

Gravitational induced particle production through a nonminimal curvature–matter coupling

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Abstract We consider the possibility of a gravitationally induced particle production through the mechanism of a non-minimal curvature–matter coupling. An interesting feature of this gravitational theory is that the divergence of the energy–momentum tensor is nonzero. As a first step in our study we reformulate the model in terms of an equivalent scalar–tensor theory, with two arbitrary potentials. By using the formalism of open thermodynamic systems, we interpret the energy balance equations in this gravitational theory from a thermodynamic point of view, as describing irreversible matter creation processes. The particle number creation rates, the creation pressure, and the entropy production rates are explicitly obtained as functions of the scalar field and its potentials, as well as of the matter Lagrangian. The temperature evolution laws of the newly created particles are also obtained. The cosmological implications of the model are briefly investigated, and it is shown that the late-time cosmic acceleration may be due to particle creation processes. Furthermore, it is also shown that due to the curvature–matter coupling, during the cosmological evolution a large amount of comoving entropy is also produced.

1 Introduction

The simplest explanation of the late-time accelerated expansion of the Universe [1, 2] is to invoke a cosmological constant, Λ , which can be associated to the vacuum energy [3]. Despite the excellent fit to observational data, the presence of Λ suffers from two serious drawbacks, namely, the cos-

mological constant problem and the coincidence problem. However, it has recently been argued that the vacuum energy is not constant but decays into other particle constituents [4]. Indeed, phenomenological models, with a variable cosmological constant [5–10], have been proposed to address the above problems. For instance, a simple and thermodynamically consistent cosmology with a phenomenological model of quantum creation of radiation due to vacuum decay was presented in [11], where the thermodynamics and Einstein's equations lead to an equation in which H is determined by the particle number N . It was shown that the evolution equation for H has a remarkably simple exact solution, in which a non-adiabatic inflationary era exits smoothly to the radiation era, without a reheating transition. In [12], a new accelerating flat model without dark energy that is fully dominated by cold dark matter (CDM) was investigated. It was shown that the number of CDM particles is not conserved and the present accelerating stage is a consequence of the negative pressure describing the irreversible process of gravitational particle creation.

In [4], the correspondence between cosmological models powered by a decaying vacuum energy density and gravitationally induced particle production was explored. Although being physically different, it was shown that under certain conditions both classes of cosmologies can exhibit the same dynamical and thermodynamical behavior. By using current type Ia supernovae data, recent estimates of the cosmic microwave background shift parameter and baryon acoustic oscillations measurements, the authors performed a statistical analysis to test the observational viability of the models and the best-fit of the free parameters was also obtained. Furthermore, the particle production cosmologies [and the associated decaying $\Lambda(t)$ -models] were modeled in the framework of field theory by a phenomenological scalar field model.

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In this context, a new cosmic scenario with gravitationally induced particle creation was proposed [13], where the Universe evolves from an early to a late-time de Sitter era, with the recent accelerating phase driven only by the negative creation pressure associated with the CDM component. The model can be interpreted as an attempt to reduce the so-called cosmic sector (dark matter plus dark energy) and relate the two cosmic accelerating phases (early- and late-time de Sitter expansions). A detailed thermodynamic analysis including possible quantum corrections was also carried out. For a very wide range of the free parameters, it was found that the model presents the expected behavior of an ordinary macroscopic system in the sense that it approaches thermodynamic equilibrium in the long run (i.e., as it nears the second de Sitter phase). Moreover, an upper bound was found for the Gibbons–Hawking temperature of the primordial de Sitter phase [14]. Finally, when confronted with the recent observational data, the current ‘quasi’-de Sitter era, as predicted by the model, it was verified to pass the cosmic background tests very comfortably.

In this work, we consider an alternative mechanism for the gravitational particle production, namely, through a nonminimal curvature–matter coupling, in modified theories of gravity. A general property of these theories [15–20] is the non-conservation of the energy–momentum tensor (for a recent review of modified gravity models with curvature–matter coupling see [21]). Thus, the coupling between the matter and the higher derivative curvature terms may be interpreted as an exchange of energy and momentum between the two. The latter mechanism induces a gravitational particle production. We note that the generalized energy balance equations in these gravitational theories have been interpreted from a thermodynamic point of view as describing irreversible matter creation processes in [22]. Thus, the coupling between matter and geometry generates an irreversible energy flow from the gravitational field to newly created matter constituents, with the second law of thermodynamics requiring that the geometric curvature transforms into matter.

Here we extend and refine the analysis initiated in [22] by investigating in detail the thermodynamic interpretation of the curvature–matter gravitational coupling for the so-called linear version of the $f(R, L_m)$ gravity theory, where R is the Ricci scalar and L_m is the matter Lagrangian, which we call $L_f(R, L_m)$ theory, with the gravitational Lagrangian given by $f_1(R)/2 + [1 + \lambda f_2(R)] L_m$, where $f_1(R)$ and $f_2(R)$ are arbitrary functions of R , and λ is a coupling constant. We note that in the $L_f(R, L_m)$ theory, the gravitational action is *linear in the matter Lagrangian* L_m , and not in the Ricci scalar R . As a first step in our study we introduce the equivalent scalar–tensor description of the theory [23], in which the action is equivalent to a Brans–Dicke type theory, with a single scalar field ψ , a vanishing Brans–Dicke parameter ω , and a coupling $U(\psi)$ between the scalar field and matter.

By using the formalism of open thermodynamic systems [24–27], we interpret the energy balance equation of the theory as describing a matter creation process. Indeed, the irreversible thermodynamics of open systems, and its implications for cosmology have been extensively analyzed [28–40]. Here, we obtain the equivalent particle number creation rates, the creation pressure and the entropy production rates as functions of the scalar field, of the two scalar potentials, and of the matter Lagrangian, respectively. The temperature evolution of the newly created particles is also obtained. Due to the curvature–matter coupling, during the cosmological evolution a large amount of comoving entropy could be produced. The cosmological implications of the theory in its scalar–tensor representation are also investigated.

The present paper is organized as follows. In Sect. 2, we present the action and the field equations of the modified gravity model with a linear curvature–matter coupling in both their standard and scalar–tensor representations. The thermodynamic interpretation of the theory is developed in Sect. 3, where the particle creation rates, the creation pressure, and the entropy production are analyzed in detail. In Sect. 4, the cosmological implications of the theory in its scalar–tensor representation are considered. In Sect. 5, the behavior of the entropy of the Universe in the $L_f(R, L_m)$ gravity theory, with the horizon entropy included, is analyzed. We discuss and conclude our results in Sect. 6.

2 Nonminimal curvature–matter coupling

2.1 General formalism

The action of $f(R)$ gravity can be generalized with the introduction of a linear nonminimal coupling between matter and curvature. The corresponding gravitational theory, a particular case of the general $f(R, L_m)$ theory [17], and which we denote $L_f(R, L_m)$, has the action given by [15],

$$S = \int \left\{ \frac{1}{2} f_1(R) + [1 + \lambda f_2(R)] L_m(g_{\mu\nu}, \Phi) \right\} \sqrt{-g} d^4x \quad (1)$$

where the factors $f_i(R)$ (with $i = 1, 2$) are arbitrary functions of the Ricci scalar R . The coupling constant λ determines the strength of the interaction between $f_2(R)$ and the matter Lagrangian. L_m is the matter Lagrangian density, which is a function of the metric $g_{\mu\nu}$ and of the matter fields Φ .

Now, varying the action with respect to the metric $g_{\mu\nu}$ provides the following field equations:

$$\Theta R_{\mu\nu} - \frac{1}{2} f_1(R) g_{\mu\nu} + \hat{P}_{\mu\nu} \Theta = [1 + \lambda f_2(R)] T_{\mu\nu}, \quad (2)$$

where $F_i(R) = f'_i(R)$, $i = 1, 2$, and the prime represents a derivative with respect to the scalar curvature R . We have defined

$$\Theta = F_1(R) + 2\lambda F_2(R)L_m \tag{3}$$

and

$$\hat{P}_{\mu\nu} = (g_{\mu\nu}\square - \nabla_\mu\nabla_\nu), \tag{4}$$

for notational simplicity.

