja, Nigeria

[7] Awoyemi, D. O, Adesanya, A.O, Ogunyebi, S. N (2009): Construction of self Starting Numerov method for the solution of initial value problems of general second order ordinary differential equation. Int. j. Num. Math, University of Ado Ekiti, 4(2), 269-278

[8] Fatunla, S. O. (1991): Block method for second order ordinary differential equation. International journal of computer mathematics, 41(1 & 2), 55-63

[9] Fatunla, S. O. (1992): Parallel method for second order ordinary differential equation. Proceedings of the National conference of computational mathematics held at University of Benin, Nigeria, 87-99

[10] Fatunla, S. O (1995): A class of block method for second order initial value problems. International journal of computer mathematics, 55(1 & 2), 119-133

[11] Jator, S. N. (2007): A sixth order linear multistep method for the direct solution of International journal of pure and applied Mathematics, 40(1), 457-472

[12] Jator, S. N, Li, J (2009): A self starting linear multistep method for a direct solution of the general second order initial value problems. Intern J. Comp. Math. 86(5), 827-836.

[13] Olabode, B. T (2009): An accurate scheme by block method for the third order ordinary differential equation. Pacific journal of science and technology 10(1) http:/www.okamaiuniversity.us/pjst.htm

[14] Lambert, J. D. (1973): computational methods in ordinary differential equation, John Wiley, New York

[15] Wend, V. V David (1969): Existence and Uniqueness of solution of ordinary differential equation. Proceedings of the American Mathematical Society, 23(1), 27-23

[16] Wend, V. V David (1967): Uniqueness of solution of ordinary differential equation. The American Mathematical monthly, 74(8), 27-33

[17] Yahaya, Y. A (2007): A note on the construction of Numerov method through a quadratic continuous polynomial for the general second order ordinary differential equation. Journal of Nigerian Association of mathematical physics, II, 253-260

[18] Badmus, A. M., Yahaya, Y. A. (2009): An accurate uniform order 6 method for direct solution of general second order ordinary differential equation. The pacific journal of science and technology, 10(2), 248-254.http://www.akamaiuniversity.us/PJST.htm

[19] Okunuga, S. O: one leg multistep method for numerical integration of periodic second order initial value problems. Journal of Nigerian Association of Mathematical Physics 13, 63-68, 2008