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A One Step Method for the Solution of General Second Order Ordinary Differential Equations

Timothy A. Anake[1],David O. Awoyemi[2] ,Adetola A. Adesanya[3]
[1,3] Department of Mathematics, Covenant University, Ota Ogun State
[2]Department of Mathematical Sciences, Federal University of Technology, Akure, Nigeria

ABSTRACT

In this paper, an implicit one step method for the numerical solution of second order initial value problems of ordinary differential equations has been developed by collocation and interpolation technique. The introduction of an o step point guaranteed the zero stability and consistency of the method. The implicit method developed was implemented as a block which gave simultaneous solutions, as well as their rst derivatives, at both o step and the step point. A comparison of our method to the predictor-corrector method after solving some sample problems reveals that our method performs better.

Keywords: *Implicit, one step, hybrid, offstep, collocation, interpolation, block method*

AMS Mathematics Subject Classification: 65L05, 65L06, 65L12

1. INTRODUCTION

The general second order initial value problem of ordinary differential equations (ODE) of the form

$$\begin{cases} y''=f(x,y,y') & x\in[a,b] \\ y(a)=\zeta_0,\ y'(a)=\zeta_1 \end{cases}, \qquad (1)$$

where f is continuous in $[a,b]$, is often encountered in areas such control theory chemical kinetics, circuit theory and biology.

The fact that most often, this class of equations cannot be solved analytically has led to the development of several numerical methods to approximate the solution of problem (1). A conventional approach of solving problem (1) is to reduce it to an equivalent system of first order equations which are then solved by existing first order methods, [1, 2]. This approach has been reported to increase the dimension of the problem and therefore results in more computation, [3, 4]. The alternative is to solve (1) directly. Some approaches to this alternative method include the Nystrom type methods, [4]; the self-starting Runge-Kutta type methods which involve several function evaluations per step, [1, 2] and the linear multistep methods, particularly the implicit meth-ods, which though not self-starting, re-quire fewer function evaluations per step, [5, 6, 7].

Conventionally, implicit linear multistep methods are implemented in the predictor-corrector mode which is prone to error propagation as the integration process progresses. Indeed, [8] noted that this method is cumbersome and results in longer computer time.

The disadvantages associated with the predictor-corrector method led to the development of block methods from linear multistep method. Apart from being being self-starting, the method does not re-quire the development of predictors separately, and evaluates fewer functions per step when compared to the Runge-Kutta type methods. Furthermore, it can be applied as simultaneous integrators over non-overlapping subintervals of integration, (see [9, 11]).

We note that all of these methods are governed by the Dahlquist's barrier conditions [12]. However, this barrier conditions have been circumvented in [13, 1, 14, 5], by the application of hybrid methods in which collocation could be done at o step points.

Our aim in this paper is to construct a zero stable, continuous implicit one-step method for the solution of initial value problems of general second order ordinary differential equations. To achieve this, we will collocate and interpolate a power series approximate solution at both the step points and the off step points which are incorporated to augment the procedure. The implementation will be by a simultaneous application of the method to pro-vide approximations to the solutions of (1) at a block of points $x_n, x_{n+\frac{1}{2}}, x_{n+1}; n=0,1,\ldots,N-1$ on a partition of $[a,b]$.

In section two, we will discuss the development of our method. Block method is discussed in section three and in section three, the analysis of our method for ac-curacy and stability is done in section four and the efficiency of our method is tested with some sample problems in section ve. Finally, the results obtained are discussed in section six and conclusions are made.

2. ONE-STEP METHOD

In this section, we intend to derive a continuous representation of a one-step method which will be used to generate the main method and other methods required to set up the block method. We set out by approximating the analytical solution of problem (1) with a power series polynomial of the form

$$Y(x)=\sum_{j=0}^{m} a_i t^i \qquad (2)$$

on the partition

$\Delta_N : a = x_0 < x_1 < \cdots < x_n < x_{n+1} < \cdots < x_N = b$ of the integration interval [a, b], with a constant step size h, given by $h = x_{n+1} - x_n;\ n = 0,1,\cdots,N-1$.