The matter energy–momentum tensor is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_m)}{\delta(g^{\mu\nu})}. \tag{5}$$

An important property of any gravitational theory is its stability with respect to local perturbations. In the standard $f(R)$ gravity, a fatal instability (the Dolgov–Kawasaki instability) appears once the condition $f''(R) < 0$ [41]. The instability develops on time scales of the order of 10^{26} s. The stability properties of the gravitational models described by the action given by Eq. (1) were studied in [42], and it turns out that the corresponding stability criterion is $f'_1(R) + 2\lambda f'_2(R) > 0$. In order to obtain the stability condition one expands the parameters of the model as the sum of a background field with constant curvature, and a small perturbation, so that $R = R_0 + R_1$, $T = T_0 + T_1$, $f_1(R) = R_0 + R_1 + \epsilon\phi(R_0) + \epsilon\phi'(R_0)R_1 + \dots$, $f'_1(R) = 1 + \epsilon\phi'(R_0) + \epsilon\phi''(R_0)R_1 + \dots$. With the use of the linearized field equations we obtain the stability condition $\epsilon\phi''(R) + 2\lambda f'_2(R) > 0$ [42], generalizing the stability condition $f''(R) = \epsilon\phi''(R) > 0$, found in $f(R)$ gravity [41]. From a physical point of view the reason for stability is once the stability conditions are satisfied the effective mass m_{eff} of the dynamical degree of freedom associated to the small perturbation R_1 of the background curvature is non-negative.

A general property of these nonminimal curvature–matter coupling theories is the non-conservation of the energy–momentum tensor. This can easily be verified by taking into account the covariant derivative of the field Eq. (2), the Bianchi identities, $\nabla^\mu G_{\mu\nu} = 0$, and the following identity:

$$(\square\nabla_\nu - \nabla_\nu\square)F_i = R_{\mu\nu}\nabla^\mu F_i, \tag{6}$$

which then implies the following relationship:

$$\nabla^\mu T_{\mu\nu} = \frac{\lambda F_2}{1 + \lambda f_2} [g_{\mu\nu}L_m - T_{\mu\nu}]\nabla^\mu R. \tag{7}$$

Note that in the absence of the coupling, $\lambda = 0$, one obtains the conservation of the energy–momentum tensor [45], which can also be verified through the diffeomorphism invariance of the matter part of the action. The conservation

of the energy–momentum tensor also follows from Eq. (7), if $f_2(R)$ is a constant or the matter Lagrangian is not an explicit function of the metric.

In order to test the motion in our model, we consider for the energy–momentum tensor of matter a perfect fluid

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu - pg_{\mu\nu}, \tag{8}$$

where ρ is the total energy density and p , the pressure, respectively. The four-velocity, U_μ , satisfies the conditions $U_\mu U^\mu = 1$ and $\nabla_\nu U^\mu U_\mu = 0$. We also introduce the projection operator $h_{\mu\lambda} = g_{\mu\lambda} - U_\mu U_\lambda$ from which one obtains $h_{\mu\lambda} U^\mu = 0$.

From Eq. (7), we deduce the equation of motion for a fluid element:

$$\frac{DU^\alpha}{ds} \equiv \frac{dU^\alpha}{ds} + \Gamma^\alpha_{\mu\nu} U^\mu U^\nu = f^\alpha \tag{9}$$

where the extra force is given by

$$f^\alpha = \frac{1}{\rho + p} \left[\frac{\lambda F_2}{1 + \lambda f_2} (L_m + p) \nabla_\nu R + \nabla_\nu p \right] h^{\alpha\nu}. \tag{10}$$

An intriguing feature is that the extra force depends on the form of the Lagrangian density. Note that considering the Lagrangian density to be $L_m = -p$, where p is the pressure, the contribution of the nonminimal curvature–matter vanishes [46]. It has been argued that this is not the unique choice for the matter Lagrangian density and that more natural forms for L_m , such as $L_m = \rho$, do not imply the vanishing of the extra force. Indeed, in the presence of the nonminimal coupling, they give rise to two distinct theories with different predictions [47,48].

2.2 Scalar–tensor representation of the linear curvature–matter coupling

As has been shown in [49–52], $f(R)$ gravity is equivalent to a scalar–tensor theory. In this context, the equivalence between the modified gravity models to a linear curvature–matter coupling and scalar–tensor gravity models was also established in [23]. More specifically, it was shown that the action given by Eq. (1) is equivalent with a two-potential scalar–tensor Brans–Dicke type theory, with a single scalar field, a vanishing Brans–Dicke parameter ω , and an unusual coupling of the second potential $U(\psi)$ of the theory to matter.

As a first step in the scalar–tensor formulation of the theory we introduce a new field ϕ , and reformulate the action (1) as

$$S = \int d^4x \sqrt{-g} \left\{ \frac{f_1(\phi)}{2} + \frac{1}{2} \frac{df_1}{d\phi} (R - \phi) + [1 + \lambda f_2(\phi)] L_m \right\}. \tag{11}$$

Next, we introduce the second field $\psi(\phi) \equiv f'_1(\phi)$ (with a prime denoting a differentiation with respect to ϕ), and thus we obtain for the action the final expression

$$S = \int d^4x \sqrt{-g} \left[\frac{\psi R}{2} - V(\psi) + U(\psi)L_m \right], \tag{12}$$

where the two potentials $V(\psi)$ and $U(\psi)$ of the theory are defined as

$$V(\psi) = \frac{\phi(\psi)f'_1[\phi(\psi)] - f_1[\phi(\psi)]}{2}, \tag{13}$$

and

$$U(\psi) = 1 + \lambda f_2[\phi(\psi)], \tag{14}$$

respectively. The function $\phi(\psi)$ must be obtained by inverting $\psi(\phi) \equiv f'_1(\phi)$. The actions (1) and (12) are equivalent when $f''_1(R) \neq 0$ [23], similarly to the case of pure $f(R)$ gravity [49–52].

The action given by Eq. (12) can be written, via a conformal rescaling $g_{\mu\nu} = \exp(-\alpha\phi/2) \hat{g}_{\mu\nu}$, $\alpha = \text{constant}$, as a four-dimensional dilaton gravity whose action, in the ‘‘Einstein frame’’, has the form [42]

$$S_E = \int d^4x \sqrt{-g} \left(\frac{\hat{R}}{2} - \nabla_\mu \phi \nabla^\mu \phi - e^{-\alpha\phi} L_m \right). \tag{15}$$

In the Einstein frame representation of the modified gravity with linear coupling between matter and geometry the extra force is due to the coupling between the matter Lagrangian and the Brans–Dicke-like scalar ϕ . On the other hand it is important to mention that in the $Lf(R, L_m)$ theory there is no scaling of units with some powers of the conformal factor of the conformal transformation [42]. For this reason it is impossible to reduce $Lf(R, L_m)$ gravity to a standard scalar–tensor theory, or to find a string gravity equivalent.

By varying the action (12) with respect to $g_{\mu\nu}$ one provides the gravitational field equations of the scalar–tensor theory as

$$\psi \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + \hat{P}_{\mu\nu} \psi = U(\psi) T_{\mu\nu} + V(\psi) g_{\mu\nu}, \tag{16}$$

while the variation of the action with respect to the field ψ gives the relation

$$\frac{R}{2} - V'(\psi) + U'(\psi)L_m = 0. \tag{17}$$

The contraction of the field equation (16) yields the scalar relation

$$-\psi R + 3\Box\psi = U(\psi)T + 4V(\psi), \tag{18}$$

where $T = T^\mu_\mu$ is the trace of the energy–momentum tensor. By combining Eqs. (17) and (18) we obtain the field equation of the field ψ as

$$\Box\psi = \frac{1}{3}U(\psi)T + \frac{4}{3}V(\psi) + \frac{2}{3}\psi V'(\psi) - \frac{2}{3}\psi U'(\psi)L_m. \tag{19}$$

By eliminating the term $\Box\psi$, and taking into account Eq. (19) the field equations (16) provide

$$\psi \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) - \nabla_\mu \nabla_\nu \psi = U(\psi) \left(T_{\mu\nu} - \frac{1}{3} g_{\mu\nu} T \right) - \left[\frac{1}{3}V(\psi) + \frac{2}{3}\psi V'(\psi) - \frac{2}{3}\psi U'(\psi)L_m \right] g_{\mu\nu}. \tag{20}$$

Now, taking the covariant divergence of the field equation (16), with the use of Eq. (6), we obtain first

$$-\left[\frac{R}{2} + V'(\psi) \right] \nabla_\nu \psi = U'(\psi) \nabla_\mu \psi T^\mu_\nu + U(\psi) \nabla_\mu T^\mu_\nu. \tag{21}$$

