Mathematical Problems in Engineering, 2002, Vol. 8(4–5), pp. 413–438 Taylor & Francis
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Higher-Order Techniques for Some Problems of Nonlinear Control

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(Received 7 February 2002; In final form 18 February 2002)

A natural first step when dealing with a nonlinear problem is an application of some version of *linearization principle*. This includes the well known linearization principles for controllability, observability and stability and also first-order optimality conditions such as Lagrange multipliers rule or Pontryagin's maximum principle. In many interesting and important problems of nonlinear control the linearization principle fails to provide a solution. In the present paper we provide some examples of how higher-order methods of differential geometric control theory can be used for the study nonlinear control systems in such cases. The presentation includes: nonlinear systems with impulsive and distribution-like inputs; second-order optimality conditions for bang–bang extremals of optimal control problems; methods of high-order averaging for studying stability and stabilization of time-variant control systems.

Key words: Nonlinear control; Optimal control; Generalized inputs; Stability and stabilization; Averaging; Geometric control

1 INTRODUCTION

This paper contains a survey of some results concerning nonlinear control systems. A natural first step when dealing with a nonlinear problem is an application of some version of *linearization principle*. In a broad sense this includes the well known linearization principles for controllability, observability and stability and also first-order optimality conditions such as Lagrange multipliers rule or Pontryagin's maximum principle which deal with the first variation (linearization) of the problem.

Obviously for many interesting and important problems of nonlinear control the linearization principle fails to provide a solution. These are the cases of systems with noncontrollable or nonobservable linearizations, critical equilibria, singular extremals in optimal control problems. In this presentation we predominantly deal with this kind of problems.

Over the last three decades methods of differential geometric control theory proved to be effective in nonlinear control especially in the critical cases. In our presentation we will show how these methods can be applied to the study of nonlinear control systems, in particular to the study of:

- nonlinear systems with impulsive and distribution-like inputs;
- necessary/sufficient high-order optimality conditions for bang–bang extremals in optimal control problems;

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ISSN 1024-123X print; ISSN 1563-5147 online

– high-order averaging and study of stability and stabilization for time-variant systems.

We keep this introduction short providing instead introductory subsections for each section.

This paper has been submitted a few weeks before two birthdays of two people who taught me most of what I managed to learn in mathematics. Revaz Valerianovich Gamkrelidze will be 75 and Andrey Alexandrovich Agrachev will be 50. Happy birthday to you both!

2 CONTROL-AFFINE SYSTEMS WITH IMPULSIVE AND DISTRIBUTION-LIKE INPUTS

2.1 Introduction

In this section we will work with control affine nonlinear systems of the form:

$$\dot{x}(\tau) = f_\tau(x(\tau)) + G_\tau(x(\tau))u(\tau), \quad x \in R^n, \ u \in R^\tau, \ G_\tau(x) \in R^{n\times r} \tag{2.1}$$

where $G_\tau(x) = (g_\tau^1(x), \dots, g_\tau^r(x))$ and $f_\tau(x), g_\tau^i(x)$ $(i = 1, \dots, \tau)$ are time-variant vector fields in R^n. We develop a formalism for dealing with distribution-like inputs for the system (2.1).

There are various difficulties arising when one tries to define trajectory corresponding to a distribution-like input u. For example if the input u of a system $\dot{x}(\tau) = g_\tau(x(\tau))u(\tau)$, $x(0) = x_0$, is a Dirac measure $\delta(\tau - \tau_0)$, then it is natural to expect that the corresponding trajectory $x(\cdot)$ will 'jump' at τ_0. Transforming the differential equation into integral one $x(t) = x_0 + \int_0^t g_\tau(x(\tau))u(\tau)\,d\tau$ we encounter a necessity to integrate an apparently discontinuous function $g_\tau(x(\tau))$ with respect to a measure $\delta(\tau - \tau_0)$, which contains an atom exactly at the point of discontinuity. Such an integration is not defined properly.

Here we describe an approach to a construction of generalized trajectories for the system (2.1). The idea (which is close to the one represented in [1, 2]) amounts to furnishing the space of 'ordinary', say, integrable, inputs $u(\cdot)$ (say $\mathcal{U} = L_1^r[0, T]$) and of trajectories $x(\cdot)$ with weak topologies for which the input–trajectory map $u(\cdot) \mapsto x(\cdot)$ is still (uniformly) continuous. In this case one can extend this map by continuity onto a completion of the space of inputs, which may contain distributions.

The core issue of this approach is proving the continuity of the input/trajectory map. It is convenient to introduce topology in the space of inputs as an induced one by a topology in the space of their primitives. Note that for the integrable inputs their primitives and also the corresponding trajectories belong to $W_{1,1}[0, T]$ – the space of absolutely continuous functions.