Conventionally, we need to interpolate at atleast two points to be able to approximate (1); this is obviously not possible with a one step method. To make this happen, we proceed by arbitrarily selecting an offstep point, $x_{n+\nu},\ \nu \in (0,1)$, in $\{x_n, x_{n+1}\}$ in such a manner that the zero stability of the main method is guaranteed. Then, (2) is interpolated at $x_{n+i},\ i = 0, \nu$ and collocated at $x_{n+i},\ i = 0,\nu,1$, so that we obtain a system of six equations each of degree four, i.e m = 4, as follows:

$$\sum_{j=0}^{4} a_j x_{n+i}^j = y_{n+i}, \qquad i = 0,\nu \qquad (3)$$

$$\sum_{j=0}^{4} j(j-1) a_j x_{n+i}^{j-2} = f_{n+i} \qquad (4)$$

In what follows, let us arbitrarily set $\nu = \frac{1}{2}$. Then solving the system of equations (3) – (4) yields values for the unknown parameters $a_j;\ j = 0,1,\ldots,4$ which when substituted into (2) gives our continuous implicit hybrid one-step method in the form

$$Y(x)=\alpha_0(x)y_n+\alpha_{\frac{1}{2}}(x)y_{n+\frac{1}{2}} + h^2\left[\sum_{j=0}^{1}\beta_j(x)f_{n+j}+\beta_{\frac{1}{2}}f_{n+\frac{1}{2}}\right] \qquad (5)$$

where $\alpha_j(x)$ and $\beta_j(x)$ are continuous coefficients, $y_{n+j} = y(x_n + jh)$ is the numerical approximation of the analytical solution at x_{n+j} and $f_{n+j} = f\left(x_{n+j}, y_{n+j}, y'_{n+j}\right)$. Indeed, by evaluating (5) at x_{n+j} , the main method is obtained as follows:

$$y_{n+1}-2y_{n+\frac{1}{2}}+y_n=\frac{h^2}{48}\left[f_{n+1}+10f_{n+\frac{1}{2}}+f_n\right] \qquad (6)$$

To derive our block method, additional equations are needed since (6) alone will not be sufficient if the solution at $x_{n+\frac{1}{2}}, x_{n+1}$ are to be obtained simultaneously. The additional methods can be obtained from evaluating the first derivative of (5):

$$Y'(x)=\frac{1}{h}\left(\alpha_0(x)y_n+\alpha_{\frac{1}{2}}(x)y_{n+\frac{1}{2}}\right) + h\left[\sum_{j=0}^{1}\beta'_j(x)f_{n+j}+\beta'_{\frac{1}{2}}f_{n+\frac{1}{2}}\right] \qquad (7)$$

at $x_n, x_{n+\frac{1}{2}}$, and x_{n+1} respectively. This yields the following discrete derivative schemes:

$$48hy'_n-96y_{n+\frac{1}{2}}+96y_n=h^2\left[f_{n+1}-6f_{n+\frac{1}{2}}-7f_n\right] \qquad (8)$$

$$48hy'_{n+\frac{1}{2}}-96y_{n+\frac{1}{2}}+96y_n=h^2\left[-f_{n+1}+10f_{n+\frac{1}{2}}+3f_n\right] \qquad (9)$$

$$48hy'_{n+1}-96y_{n+\frac{1}{2}}+96y_n=h^2\left[9f_{n+1}+26f_{n+\frac{1}{2}}+f_n\right] \qquad (10)$$

3. BLOCK METHOD

Block method, (see [16,11]), is adopted with modification for the implementation of our scheme. The modified definition is given in vector notation, as:

$$h^{\lambda}\bar{A}Y_m = h^{\lambda}\bar{E}y_m + h^{\mu-\lambda}\left[\bar{D}F(y_m)+\bar{B}F(Y_m)\right] \qquad (11)$$

where $\bar{A},\bar{B},\bar{C},\bar{D}$ are constant coefficient matrices;

$$Y_m=\left(y_{n+\frac{1}{2}}, y_{n+1}, y'_{n+\frac{1}{2}}, y'_{n+1}\right)^T,$$
$$y_m=\left(y_{n-1}, y_n, y'_{n-1}, y'_n\right)^T,$$
$$F(Y_m)=\left(f_{n-1}, f_{n-\frac{1}{2}}, f_{n+\frac{1}{2}}, f_{n+1}\right)^T,$$
$$F(y_m)=(f_n),$$

λ is the power of the derivative in (7) and μ is the order of problem (1).