Then, by eliminating $R/2$ with the help of Eq. (17) we obtain for the divergence of the energy–momentum tensor

$$\nabla_\mu T^\mu_\nu = -[\nabla_\mu \ln U(\psi)] T^\mu_\nu - \frac{2V'(\psi) - U'(\psi)L_m}{U(\psi)} \nabla_\nu \psi. \tag{22}$$

Equation (22) allows the formulation of the energy and momentum balance equations in the scalar–tensor representation of the modified theory of gravity with a linear coupling between matter and geometry. By assuming that the energy–momentum tensor has the perfect fluid form given by Eq. (8), then Eq. (22) can be written in the equivalent form

$$\begin{aligned} &(\nabla^\mu \rho + \nabla^\mu p)U_\mu U_\nu + (\rho + p) U_\nu \nabla^\mu U_\mu - \nabla^\mu p g_{\mu\nu} \\ &+ (\rho + p) U_\mu \nabla^\mu U_\nu + \nabla^\mu [\ln U(\psi)] T_{\mu\nu} \\ &+ \frac{2V'(\psi) - U'(\psi)L_m}{U(\psi)} \nabla_\nu \psi = 0. \end{aligned} \tag{23}$$

By multiplying Eq. (23) with U^ν we obtain the energy balance equation in the scalar–tensor representation of the linear curvature–matter coupling given by

$$\dot{\rho} + 3H(\rho + p) + \rho \frac{d}{ds} \ln U(\psi) + \frac{2V'(\psi) - U'(\psi)L_m}{U(\psi)} \dot{\psi} = 0, \tag{24}$$

where we have introduced the Hubble function $H = (1/3)\nabla^\mu U_\mu$, and we have denoted $\dot{} = U^\mu \nabla_\mu = d/ds$, respectively, where ds is the line element corresponding to the metric $g_{\mu\nu}$, $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. After acting on Eq. (23)

with the projection operator $h^{\nu\lambda}$, provides the momentum balance equation for a perfect fluid as

$$U^\mu \nabla_\mu U^\alpha = \frac{d^2 x^\alpha}{ds^2} + \Gamma_{\mu\nu}^\alpha U^\mu U^\nu - f^\alpha = 0 \tag{25}$$

where the extra force is given by

$$f^\alpha = h^{\mu\alpha} \left[p \nabla_\mu \ln U(\psi) - \frac{2V'(\psi) - U'(\psi)L_m}{U(\psi)} \nabla_\mu \psi \right]. \tag{26}$$

It is important to note that in the scalar–tensor representation of modified gravity with a linear curvature–matter coupling, the extra force acting on test fluids is non-zero independently of the choice for the matter Lagrangian.

3 Gravitationally induced particle creation

In the present section, we analyze the physical interpretation of the curvature–matter coupling in the scalar–tensor representation by adopting the point of view of the thermodynamics of open systems, in which matter creation irreversible processes may take place at a cosmological scale [24–27]. As we have already seen in the previous section, the energy conservation equation of the curvature–matter coupling, given by Eq. (24), contains, as compared to the standard adiabatic conservation equation, an extra term, which can be interpreted in the framework of the open thermodynamic systems as an irreversible matter creation rate. According to irreversible thermodynamics, matter creation also represents an entropy source, generating an entropy flux, and thus leading, in the presence of the curvature–matter coupling, to a modification in the temperature evolution.

In the following, we investigate only the case in which all the non-diagonal components of the energy–momentum tensor of the matter are equal to zero, so that $T_{\mu\nu} = 0, \mu \neq \nu$. Generally, the energy–momentum tensor for a viscous dissipative fluid in the presence of heat conduction is given by $T_{\mu\nu} = (\rho + p + \Pi) - (p + \Pi)g_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu + \pi_{\mu\nu}$, where Π is the bulk-viscous pressure, q_ν is the heat flux, and $\pi_{\mu\nu}$ is the tensor of viscous dissipation [43, 44]. q_μ and $\pi_{\mu\nu}$ must satisfy the conditions $q_\mu u^\mu = 0$ and $\pi_{\mu\nu} u^\nu = \pi_\mu^\mu = 0$, respectively. In the following we neglect the viscous effects in the cosmological fluid, thus assuming $\Pi \equiv 0$ and $\pi_{\mu\nu} \equiv 0$. In a comoving reference frame with $u^\mu = (1, 0)$, the form of the heat flux vector is fixed by the normalization condition as $q^\mu = (0, \mathbf{q})$. Therefore in a comoving reference frame all components of the form $u_\mu q_\nu$ of the energy–momentum tensor are identically equal to zero, and $T_{\mu\nu}$ is a diagonal tensor. From the point of view of the thermodynamics of the irreversible processes, this condition implies the impossibility of

heat transfer in the considered gravitational system. In particular, this condition is always satisfied in homogeneous and isotropic cosmological models, described by the Friedmann–Robertson–Walker geometry, since in these models the condition $T_{0i} \equiv 0, i = 1, 2, 3$ must always hold.

3.1 Matter creation rates and the creation pressure

We assume that the cosmological metric is given by the flat isotropic and homogeneous Friedmann–Robertson–Walker (FRW) metric,

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2), \tag{27}$$

where $a(t)$ is the scale factor, describing the expansion of the Universe. In this geometry the cosmological matter is comoving with the cosmological expansion, and therefore the four-velocity of the cosmological fluid is $U^\mu = (1, 0, 0, 0)$, while the Hubble function takes the form $H = \dot{a}/a$, since $U^\mu \nabla_\mu = \dot{}$.

To investigate the thermodynamical implications at the cosmological scale with a curvature–matter coupling we consider that the Universe contains N particles in a volume V , with an energy density ρ and a thermodynamic pressure p , respectively. For such a cosmological system, the second law of thermodynamics, in its most general form, is given by [25]

$$\frac{d}{dt}(\rho a^3) + p \frac{d}{dt} a^3 = \frac{dQ}{dt} + \frac{\rho + p}{n} \frac{d}{dt}(na^3), \tag{28}$$

where dQ is the heat received by the system during time dt , and $n = N/V$ is the particle number density, respectively. Due to our choice of the geometry of the Universe, and of the cosmological principle, only adiabatic transformations, defined by the condition $dQ = 0$, are possible. Therefore in the following we ignore proper heat transfer processes in the Universe. However, as one can see from Eq. (28), under the assumption of adiabatic transformations the second law of thermodynamics contains the term $[(\rho + p)/n]d(na^3)/dt$, which explicitly takes into account the time variation of the cosmological particles in a given volume V . Hence, in the irreversible thermodynamics description of open systems, even for adiabatic transformations $dQ = 0$, one can consider the “heat” (internal energy) received/lost by the system, due to the change in the particle number n . For modified gravity with a curvature–matter coupling the change in the particle number is due to the transfer of energy from gravity to matter. Thus, via matter creation, gravity acts as a source of internal energy and of entropy. For adiabatic transformations $dQ/dt = 0$, we can reformulate Eq. (28) in an equivalent form as

$$\dot{\rho} + 3(\rho + p)H = \frac{\rho + p}{n} (\dot{n} + 3Hn). \tag{29}$$

Therefore, from the point of view of the thermodynamics of open systems, Eq. (24), giving the energy balance in the presence of a curvature–matter coupling, can be interpreted as describing particle creation in an homogeneous and isotropic geometry, with the time variation of the particle number density obtained as

$$\dot{n} + 3nH = \Gamma n, \tag{30}$$

where the particle creation rate Γ is a non-negative quantity defined as

$$\Gamma = -\frac{1}{\rho + p} \left\{ \rho \frac{d}{dt} \ln U(\psi) + \frac{2V'(\psi) - U'(\psi)L_m}{U(\psi)} \dot{\psi} \right\}. \tag{31}$$

Therefore, the energy conservation equation can be reformulated in the alternative form

$$\dot{\rho} + 3(\rho + p)H = (\rho + p)\Gamma. \tag{32}$$

As proven initially in [25], for adiabatic transformations Eq. (28), describing irreversible particle creation in an open thermodynamic system, can be rewritten as an effective energy conservation equation,

$$\frac{d}{dt}(\rho a^3) + (p + p_c) \frac{d}{dt}a^3 = 0, \tag{33}$$

or, in an equivalent form, as,

$$\dot{\rho} + 3(\rho + p + p_c)H = 0, \tag{34}$$

where we have introduced a new thermodynamic quantity, p_c , denoted the creation pressure and defined as [25]

$$\begin{aligned} p_c &= -\frac{\rho + p}{n} \frac{d(na^3)}{da^3} \\ &= -\frac{\rho + p}{3nH} (\dot{n} + 3nH) = -\frac{\rho + p}{3} \frac{\Gamma}{H}. \end{aligned} \tag{35}$$