Let us survey briefly the existing results. In the early 70's Krasnosel'sky and Pokrovsky [1] considered C^0-metric in the space $W_{1,1}[0, T]$ of the primitives and of the trajectories, and established continuity (called by them vibrocorrectness) of the input/trajectory map. They proved the extensibility of the input–trajectory map onto the space of continuous measures – generalized derivatives of continuous (but not absolutely continuous) functions. Orlov [2] used similar method to prove extensibility of the input–trajectory map to the space of Radon measures (the generalized derivatives of the functions of bounded variation). Our method [3–5] allows not only to extend the input–trajectory map onto a larger space $W_{-1,\infty}$ of generalized derivatives of measurable essentially bounded functions, but also to obtain a representation of the generalized trajectories via the generalized primitives of the inputs. About the same time Bressan proved [6] extensibility of the input–trajectory map on the same space.

The key tool of our approach is a class of representation formulae for the trajectories of the system (2.1). These formulae are multiplicative analogies of the classical integration by parts formula. They allow to represent the (generalized) trajectories via solutions of ODE, involving the (generalized) primitives of the (generalized) inputs.

There is another approach to the construction of generalized trajectories corresponding to the distribution-like inputs – the one based on completion and reparametrization of graphs of discontinuous functions. It allows to deal with the systems for which the 'commutativity assumption' fails, but also the continuity of the input/trajectory map is not maintained. We refer to the publications of Miller [7], Bressan, Rampazzo [8] and to the bibliography therein for the detailed description of this approach.

2.2 Multiplicative Analogy of Integration by Parts Formula

First consider control-linear (without a drift term) system

$$\dot{x}(\tau) = Y_\tau(x(\tau))u(\tau), \quad \tau \in [0, T], \ x \in R^n, \ u \in R. \tag{2.2}$$

For a moment assume the control $u \in R$ to be scalar-valued, $u(\cdot) \in L_1[0, T]$. Let the right-hand side Y_τ be differentiable with respect to x and C^1 with respect to τ. If for a given $u(\cdot)$ the solution (the flow) generated by Eq. (2.2) exists for $\tau \in [0, T]$, we will denote it (following [9]) by $P_t = \overrightarrow{\exp} \int_0^t Y_\tau u(\tau)\, d\tau$, $t \in [0, T]$ and call it right chronological exponential. The following proposition provides an expression for P_t in terms of the primitive $v(\cdot) = \int_0^{\cdot} u(\xi)\, d\xi$ of $u(\cdot)$.

We make an agreement concerning the notation. If a composition of diffeomorphisms $P \circ Q$ is applied to a point x^0 this means that first P and then Q is applied. In general a result of application of a diffeomorphism P to a point x^0 will be denoted by $x^0 \circ P$.

PROPOSITION 2.1 *If the solution of Eq.* (2.2) *and the diffeomorphisms* $e^{Y_t v(t)}$ *exist for all* $t \in [0, T]$ *then the following equality holds*:

$$P_t = \overrightarrow{\exp} \int_0^t Y_\tau u(\tau)\, d\tau = \overrightarrow{\exp} \int_0^t \left(- \int_0^1 (e^{-\xi Y_\tau v(\tau)})_* \dot{Y}_\tau v(\tau)\, d\xi \right) d\tau \circ e^{Y_t v(t)}. \tag{2.3}$$

Remark The diffeomorfism $e^{Y_t v(t)} = \overrightarrow{\exp} \int_0^1 Y_t v(t)\, d\xi$ in the formula (2.3) is generated by the time-invariant vector field $Y_t v(t)$ with t fixed. The notation $(e^{-\xi Y_\tau v(\tau)})_* \dot{Y}_\tau v(\tau)$ stays for the pullback of the vector field $\dot{Y}_\tau v(\tau)$ by the differential of the diffeomorfism $e^{-\xi Y_\tau v(\tau)}$ with τ fixed.

We relate (2.3) to the integration by parts formula due to the following reason. If for all $\tau \in [0, t]$ the vector fields Y_τ and $\dot{Y}_\tau$ commute, then $e^{\xi \mathrm{ad} Y_\tau v(\tau)} \dot{Y}_\tau = \dot{Y}_\tau, \forall \tau \in [0, t]$, and the formula (2.3) takes form

$$\overrightarrow{\exp} \int_0^t Y_\tau u(\tau)\, d\tau = \overrightarrow{\exp} \int_0^t (-\dot{Y}_\tau v(\tau))\, d\tau \circ e^{Y_t v(t)}, \tag{2.4}$$

becoming a multiplicative analogy of the integration by parts formula

$$\int_0^t Y_\tau u(\tau)\, d\tau = \int_0^t Y_\tau\, dv(\tau) = - \int_0^t \dot{Y}_\tau v(\tau)\, d\tau + Y_t v(t).$$

The result of the Proposition 2.1 can be reformulated for the multiinput system with $Y_\tau(\tau) = \sum_{i=1}^r Y_\tau^i u_i(\tau)\, d\tau$ under one crucial additional assumption.