To set up our block method, (6) is combined with equations (8) – (10) to form a block from where the constant coefficient matrices $\bar{A},\bar{B},\bar{C},\bar{D}$ are obtained as follows:

$$\bar{E}=\begin{bmatrix}0&1&0&\frac{1}{2}\\0&1&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix} \qquad \bar{D}=\begin{bmatrix}\frac{7}{96}\\\frac{1}{6}\\\frac{5}{24}\\\frac{1}{6}\end{bmatrix} \text{ and}$$

$$\bar{B}=\begin{bmatrix}\frac{1}{16}&-\frac{1}{96}\\\frac{1}{3}&0\\\frac{1}{3}&-\frac{1}{24}\\\frac{2}{3}&\frac{1}{6}\end{bmatrix}$$

A single application of the formula guarantees simultaneously, the approximate solutions and their

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derivatives, $\left\{y_{n+\frac{1}{2}}, y_{n+1}, y'_{n+\frac{1}{2}}, y'_{n+1}\right\}$, at the points $x_{n+\frac{1}{2}}, x_{n+1}$ respectively, as the following discrete schemes:

$$\begin{aligned} y_{n+\frac{1}{2}} &= y_n + \frac{h}{2}y'_n + \frac{h^2}{96}\left[-f_{n+1} + 6f_{n+\frac{1}{2}} + 7f_n\right] \\ y_{n+1} &= y_n + hy'_n + \frac{h^2}{6}\left[2f_{n+\frac{1}{2}} + f_n\right] \\ y'_{n+\frac{1}{2}} &= y'_n + \frac{h}{24}\left[-f_{n+1} + 8f_{n+\frac{1}{2}} + 5f_n\right] \\ y'_{n+1} &= y'_n + \frac{h}{6}\left[f_{n+1} + 4f_{n+\frac{1}{2}} + f_n\right] \end{aligned} \tag{12}$$

The one-step block method is implemented as a simultaneous integrator, without requiring other methods to supply starting values or for the development of predictors, over the subintervals, $[x_0, x_1], \ldots, [x_{N-1}, x_N]$ of the partition $\Delta_N : a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$. This way, the initial conditions are obtained at $x_{n+1},\ n = 0,1,\ldots,N-1$.

4. ANALYSIS OF THE BLOCK METHOD

In this section, fundamental properties of the one-step block method are discussed.

4.1 Order and Error Constant

In what follows, we will define, in the spirit of Awoyemi, et al [19], the linear difference operator associated with the one-step block method with some modifications. We will proceed by first of all, recasting (12) as:

$$\sum_{ij} \bar{\alpha}_{ij}^{\lambda} y_{n+j}^{\lambda} = h^2 \sum_{ij} \beta_{ij}^{\lambda} f_{n+j} \tag{13}$$

where $i, j = 0, v, 1$ and λ is the degree of the derivative in (7).

Definition 1: *The linear difference operator L associated with (13) is defined as:*

$$L\left[y(x);h\right] = \sum_{ij}\left[\bar{\alpha}_{ij}^{\lambda} y(x_n + jh) - h^2 \bar{\beta}_{ij}^{\lambda} y''(x_n + jh)\right] \tag{14}$$

where $i, j = 0, v, 1;\ y(x)$ *is an arbitrary test function which is continuously differentiable on [a,b].*

Expanding $y(x_n + jh)$ and $y''(x_n + jh)$ in Taylor's series and collecting like terms in powers of h yields the linear equation:

$$\begin{aligned} L[y(x);h] &= \bar{C}_0 y(x) + \bar{C}_1 h y^{(1)}(x) + \ldots + \bar{C}_p h^p y^{(p)}(x) \\ &+ \bar{C}_{p+1} h^{p+1} y^{(p+1)}(x) + \bar{C}_{p+2} h^{p+2} y^{(p+2)}(x) + \ldots \end{aligned} \tag{15}$$

where the $\bar{C}_i;\ i = 0,1,\ldots$ are vectors.