Therefore in modified gravity with a linear curvature–matter coupling the creation pressure is given by

$$p_c = -\frac{1}{3H} \left\{ \rho \frac{d}{dt} \ln U(\psi) + \frac{2V'(\psi) - U'(\psi)L_m}{U(\psi)} \dot{\psi} \right\}. \tag{36}$$

Note that from Eq. (7), the coupling between the matter and the higher derivative curvature terms may be interpreted as an exchange of energy and momentum between both. In the standard formulation of the linear curvature–matter coupling,

by taking into account a FRW background, and from Eq. (7) we obtain the energy balance equation as

$$\dot{\rho} + 3H(\rho + p) = \frac{\lambda F_2(R)}{1 + \lambda f_2(R)} (\alpha - 1)\rho \dot{R}. \tag{37}$$

Hence, by considering that the mechanism for gravitational particle production is through the nonminimal curvature–matter coupling, so that comparing Eqs. (35) and (37), we have for the creation pressure

$$p_c = \frac{\lambda F_2(R)}{1 + \lambda f_2(R)} \frac{1}{3H} (1 - \alpha)\rho \dot{R}, \tag{38}$$

with the requirement that p_c be negative. We will only consider the case of $L_m = -p$, i.e., $\alpha = -\omega$, in order to have a non-vanishing creation pressure.

3.2 Entropy and temperature evolution

According to the basic principles of the thermodynamics of open systems the entropy change consists of two components: the entropy flow term $d_e S$, and the entropy creation term $d_i S$. The total entropy S of an open thermodynamic system can be represented as [24,25]

$$dS = d_e S + d_i S, \tag{39}$$

where by definition $d_i S > 0$. Both the entropy flow and the entropy production can be obtained from the total differential of the entropy given by [25],

$$T d(\bar{s} a^3) = d(\rho a^3) + p da^3 - \mu d(na^3), \tag{40}$$

where T is the temperature of the open thermodynamic system, $\bar{s} = S/a^3$ is the entropy per unit volume, and μ is the chemical potential, defined as

$$\mu n = h - T\bar{s}, \tag{41}$$

where $h = \rho + p$ is the enthalpy of the system.

In the case of a closed thermodynamic system and for adiabatic transformations we have $dS = 0$ and $d_i S = 0$. However, in the presence of a curvature–matter coupling, leading to effective matter creation, there is a non-zero contribution to the total entropy. For a homogeneous and isotropic Universe the entropy flow term $d_e S$ vanishes, so that $d_e S = 0$. On the other hand matter creation also represents a source for entropy creation, and the time variation of the corresponding entropy is obtained as [25]

$$\begin{aligned} T \frac{d_i S}{dt} &= T \frac{dS}{dt} = \frac{h}{n} \frac{d}{dt}(na^3) - \mu \frac{d}{dt}(na^3) \\ &= T \frac{\bar{s}}{n} \frac{d}{dt}(na^3) \geq 0. \end{aligned} \tag{42}$$

Equation (42) gives the time variation of the entropy as

$$\frac{dS}{dt} = \frac{S}{n} (\dot{n} + 3Hn) = \Gamma S \geq 0, \tag{43}$$

so that the entropy increase due to particle production yields the expression

$$S(t) = S_0 e^{\int_0^t \Gamma(t') dt'}, \tag{44}$$

where $S_0 = S(0)$ is a constant. With the use of Eq. (43), we obtain for the entropy creation in the scalar–tensor representation of the linear coupling between matter and geometry the following equation:

$$\frac{1}{S} \frac{dS}{dt} = -\frac{1}{\rho + p} \left\{ \rho \frac{d}{dt} \ln U(\psi) + \frac{2V'(\psi) - U'(\psi)L_m}{U(\psi)} \dot{\psi} \right\}. \tag{45}$$

The entropy flux four-vector S^μ is defined as [26]

$$S^\mu = n\sigma U^\mu, \tag{46}$$

where $\sigma = S/N$ is the specific entropy per particle. S^μ must satisfy the second law of thermodynamics, which imposes the constraint $\nabla_\mu S^\mu \geq 0$ on its four-dimensional divergence. The Gibbs relation [26],

$$nT d\sigma = d\rho - \frac{h}{n} dn, \tag{47}$$

together with the definition of the chemical potential μ of the open thermodynamic system,

$$\mu = \frac{h}{n} - T\sigma, \tag{48}$$

yields

$$\begin{aligned} \nabla_\mu S^\mu &= (\dot{n} + 3nH)\sigma + nU^\mu \nabla_\mu \sigma \\ &= \frac{1}{T} (\dot{n} + 3Hn) \left(\frac{h}{n} - \mu \right), \end{aligned} \tag{49}$$

where we have used the relation

$$nT\dot{\sigma} = \dot{\rho} - \frac{\rho + p}{n} \dot{n} = 0, \tag{50}$$

which immediately follows from Eq. (29). With the use of Eq. (30) we obtain the entropy production rate due to the particle creation processes given by

$$\begin{aligned} \nabla_\mu S^\mu &= \Gamma \frac{n}{T} \left(\frac{h}{n} - \mu \right) = -\frac{n}{T(\rho + p)} \left\{ \rho \frac{d}{dt} \ln U(\psi) \right. \\ &\quad \left. + \frac{2V'(\psi) - U'(\psi)L_m}{U(\psi)} \dot{\psi} \right\} \left(\frac{h}{n} - \mu \right). \end{aligned} \tag{51}$$

A general thermodynamic system is described by two fundamental thermodynamic variables, the particle number density n , and the temperature T , respectively. If the system is in an equilibrium state, the energy density ρ and the thermodynamic pressure p are obtained, in terms of n and T , from the equilibrium equations of state of the matter,

$$\rho = \rho(n, T), \quad p = p(n, T). \tag{52}$$

Therefore the energy conservation equation (32) can be obtained in the following general form:

$$\frac{\partial \rho}{\partial n} \dot{n} + \frac{\partial \rho}{\partial T} \dot{T} + 3(\rho + p)H = \Gamma n. \tag{53}$$

By using the general thermodynamic relation [26]

$$\frac{\partial \rho}{\partial n} = \frac{h}{n} - \frac{T}{n} \frac{\partial p}{\partial T}, \tag{54}$$

it follows that the temperature evolution of the newly created particles due to the curvature–matter coupling is given by the expression

$$\frac{\dot{T}}{T} = c_s^2 \frac{\dot{n}}{n} = c_s^2 (\Gamma - 3H), \tag{55}$$

where the speed of sound c_s is defined as $c_s^2 = \partial p / \partial \rho$. If the geometrically created matter satisfies a barotropic equation of state of the form $p = (\gamma - 1)\rho$, $1 \leq \gamma \leq 2$, the temperature evolution follows the simple equation

$$T = T_0 n^{\gamma-1}. \tag{56}$$

3.3 Bulk-viscosity description of matter creation processes with a curvature–matter coupling

An alternative physical interpretation of particle creation processes in cosmology was suggested by Zeldovich [53], and later on by Murphy [54] and Hu [55]. According to this interpretation, the viscosity of the cosmological fluid represents a phenomenological description of the effect of the creation of particles by the non-stationary gravitational field of the expanding universe. Therefore, from a physical point of view, a non-vanishing particle production rate is equivalent to the introduction of a bulk-viscous pressure in the energy–momentum tensor of the cosmological fluid. From a quantum mechanical point of view, such a viscous pressure can also be related to the viscosity of the vacuum [53–55]. This physical interpretation follows from the simple circumstance that any source term in the energy balance equation of a general relativistic fluid may be formally rewritten in terms of an effective bulk viscosity [43, 44].