COMMUTATIVITY ASSUMPTION *The vector fields $Y^1_\tau, \ldots, Y^r_\tau$ are pairwise commuting for each $\tau : [Y^i_\tau, Y^j_\tau] = 0, \forall i,j = 1, \ldots, r; \forall \tau \in [0, T]$.*

Krasnoselsky and Pokrovsky proved [1], that this condition is necessary for vibro-correctness, or in other words for the extensibility by continuity of the input/trajectory map.

This condition is equivalent to the Frobenius integrability condition for the differential (Pfaffian) systems span $\{Y^i_\tau: i = 1, \ldots, r\}$ with arbitrary fixed τ. These systems are called 'distributions' in differential geometry and global analysis; we keep the name 'distribution' for the generalized inputs.

PROPOSITION 2.2 *If the commutativity assumption is verified then the formula* (2.3) *holds for the flow $\overrightarrow{\exp} \int_0^t Y_\tau u(\tau)\,d\tau$ generated by the multiinput system $\dot{x} = Y_\tau(x)u(\tau)$.*

In [4] more versions of the integration by parts formula can be found, The following result provides representation formula for the flow generated by control-affine nonlinear systems.

THEOREM 2.3 *Let vector fields f_τ, $g^i_\tau (i = 1, \ldots, r)$ be differentiable with respect to x, continuous with respect to τ and g^i_τ be C^1 with respect to τ. Let the vector fields $g^i_\tau (i = 1, \ldots, r)$ satisfy the commutativity assumption for all $\tau \in [0, t]$. If for the input $u(\cdot) \in L^r_1[0, t]$ the solution $\overrightarrow{\exp} \int_0^t (f_\tau + G_\tau u(\tau))\,d\tau$ of Eq.* (2.1) *and the diffeomorfisms $e^{G_t v(t)}$ exist for all $t \in [0, T]$, then*

$$\overrightarrow{\exp} \int_0^t (f_\tau + G_\tau u(\tau))\,d\tau = \overrightarrow{\exp} \int_0^t \left((e^{-G_\tau v(\tau)})_* f_\tau - \int_0^1 (e^{-\xi G_\tau v(\tau)})_* \dot{G}_\tau v(\tau)\,d\xi \right) d\tau \circ e^{G_t v(t)}, \tag{2.5}$$

where $v(\cdot) = \int_0^{\cdot} u(\eta)\,d\eta$.

Up to the end of this section we assume the commutativity assumption to hold for the vector fields $g^i_\tau (i = 1, \ldots, r)$.

2.3 Continuity of the Input–Trajectory Map. Generalized Inputs and Trajectories

As long as we have obtained the formulae for trajectories in terms of the primitives of the inputs, the extensibility of the input/trajectory map follows rather easily from standard results on continuous dependence of solutions of ODE on the right-hand side.

Let us fix the initial point x_0 of our trajectories (we do it for the sake of simplicity of presentation; in [3] it is done for flows). Consider an input $u(\cdot) \in L^r_1[0, T]$, its primitive $v(\cdot) = \int_0^{\cdot} u(\xi)\,d\xi$ and the vector field

$$F_\tau(v) = (e^{-G_\tau v(\tau)})_* f_\tau - \int_0^1 (e^{-\xi G_\tau v(\tau)})_* \dot{G}_\tau v(\tau)\,d\xi. \tag{2.6}$$

According to (2.5), the trajectory corresponding to the input $u(\cdot)$ can be represented as

$$P_t(u(\cdot)) = Q_t(v(\cdot)) = x_0 \circ \overrightarrow{\exp} \int_0^t F_\tau(v(\tau))\,d\tau \circ e^{G_t v(t)}.$$

Consider the triple of maps $u(\cdot) \overset{J}{\mapsto} v(\cdot) \mapsto Q_t(v(\cdot))$, where $u(\cdot) \in L_1^r[0, T]$, $v(\cdot) \in W_{1,1}^r[0, T]$. Introduce L_1^τ-norm in the space $W_{1,1}^r[0, T]$ of $v(\cdot)$'s. The induced norm in the space of inputs will be denoted by DL_1:

$$\|u(\cdot)\|_{DL_1} = \left\| \int_0^{\cdot} u(\eta)\, d\eta \right\|_{L_1}.$$

As long as $W_{1,1}^r[0, T]$ is dense subspace of $L_1^r[0, T]$, then the completion of the space $L_1^r[0, T]$ of inputs with respect to the DL_1-norm coincides with the space of distributions, which are generalized derivatives of the functions from $L_1^r[0, T]$. With some abuse of notation we denote this space of distributions by $W_{-1,1}^r[0, T]$. (Recall that the smaller space of generalized derivatives of the square-integrable functions is Sobolev space denoted by $W_{-1,2}^r[0, T]$ or $H_{-1}^r[0, T]$).