Definition 2: *The one-step block method (11) and the associated linear difference operator (14) are said to have order p if* $\bar{C}_0 = \bar{C}_1 = \cdots = \bar{C}_{p+1} = 0$ *and* $\bar{C}_{p+2} \neq 0$.

Definition 3: *The term* $\bar{C}_{p+2}$ *called the error constant implies that the one-step block method (13) has local truncation error given by*

$$t_{n+k} = \bar{C}_{p+2} h^{p+2} y^{(p+2)}(x) + O\left(h^{p+3}\right) \tag{16}$$

From our calculation, our block method has order of accuracy $p = (4,4,4,4)^T$ and error term given as $\bar{C}_{p+2} = \left(\frac{3}{10240}, \frac{1}{1920}, \frac{13}{11520}, \frac{17}{8640}\right)^T$.

4.2 Zero Stability, Consistency and Convergence

Definition 4: *The one-step block method (11) is said to be zero stable as* $h \to 0$ *if its first characteristic polynomial* $\bar{\rho}(z)$ *satisfies*

$$\begin{aligned} \bar{\rho}(z) &= \det\left[z\bar{A} - \bar{E}\right] \\ &= z^{r-\mu}(z-1)^{\mu} \\ &= 0 \end{aligned} \tag{17}$$

where r *is the order of the matrices* $\bar{A}, \bar{E}$ *and the roots* $z_s,\ s = 1,\ldots,4$ *of (18v) satisfies the condition* $|z_s| \leq 1$. *Furthermore, those roots with* $|z_s| = 1$ *have multiplicity not exceeding two.*

By definition 4, our one-step block method with $r = 4$ and $\mu = 2$ yields

$$\bar{\rho}(z) = z^2(z-1)^2 = 0$$

Clearly, the conditions of (18) are satisfied hence, the method is zero stable.
The consistency of the method follows from the fact that the order of the block is greater than one.
Following [20], our method is also convergent.

4.3 Region of Absolute Stability of the Block Method

The stability polynomial of our one-step block method is obtained by applying the scalar test problem

$$y'' = -\lambda^2 y \tag{18}$$

to the block formula (11), in the spirit of [2], such that

$$Y_m = W\left(\bar{h}\right) y_m \tag{19}$$

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where $\bar{h} = \lambda^2 h^2$ and $W(\bar{h}) = (\bar{A} - \bar{h}\bar{B})^{-1}(\bar{E} + \bar{h}\bar{D})$ is called the amplification matrix.

Definition 5: *The interval* $(0, \bar{h}_0)$ *of the real line is said to be the interval of absolute stability if in this interval* $\phi(\bar{h}) < 1$, *where* $\phi(\bar{h})$ *is the spectral radius of* $W(\bar{h})$, *(see [5]).*

Our block method is found to satisfy the condition $\phi(\bar{h}) < 1$ if $\bar{h} \in (0, 16128)$.

5. NUMERICAL EXAMPLES

In this section, the efficiency and accuracy of our one-step method implemented as a block method is tested on some numerical examples. The absolute errors computed are compared with those obtained in [18], which used a numerical scheme implemented in the predictor corrector mode. Each of the following examples is tested using step size $h = \frac{1}{320}$. The tables of results for the problems are given in Tables 1, 2 and 3 respectively.