The energy–momentum tensor of a general relativistic fluid with bulk viscosity as the only dissipative process can be written as [43,44]

$$T_{\mu\nu} = (\rho + p + \Pi) U_\mu U_\nu - (p + \Pi) g_{\mu\nu}, \tag{57}$$

where Π is the bulk-viscous pressure. The particle flow vector N^μ is defined as $N^\mu = nU^\mu$. In the framework of causal thermodynamics the entropy flow vector S^μ takes the form [56–58]

$$S^\mu = {}_s N^\mu - \frac{\tau \Pi^2}{2\xi T} U^\mu, \tag{58}$$

where τ is the relaxation time, and ξ is the coefficient of bulk viscosity. In Eq. (58), we have limited ourselves to considering only second-order deviations from equilibrium. In the case of homogeneous and isotropic geometries, in the presence of bulk-viscous dissipative phenomena, the energy conservation equation is obtained as

$$\dot{\rho} + 3(\rho + p + \Pi) H = 0. \tag{59}$$

By comparing Eq. (59), giving the energy conservation equation for a cosmological fluid in the presence of bulk viscosity, with Eq. (34), which includes in the energy balance the creation of particles from the gravitational field due to the curvature–matter coupling, it follows that these two equations are equivalent if

$$p_c = \Pi = -\frac{1}{3H} \left\{ \rho \frac{d}{dt} \ln U(\psi) + \frac{2V'(\psi) - U'(\psi)L_m}{U(\psi)} \dot{\psi} \right\}. \tag{60}$$

Therefore particle creation can be indeed described from a phenomenological point of view by introducing an effective bulk-viscous pressure in the energy–momentum tensor of the cosmological fluid. Hence it follows that the causal bulk-viscous pressure Π acts as a creation pressure.

Hence, it would be interesting to investigate matter creation processes in modified gravity with a curvature–matter coupling from the point of view of bulk-viscous thermodynamic processes. Hence we shall consider in the following that there is a change in the number of particles, due to matter creation processes, with bulk-viscous pressure playing the role of the creation pressure. We introduce a simple toy model in which the newly created particles obey, as a function of the particle number density n , an equation of state of the form

$$\rho = \rho_0 \left(\frac{n}{n_0} \right)^\gamma = kn^\gamma, \quad p = (\gamma - 1)\rho, \tag{61}$$

where ρ_0 , n_0 and γ are constants, we have denoted $k = \rho_0/n_0^\gamma$, and $1 \leq \gamma \leq 2$, respectively. Using Eq. (61), then Eq. (59) takes the form of a particle balance equation,

$$\dot{n} + 3Hn = \Gamma n, \tag{62}$$

where

$$\Gamma = -\frac{\Pi}{\gamma H} \tag{63}$$

is the particle production rate, proportional to the bulk-viscous pressure. Combining the equation of state Eq. (61) with the Gibbs relation $Td\bar{s} = d(\rho/n) + pd(1/n)$ we obtain $\bar{s} = \bar{s}_0 = \text{constant}$, that is, particles are created with constant entropy density.

However, there is a major difference between the particle creation irreversible processes in open thermodynamic systems and bulk-viscous processes, and this difference is related to the expression for entropy production rate. While the entropy production rate associated to particle creation is given by [26]

$$\nabla_\mu S^\mu = -\frac{3Hp_c}{T} \left(1 + \frac{\mu\Gamma n}{3Hp_c} \right) \geq 0, \tag{64}$$

in the presence of bulk-viscous dissipative processes the entropy production rate can be obtained as [43,44]

$$\nabla_\mu S^\mu = -\frac{\Pi}{T} \left[3H + \frac{\tau}{\xi} \dot{\Pi} + \frac{\tau}{2\xi} \Pi \left(3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T} \right) \right]. \tag{65}$$

In the particle creation model in open thermodynamics systems the entropy production rate is proportional to the creation pressure, while in the viscous dissipative processes thermodynamic interpretation $\nabla_\mu S^\mu$ is quadratic in the creation pressure, $\nabla_\mu S^\mu \propto p_c^2/\xi T$, and, moreover, involves a new dynamical variable, the bulk-viscosity coefficient.

4 Cosmological applications

In the present Section, we consider several cosmological applications of the scalar–tensor formulation of modified gravity with a linear curvature–matter coupling, as its interpretation as a particle creation theory. For a homogeneous and isotropic geometry the gravitational field equations (17)–(20) take the form

$$3H^2 = \frac{\ddot{\psi}}{\psi} + \left(\frac{2}{3}\rho + p \right) \frac{U(\psi)}{\psi} - \frac{1}{3} \left[\frac{V(\psi)}{\psi} + 2V'(\psi) - 2U'(\psi)L_m \right], \tag{66}$$

$$2\dot{H} + 3H^2 = -\frac{\dot{\psi}}{a\psi} - \frac{1}{3}\rho\frac{U(\psi)}{\psi} - \frac{1}{3}\left[\frac{V(\psi)}{\psi} + 2V'(\psi) - 2U'(\psi)L_m\right], \tag{67}$$

and

$$-3(\dot{H} + 2H^2) - V'(\psi) + U'(\psi)L_m = 0, \tag{68}$$

respectively.

Equation (30) can be integrated to give the particle number time variation as

$$n(t) = \frac{n_0}{a^3} e^{\int \Gamma(t) dt}, \tag{69}$$

where n_0 is an arbitrary constant of integration, while, by assuming a barotropic equation of state of the form $p = p(\rho)$ we obtain for the density evolution

$$\int \frac{d\rho}{\rho + p(\rho)} = -3 \ln a + \int \Gamma(t) dt + \ln \rho_0, \tag{70}$$

where ρ_0 is an integration constant.

4.1 Cosmological models satisfying the condition

$$2V'(\psi) - U'(\psi)L_m = 0.$$

The linear curvature–matter coupling depends on the two arbitrary (and independent) potentials $U(\psi)$ and $V(\psi)$. As a simple toy model, in the following, we assume that the two potentials U and V are related to the matter Lagrangian via the relation

$$2V'(\psi) - U'(\psi)L_m = 0. \tag{71}$$

Moreover, we restrict our analysis to the case of dust, with negligible thermodynamic pressure $p = 0$. With the choice of Eq. (71), the particle creation rate, given by Eq. (31), takes the form

$$\Gamma = -\frac{d}{dt} \ln U(\psi(t)) = -\frac{U'(\psi)}{U(\psi)} \dot{\psi} \geq 0. \tag{72}$$

Therefore, in this approach the matter creation rate is determined by the function $U(\psi)$, which describes the coupling between the matter Lagrangian and the scalar field, its derivative, and the time variation of the scalar field ψ only.

With this choice, the variations of the particle number and of the matter energy density are given by

$$n(t) = \frac{n_0}{a^3 U(\psi)}, \quad \rho(t) = \frac{\rho_0}{a^3 U(\psi)}. \tag{73}$$

With the assumption of Eq. (71) on the potentials, the gravitational field equations (66)–(68) take the form

$$3H^2 = \frac{\ddot{\psi}}{\psi} + \frac{2}{3}\rho\frac{U(\psi)}{\psi} - \frac{1}{3}\left[\frac{V(\psi)}{\psi} - U'(\psi)L_m\right], \tag{74}$$

$$2\dot{H} + 3H^2 = -\frac{\dot{\psi}}{a\psi} - \frac{1}{3}\rho\frac{U(\psi)}{\psi} - \frac{1}{3}\left[\frac{V(\psi)}{\psi} - U'(\psi)L_m\right], \tag{75}$$

$$6(\dot{H} + 2H^2) = U'(\psi)L_m, \tag{76}$$

respectively.

In the following, we consider only de Sitter type accelerating solutions of the system given by Eqs. (74)–(76), with $H = H_0 = \text{constant}$ and $a(t) = \exp(H_0 t)$. Then we obtain first

$$U'(\psi)L_m = 12H_0^2, \tag{77}$$

while the evolution equation for the scalar field ψ , which can be obtained from Eqs. (73)–(75), is given by

$$\ddot{\psi} + H_0\dot{\psi} + \frac{\rho_0}{e^{3H_0 t}} = 0, \tag{78}$$

with the general solution given by

$$\psi(t) = -\frac{1}{6H_0^2}[\rho_0 e^{-3H_0 t} - 3\rho_0 e^{-H_0(t+2t_0)} + 2\rho_0 e^{-3H_0 t_0} + 6H_0\psi_{01} e^{-H_0(t-t_0)} - 6H_0(H_0\psi_0 + \psi_{01})], \tag{79}$$

where we have used the initial conditions $\psi(t_0) = \psi_0$, and $\psi'(t_0) = \psi_{01}$, respectively. Since for the dust fluid the matter Lagrangian is $L_m = \rho$, Eq. (77) yields

$$\frac{U'(\psi)}{U(\psi)} = 12\frac{H_0^2}{\rho_0} e^{3H_0 t}, \tag{80}$$

or, equivalently,

$$\frac{1}{U(t)} \frac{dU(t)}{dt} = 12\frac{H_0^2}{\rho_0} e^{3H_0 t} \dot{\psi}. \tag{81}$$

The above equation determines the time variation of the potential $U(\psi)$ as

$$U(t) = U_0 \exp\left[\frac{6H_0\psi_{01} e^{H_0(2t+t_0)}}{\rho_0} - 3e^{2H_0(t-t_0)} + 6H_0 t\right], \tag{82}$$

where U_0 is an arbitrary constant of integration. The time variation of the particle creation rate is obtained as

$$\Gamma(t) = \frac{6H_0U_0}{\rho_0} [2H_0|\psi_{01}|e^{H_0(2t+t_0)} + \rho_0e^{2H_0(t-t_0)} - \rho_0], \tag{83}$$

where we have assumed $\psi_{01} < 0$. The particle creation rate is non-negative for all times t , and is a monotonically increasing function of time for all $t \geq t_0$, and its initial value is given by $\Gamma(t_0) = 12H_0^2|\psi_{01}|/\rho_0 \geq 0$.

Therefore we obtain for the time variation of the matter energy density the equation

$$\rho(t) = \frac{\rho_0}{U_0} \exp \left[\frac{3e^{2H_0(t-t_0)} (2H_0|\psi_{01}|e^{3H_0t_0} + \rho_0)}{\rho_0} - 9H_0t \right]. \tag{84}$$

For the time variation of the potential $V(t)$ we obtain the evolution equation

$$2 \frac{dV}{dt} = \rho \frac{dU}{dt}, \tag{85}$$

which gives the potential V in an integral form as

$$V(t) = V_0 + 3H_0U_0 \int e^{H_0 \left[\frac{12|\psi_{01}|e^{H_0(2t+t_0)}}{\rho_0} - 3t \right]} \times [-\rho_0e^{2H_0(t-t_0)} + 2H_0|\psi_{01}|e^{H_0(2t+t_0)} + \rho_0] dt, \tag{86}$$

where V_0 is an arbitrary constant of integration. The creation pressure, defined as $p_c = -\rho\Gamma/(3H)$, is given by

$$p_c = 2U_0[-\rho_0e^{2H_0(t-t_0)} - 2H_0|\psi_{01}|e^{H_0(2t+t_0)} + \rho_0] \times \exp \left[\frac{3e^{2H_0(t-t_0)} (2H_0|\psi_{01}|e^{3H_0t_0} + \rho_0)}{\rho_0} - 9H_0t \right]. \tag{87}$$

Finally, for the comoving entropy of the de Sitter type expanding Universe in the linear curvature–matter coupling theory we obtain

$$S(t) = \frac{S_0}{U(t)} = \frac{S_0}{U_0} \exp \left[3e^{2H_0(t-t_0)} - 6H_0t + \frac{6H_0|\psi_{01}|e^{H_0(2t+t_0)}}{\rho_0} \right]. \tag{88}$$

The entropy is a monotonically increasing function of time, with the property $\dot{S}(t) \geq 0, \forall t \geq t_0$. In the first order approximation, and for small times we obtain for the entropy the following expression:

$$S(t) \propto \exp \left(\frac{12H_0^2|\psi_{01}|}{\rho_0} t \right). \tag{89}$$

Therefore, the curvature–matter coupling allows the production of a large amount of entropy during a de Sitter type evolutionary phase of the Universe.

4.2 de Sitter type expansionary models with constant matter creation rate

In the following, in the scalar–tensor representation of $L_f(R, L_m)$ gravity, we consider a second simple cosmological toy model by assuming that the cosmological expansion of a dust Universe, with $p = 0$, is accelerating with $a = \exp(H_0t)$, where $H_0 = \text{constant}$, and the particle creation rate is a constant during the entire accelerating phase, and it is given by $\Gamma = \Gamma_0 = 3H_0 = \text{constant}$. Moreover, we take the matter Lagrangian as $L_m = \rho$. Then from Eq. (70) it follows immediately that the matter density of the Universe is also a constant,

$$\rho = \rho_0 = \text{constant}. \tag{90}$$

Then the constancy of the matter creation rate, given by Eq. (31), imposes the following condition on the potentials U and V :

$$2\dot{V} + 3H_0\rho_0U = 0, \tag{91}$$

while the field equation Eq. (68) yields

$$\dot{V} - \rho_0\dot{U} = -6H_0^2\dot{\psi}. \tag{92}$$

From the field equations Eqs. (66) and (67) we obtain the evolution equation for ψ as

$$\ddot{\psi} + H_0\dot{\psi} + \rho_0U(\psi(t)) = 0. \tag{93}$$

From Eqs. (91) and (92) we obtain

$$\frac{3H_0\rho_0}{2}U + \rho_0\dot{U} = 6H_0^2\dot{\psi}. \tag{94}$$

By taking the time derivative of the above equation, and by eliminating $\ddot{\psi}$ with the help of Eq. (93), it follows that U satisfies the equation

$$\ddot{U} + \frac{5}{2}H_0\dot{U} + \frac{15}{2}H_0^2U = 0, \tag{95}$$

with the general solution given by

$$U(t) = \frac{e^{\frac{5}{4}H_0(t_0-t)}}{215H_0} \left\{ 215H_0U_0 \cos \left[\frac{1}{4}\sqrt{215}H_0(t-t_0) \right] \right. \\ \left. \times \sqrt{215}(5H_0U_0 + 4U_{01}) \sin \left[\frac{1}{4}\sqrt{215}H_0(t-t_0) \right] \right\}, \tag{96}$$

where we have used the initial conditions $U(t_0) = U_0$ and $U'(t_0) = U_{01}$, respectively. The time dependence of the potential V is obtained from Eq. (91) in the form

$$V(t) = \frac{\rho_0 e^{\frac{5}{4}H_0(t_0-t)}}{860H_0} \left\{ \sqrt{215}(2U_{01} - 19H_0U_0) \right. \\ \times \sin \left[\frac{1}{4}\sqrt{215}H_0(t-t_0) \right] + 43(5H_0U_0 + 2U_{01}) \\ \left. \times \cos \left[\frac{1}{4}\sqrt{215}H_0(t-t_0) \right] \right\}. \tag{97}$$

With the use of the explicit time dependence of U in Eq. (93), we obtain for $\psi(t)$ the following expression:

$$\psi(t) = \frac{1}{11610H_0^3} \left\{ -430e^{H_0(t_0-t)}(27H_0^2\psi_{01} - 3H_0\rho_0U_0 \right. \\ - 2\rho_0U_{01}) + 387(30H_0^3\psi_0 + 30H_0^2\psi_{01} - 5H_0\rho_0U_0 \\ - 2\rho_0U_{01}) + \rho_0 e^{\frac{5}{4}H_0(t_0-t)} \left[\sqrt{215}(39H_0U_0 + 14U_{01}) \right. \\ \times \sin \left(\frac{1}{4}\sqrt{215}H_0(t-t_0) \right) + 43(15H_0U_0 - 2U_{01}) \\ \left. \times \cos \left(\frac{1}{4}\sqrt{215}H_0(t-t_0) \right) \right] \left. \right\}, \tag{98}$$

where we have used the initial conditions $\psi(t_0) = \psi_0$ and $\dot{\psi}(t_0) = \psi_{01}$, respectively.

Due to the cosmological particle production the entropy of the Universe increases as

$$S(t) = S_0 e^{\int \Gamma(t) dt} = S_0 e^{3H_0 t}. \tag{99}$$

However, the specific entropy $s = S/V$ remains a constant during the cosmological evolution, $s = s_0 = \text{constant}$.

5 Total entropy behavior in $Lf(R, L_m)$ gravity with particle creation

In the present paper, we have defined the entropy through the particle production rate, given by Eq. (43), as depending on the positive particle creation rate Γ via the relation

$$\frac{\dot{S}}{S} = \Gamma \geq 0. \tag{100}$$

Therefore, in an ever expanding Universe with particle creation, the matter entropy will increase indefinitely. On the other hand, all natural systems tend to approach a state of thermodynamic equilibrium, implying that the entropy of equilibrium systems never decreases, $\dot{S} \geq 0$, and that it is concave when approaching the equilibrium state, $\ddot{S} \leq 0$. However, in the present, and several other, cosmological models, these fundamental requirements for the behavior of the entropy do not seem to be satisfied.