Problem 1:

$$y'' - x(y')^2 = 0, \quad y(0) = 1, \; y'(0) = \frac{1}{2}$$

Theoretical Solution:

$$y = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$$

Problem 2:

$$y'' - \frac{(y')^2}{2y} + 2y = 0, \quad y\left(\frac{\pi}{6}\right) = \frac{1}{4}, \; y'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

Theoretical Solution:

$$y = \sin^2 x$$

Problem 3:

$$y'' - \frac{2y'}{x} + \left(1 + \frac{2}{x^2}\right)y - xe^x = 0,$$

$$y\left(\frac{\pi}{2}\right) = \frac{\pi}{4}\left(8 + e^{\frac{\pi}{2}}\right), \; y'\left(\frac{\pi}{2}\right) = 4 - \pi + \frac{1}{4}e^{\frac{\pi}{2}}(\pi + 2)$$

Theoretical Solution:

$$y = 2x\cos x + 4x\sin x + \frac{1}{2}xe^x$$

6. DISCUSSION

In this paper, a continuous one-step method of order four is developed by the interpolation and collocation technique with the incorporation of an o step point for the approximation of the solutions of initial value problems of general second order ordinary differential equations of the form (1). The method is implemented as a block method and therefore has the capacity to generate simultaneous solutions at different gird points in a single application of the method.

Three test problems, previously solved by Awoyemi [18] using a numerical scheme developed in the predictor corrector mode, have been solved to test the efficiency and accuracy of our new method. The absolute errors obtained from the computed solutions of these problems using our new method, with three function evaluations per iteration, are compared with those obtained by the four-step sixth or-der predictor-corrector method in [18].

It is obvious from Tables 1, 2 and 3 that our new method is more efficient and accurate, especially when one considers the error term and interval of absolute stability of our new method as reported in section 4.

Notice that our method has not been compared to Jator's three-step seventh order hybrid linear multistep method [5], because it performs better. However our consolation is in the fact that our investigation reveals the viability of this approach adopted to solve higher order problems. In view of this, we intend to extend the research in order to improve our result.

In conclusion, the approach is viable for the solution of higher order initial value problems of ordinary differential equations. It is interesting and can be implemented as a block method. We there-fore recommend it for the numerical approximation of solutions of problems in the class of (1) and possibly for higher orders after a little extension.

Table 1: Comparing Absolute Errors in the New Method to Errors in [18] for Problem 1

X	Error in new method, p=4, k=1	Error in [18], p=6, k=4
0.1	0.49827253E-10	0.26075253E-09
0.2	0.41043058E-09	0.19816704E-08
0.3	0.14285815E-08	0.65074122E-08
0.4	0.35242687E-08	0.15592381E-07
0.5	0.72435324E-08	0.31504477E-07
0.6	0.13335597E-07	0.56374577E-07
0.7	0.22872871E-07	0.96164046E-07
0.8	0.37447019E-07	0.15686801E-06
0.9	0.59503708E-07	0.24869769E-06
1.0	0.92940412E-07	0.38798389E-06

Table 2: Comparing Absolute Errors in the New Method to Errors in [18] for Problem 2

X	Error in new method, p=4, k=1	Error in [18], p=6, k=4
1.1	0.66348841E-07	0.46921462E-06
1.2	0.61995628E-07	0.40802869E-06

1.3	0.31135350E-07	0.22897376E-06
1.4	0.30572108E-07	0.81287181E-07
1.5	0.12473207E-06	0.52447217E-06
1.6	0.24989906E-06	0.10897438E-05
1.7	0.40149404E-06	0.17537254E-05
1.8	0.57196191E-06	0.24814807E-05
1.9	0.75116343E-06	0.32284155E-05
2.0	0.92698387E-06	0.39430146E-05

Table 3: Comparing Absolute Errors in the New Method to errors in [18] for Problem 3

X	Error in new method, p=4, k=1	Error in [18], p=6, k=4
1.7	0.67798365E-06	0.56461918E-05
1.8	0.77776457E-06	0.76853051E-05
1.9	0.83164688E-06	0.10424724E-04
2.0	0.81943241E-06	0.13944282E-04
2.1	0.72051040E-06	0.18315672E-04
2.2	0.51423437E-06	0.23781760E-04
2.3	0.18028640E-06	0.30064597E-04
2.4	0.30097603E-06	0.37356168E-04
2.5	0.94819480E-06	0.45686289E-04
2.6	0.17787125E-06	0.55073706E-04

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