The problem of the validity of the second law of thermodynamics in cosmology was investigated in detail in [14, 59], where it was shown that the Universe approaches thermodynamic equilibrium in a de Sitter phase, if one defines the total entropy S_{tot} of the Universe as the entropy of the apparent horizon plus that of matter and radiation inside it. Then it follows that S_{tot} increases, and that it is concave, thus leading to the result that the second law of thermodynamics is still valid for the case of the cosmological expansion. In the following, we investigate the thermodynamic properties of the total entropy in a Universe with matter creation.

In the standard thermodynamic description of physical systems the time parameter t is not a thermodynamic equilibrium variable. Therefore, the variation of the thermodynamic quantities should be considered with respect to some extensive variable. In the following we will adopt, following [14, 59], as extensive variable for the cosmological system the proper volume enclosed by the apparent horizon, or, more specifically, its scale factor a . Then the relation between d/dt and d/da is simply

$$\frac{d}{dt} = aH \frac{d}{da}. \tag{101}$$

In the following we denote by a prime the derivative with respect to the extensive variable a .

We define the total entropy of a FRW Universe with dust as the sum of the entropy of the apparent horizon S_{ah} , proportional to its area, and that of the matter particles within it S_{m} [14, 59]. For practical purposes, in the case of the flat FRW model, the total entropy is

$$S_{\text{tot}} = S_{\text{ah}} + S_{\text{m}} = \frac{\pi}{H^2} + \frac{4\pi}{3H^3} n(t), \tag{102}$$

where we have used the fact that the radius of the apparent horizon is $r_{\text{ah}} = H^{-1}$ [60]. We also introduce an important observational quantity, the deceleration parameter q , defined as

$$q = \frac{d}{dt} \frac{1}{H} - 1 = -a \frac{H'}{H} - 1. \tag{103}$$

Therefore, the variation of the total entropy can be obtained as

$$\frac{S'_{\text{tot}}}{S_{\text{tot}}} = \frac{S'_{\text{ah}} + S'_m}{S_{\text{ah}} + S_m}, \tag{104}$$

$$\frac{S''_{\text{tot}}}{S_{\text{tot}}} = \frac{S''_{\text{ah}} + S''_m}{S_{\text{ah}} + S_m}, \tag{105}$$

respectively.

The particle number n satisfies Eq. (30) and is rewritten as

$$aH \frac{dn}{da} + 3nH = \Gamma n. \tag{106}$$

From their definitions the derivatives of the entropy with respect to the scale factor can be evaluated as

$$S'_{\text{tot}} = -\frac{2\pi a}{H^2(a)} H'(a) + \frac{4\pi n(a)}{H^3(a)} \left[\frac{\Gamma(a)}{3} - aH'(a) - H(a) \right], \tag{107}$$

$$S''_{\text{tot}} = \frac{2\pi}{3aH^4(a)} \{ 2H(a)[3a^2H'^2(a) + n(a)[a\Gamma'(a) - 3aH''(a) + 12H'(a)] - 6\Gamma(a)] - 3H^2(a)[a[aH''(a) + H'(a)] - 6n(a)] + 2n(a)[\Gamma(a) - 3aH'(a)]^2 \}, \tag{108}$$

respectively.

In terms of the deceleration parameter we can express the variation of the total entropy with respect to a as

$$S'_{\text{tot}} = \frac{2\pi}{H} (q + 1) + \frac{4\pi}{H^3} \left(qH + \frac{\Gamma}{3} \right) n, \tag{109}$$

$$S''_{\text{tot}} = \frac{2\pi}{3H^3(a)} \{ 2\Gamma(a)n(a)[\Gamma(a) - 3aH'(a)] + 2H(a)n(a)[a\Gamma'(a) - 6q(a)H'(a)] \times 3\Gamma(a)(q(a) - 1) - 3H^2(a)[a[q(a) + 1]H'(a) + n(a)[6q(a) - 2aq'(a)]] + 3aH(a)^3q'(a) \}. \tag{110}$$

Therefore the standard thermodynamic requirements $S'_{\text{tot}} \geq 0$ and $S''_{\text{tot}} \leq 0$ impose the following constraints on the particle creation rate Γ , and its derivative with respect to the scale factor:

$$\Gamma(a) \geq \frac{3a [H(a) + 2n(a)] H'(a)}{2n(a)} + 3H(a), \tag{111}$$

$$\Gamma(a) \geq -\frac{3 [q(a) + 1] H^2(a)}{2n(a)} - q(a)H(a), \tag{112}$$

and

$$\Gamma'(a) \leq \frac{1}{2aH(a)n(a)} \{ 6H(a)[n(a) \times [a^2H''(a) + 2\Gamma(a) - 4aH'(a)] - a^2H'^2(a)] - 2n(a)[\Gamma(a) - 3aH'(a)]^2 + 3H^2(a)[a[aH''(a) + H'(a)] - 6n(a)] \}, \tag{113}$$

$$\Gamma'(a) \leq \frac{1}{2aH(a)n(a)} \{ -2\Gamma(a)n(a)[\Gamma(a) - 3aH'(a)] + 3H^2(a)[a(q(a) + 1)H'(a) + n(a)(6q(a) - 2aq'(a))] + 6H(a)n(a)[\Gamma(a) + 2aq(a)H'(a) + \Gamma(a)(-q(a))] - 3aH^3(a)q'(a) \}, \tag{114}$$

respectively. Since in the $Lf(R, L_m)$ gravity theory the matter creation rate Γ is determined by the coupling functions U and V , the thermodynamic conditions impose some strong constraints on the allowed physical form of these functions.

A particularly interesting case is that of the de Sitter evolution of the Universe, with $H = H_0 = \text{constant}$. For this situation the total entropy of the Universe is given by

$$S_{\text{tot}} = \frac{\pi}{H_0^2} + \frac{4\pi}{3H_0^3} n(a), \tag{115}$$

giving

$$S'_{\text{tot}} = \frac{4\pi}{3H_0^3} n'(a) = \frac{4\pi}{3H_0^4} \frac{[\Gamma(a) - 3H_0]}{a} n(a) \geq 0, \tag{116}$$

$$S''_{\text{tot}} = \frac{4\pi n(a)}{3a^2H_0^5} \{ \Gamma^2(a) + H_0 [a\Gamma'(a) + 12H_0] - 7H_0\Gamma(a) \} \leq 0. \tag{117}$$

The thermodynamic condition of the non-negativity of the total entropy derivative with respect to the scale factor imposes the conditions $\Gamma \geq 3H_0$ and $\Gamma'(a) \leq [7\Gamma(a) - \Gamma^2(a)/H_0 - 12H_0]/a$ on the particle creation rate Γ . In the particular case $\Gamma = 3H_0$, we obtain $S_{\text{tot}} = \text{constant}$, showing that in this case the cosmological evolution is isentropic, with the total entropy being a constant.

6 Discussion and conclusions

In the present paper, we have considered the thermodynamic interpretation of modified theories of gravity with a linear coupling between matter and geometry, which we call $Lf(R, L_m)$ gravity. This theory represents a particular class, corresponding to a specific choice of the gravitational Lagrangian, of a very general class of theories, in which the action is an arbitrary function of the Ricci scalar and of the matter Lagrangian. An interesting characteristic of these theories is the non-conservation of the energy–momentum tensor of the matter, indicating that matter and energy fluxes can be generated by the conversion of the geometric curvature, describing the gravitational field, into matter. Hence the presence of matter and its possible coupling to geometry could modify the cosmological evolution in a way that goes far beyond the standard description of general relativity. The presence of a source term in the energy balance equation

can be naturally interpreted in the framework of the thermodynamics of open systems as describing a particle creation process, in which the “geometric energy” of the gravitational field is transferred to “real” matter. During the particle production phase a large amount of entropy is produced.

In order to estimate the effective thermodynamics quantities we have first introduced the equivalent scalar–tensor representation of the $L_f(R, L_m)$ theory, which can be formulated in terms of a scalar field ψ , with two independent potentials $V(\psi)$ and $U(\psi)$, with the potential $U(\psi)$ coupled to the matter Lagrangian. Using the scalar–tensor representation of the $L_f(R, L_m)$ theory, we have obtained the particle creation rate, the creation pressure and the entropy associated to the gravitational energy transfer to matter. The cosmological implications of the particle creation have also been investigated, by assuming a specific relation between the two potentials. The imposed condition makes the particle creation rate a function of the second scalar potential $U(\psi)$, which directly couples to the matter Lagrangian. The gravitational field equations corresponding to these choices have a de Sitter type accelerating solution, where the cosmic acceleration is triggered by the particle creation process, which generates a negative creation pressure. Thus, it was argued that the negative creation pressure is responsible for the accelerated expansion of the Universe.

Matter creation processes are supposed to play a fundamental role in the quantum field theoretical approaches to gravity, where they naturally appear. It is a standard result of quantum field theory in curved space-times that quanta of the minimally coupled scalar field are created in the expanding Friedmann–Robertson–Walker universe [61]. That is why finding an equivalent microscopic quantum description of the matter creation processes considered in the present paper could shed some light on the physical mechanisms leading to particle generation via gravity and matter geometry coupling. In the following we will briefly point out that such mechanisms do exist, and can be understood, at least qualitatively, in the framework of some semiclassical gravity models.

In semiclassical gravity it is assumed that the gravitational field remains classical, while the classical bosonic fields ϕ are quantized. In order to couple quantized fields to classical gravitational fields the quantum energy–momentum tensor $\hat{T}_{\mu\nu}$ is replaced by its expectation value with respect to some quantum state Ψ , thus leading to the effective semiclassical Einstein equation [62],

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4} \langle \Psi | \hat{T}_{\mu\nu} | \Psi \rangle. \tag{118}$$

Hence the classical energy–momentum tensor of the system $T_{\mu\nu}$ is defined as $\langle \Psi | \hat{T}_{\mu\nu} | \Psi \rangle = T_{\mu\nu}$. The semiclassical equation Eq. (118) can be obtained from the variational principle [63]

$$\delta(S_g + S_\psi) = 0, \tag{119}$$

where $S_g = (1/16\pi G) \int R\sqrt{-g}d^4x$ is the classical action of the gravitational field, and

$$S_\psi = \int [\text{Im}\langle \dot{\Psi} | \Psi \rangle - \langle \Psi | \hat{H} | \Psi \rangle + \alpha (\langle \Psi | \Psi \rangle - 1)] dt, \tag{120}$$

where \hat{H} is the Hamiltonian operator of the system, and α is a Lagrange multiplier. The variation of Eq. (119) provides the normalization condition for the wave function $\langle \Psi | \Psi \rangle = 1$, the Schrödinger equation for the wave function

$$i|\dot{\Psi}(t)\rangle = \hat{H}(t)|\Psi(t)\rangle - \alpha(t)|\Psi(t)\rangle, \tag{121}$$

as well as the semiclassical Einstein Eq. (118). In this simple case the Bianchi identities require the conservation of the energy–momentum tensor, $\nabla_\mu \langle \Psi | \hat{T}^{\mu\nu} | \Psi \rangle = 0$.

A very different set of semiclassical Einstein equations can be obtained by assuming a coupling between the quantum fields and the curvature of the space-time. In the model introduced in [63] the contribution to the total action of the geometry–quantum matter coupling term was assumed to be of the form

$$\int RF \langle (f(\phi)) \rangle_\Psi \sqrt{-g}d^4x, \tag{122}$$

where F and f are arbitrary functions, and $\langle (f(\phi)) \rangle_\Psi = \langle \Psi(t) | f[\phi(x)] | \Psi(t) \rangle$. Then, in the presence of such a geometry–matter coupling the Hamiltonian $H(t)$ in the Schrödinger Eq. (121) is modified to [63]

$$\hat{H}(t) \rightarrow \hat{H}_\Psi = \hat{H}(t) - \int NF' \langle (f(\phi)) \rangle_\Psi f(\phi) \sqrt{\gamma}d^3\xi, \tag{123}$$

where N is the lapse function, ξ^i are intrinsic coordinates, such that the normal is everywhere time-like, and $\gamma = \det \gamma_{rs}$, where γ_{rs} is the metric induced on a surface $\sigma(t)$, which gives a global slicing of the space-time into space-like surfaces. The effective semiclassical Einstein equation takes the form [63]

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 16\pi G[\langle \hat{T}_{\mu\nu} \rangle_\Psi + G_{\mu\nu}F - \nabla_\mu \nabla_\nu F + g_{\mu\nu}\square F]. \tag{124}$$

In Eq. (124) the matter energy–momentum tensor is not conserved, $\nabla_\mu \langle \hat{T}^{\mu\nu} \rangle_\Psi \neq 0$. Thus, this equation describes an effective particle production process, and can be interpreted as giving an effective semiclassical description of the quantum processes in a gravitational field. By modifying the classical part of the gravitational action we can recover the field

equations Eqs. (2) and (16), respectively, used in the present paper. Therefore the physical origin of the matter creation processes considered in the present paper can be traced back to the semiclassical approximation of the quantum field theory in a Riemannian curved geometry.

An interesting and important question is the physical nature of the particles that could be created via gravitationally induced creation processes. The most natural assumption would be that these particles are dark matter particles. It has been conjectured that dark matter may consist of ultra-light particles with masses of the order of $m \approx 10^{-24}$ eV (see [64] and references therein). From a physical point of view such a particle may represent a pseudo Nambu–Goldstone boson. Axions are other ultra-light dark matter candidates, with masses in the range $m \leq 10^{-22}$ eV [65]. Such extremely very low mass particles can be created even in very weak gravitational fields. An alternative description of dark matter is provided by the so-called scalar field dark matter models [66], in which it is assumed that dark matter is a real scalar field, minimally coupled to gravity, with the mass of the scalar field having a very small value of the order of $m < 10^{21}$ eV. For zero temperature scalar field dark matter models all particles in the system condense to the same quantum ground state, thus forming a Bose–Einstein condensate. Therefore scalar field dark matter models are equivalent to the Bose–Einstein condensate dark matter models [67–75]. This implies, from a physical point of view, that in the open irreversible thermodynamic model introduced in the present paper particle creation can take place also in the form of a scalar field. In such a model the evolution of the scalar field dark energy particles, with energy density ρ_ϕ and pressure p_ϕ , and having a particle number density n_ϕ , is governed by an equation of the form

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) + \frac{\Gamma_1(\rho_\phi + p_\phi)\rho_\phi}{n_\phi} = 0, \quad (125)$$

where Γ_1 is the particle decay rate, determined by the coupling between matter and geometry. For the energy density and pressure of the scalar field dark matter we can assume the standard form

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + U_{\text{int}}(\phi), \quad p_\phi = \frac{\dot{\phi}^2}{2} - U_{\text{int}}(\phi), \quad (126)$$

where $U(\phi)$ is the scalar field self-interaction potential.

The creation pressure corresponding to the scalar field creation processes can be obtained as

$$p_c^{(\phi)} = \frac{\Gamma_1(\rho_\phi + p_\phi)\rho_\phi}{3Hn_\phi}. \quad (127)$$

It is interesting to note that Eq. (125), which describes the creation of a scalar field as a result of the geometry–matter

coupling, can be written in an equivalent form as

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma(\phi, \dot{\phi}, U)\dot{\phi} + U'_{\text{int}}(\phi) = 0, \quad (128)$$

where we have denoted $\Gamma(\phi, \dot{\phi}, U) = \Gamma_1\rho_\phi/n_\phi$. Therefore in the scalar field dark matter model a friction term in the scalar field evolution equation Eq. (128) does appear naturally, and in a general form, as a direct consequence of the irreversible thermodynamics of open systems as applied to the dark matter case. Hence scalar field dark matter can be a result of the cosmological particle production due to the geometry–matter coupling in modified gravity theories. For gravitational models with an action given by an arbitrary function of the Ricci scalar, the matter Lagrangian density, a scalar field and a kinetic term constructed from the gradients of the scalar field, respectively; see [76].

The $L_f(R, L_m)$ gravitational theory investigated in the present paper predicts the possibility that matter creation, associated with the curvature–matter coupling, could also occur in the present-day universe, as proposed by Dirac [77] a long time ago. The late expansion of the Universe [1, 2] may be considered as empirical evidence for matter creation, and a viable alternative to the mysterious dark energy. Presently the existence of some forms of the curvature–matter coupling leading to matter creation processes cannot be fully ruled out by the existing cosmological observations or by astrophysical data. Presumably, the functional forms of the potentials $V(\psi)$ and $U(\psi)$ that completely characterize the $L_f(R, L_m)$ gravitational theory will be provided by fundamental quantum field theoretical models of the gravitational interaction, thus opening the possibility of an in depth comparison of the predictions of the $L_f(R, L_m)$ gravity with cosmological and astrophysical observational data.

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