We will need another space of generalized inputs – the one adjoint to $W_{1,1}^r[0, T]$. Recall that any linear continuous functional on $W_{1,1}^r[0, T]$ can be defined by the formula:

$$\forall z(\cdot) \in W_{1,1}^r[0, T]: \quad z(\cdot) \mapsto v_0 z(0) - \int_0^T v(\tau)\dot{z}(\tau)\, d\tau, \quad v_0 \in R^r, \ v(\cdot) \in L_\infty^\tau.$$

This adjoint space will be denoted by $W_{-1,\infty}^r$; it can be identified with $R^r \times L_\infty$. The subspace of $W_{-1,\infty}^r$ identified with $\{(0, v(\cdot))\}$ is denoted by $\overset{\circ}{W}{}_{-1,\infty}^r$; the function $v(\cdot) \in L_\infty$ will be called generalized primitive of the corresponding element from $\overset{\circ}{W}{}_{-1,\infty}^r$.

What for the space of trajectories (which are absolutely continuous) then we furnish it with the L_1-norm.

Let us take any $\alpha > 0$ and consider the set $\mathcal{U}_\alpha$ of the inputs from $L_1^r[0, T]$ whose primitives are uniformly bounded by α on $[0, T]$. The DL_1-completion of $\mathcal{U}_\alpha$ coincides with the α-ball in the space $\overset{\circ}{W}{}_{-1,\infty}^r$.

We proved in [3], that the input-trajectory map is uniformly continuous on $\mathcal{U}_\alpha$, furnished with DL_1-norm, if the space of trajectories is furnished with the L_1 norm. This means that we can (as long as $\alpha > 0$ is arbitrary) extend the input–trajectory map onto the set of generalized inputs $\overset{\circ}{W}{}_{-1,\infty}^r$. The corresponding generalized trajectories will be the functions from $L_1^n[0, T]$. They can be computed via the generalized primitives of the inputs by means of Eq. (2.5). This gives us the following result.

THEOREM 2.4 *Consider control-affine nonlinear system* (2.1)

$$\dot{x}(\tau) = f_\tau(x(\tau)) + \sum_{i=1}^r g_\tau^i(x(\tau))u(\tau), \quad q \in R^n, \ u \in R^r.$$

Let $f_\tau, g_\tau^i (i = 1, \ldots, r)$ be time-variant vector fields which are infinitely differentiable with respect to x, continuous with respect to τ. Let g_τ^i $(i = 1, \ldots, r)$ be C^1 with respect to τ and satisfy the commutativity assumption. Then for each generalized input from $\overset{\circ}{W}{}_{-1,\infty}^r$ with the generalized primitive $v(\cdot)$ the formula (2.5) *defines the DL_1-continuous extension of the input/trajectory of this system. The extension coincides with the classical input/trajectory map on the space of ordinary inputs and is continuous with respect to DL_1-norm of the space $\overset{\circ}{W}{}_{-1,\infty}^r$ and L_1-norm in the space of trajectories.*

2.4 Example: Impulsive Controls

To illustrate the previous result let us compute the trajectory of the control-affine system (2.1)

$$\dot{x}(\tau) = f_\tau(x(\tau)) + G_\tau(x(\tau))u(\tau)$$

driven by the impulsive control $u = \sum_{i=1}^{N} u_i \delta(\tau - \tau_i)$ – a linear combination of Dirac measures located on the time-axis. In principle N can be finite or infinite; in the latter case we assume the series $\sum_{i=1}^{\infty} u_i$ to be convergent.

Let N be finite and $0 = \tau_0 < \tau_1 < \cdots < \tau_N \leq T$. The primitive of u is $v(\tau) = \sum_{i=1}^{N} u_i h(\tau - \tau_i)$, $v(0) = 0$, with $h(\tau)$ being Heavyside function: $h(\tau) = 0$, for $\tau < 0$, $h(\tau) = 1$, for $\tau \geq 0$. The function $v(\tau)$ is piecewise constant and equals $v_m = \sum_{i=1}^{m} u_i$, on the interval $[\tau_m, \tau_{m+1})$, while $v(0) = 0$. The expression (2.5) can be splitted into the product

$$Q_t = \prod_{i=1}^{N} \overrightarrow{\exp} \int_{\tau_{i-1}}^{\tau_i} \left((e^{-G_\tau v_{i-1}})_* f_\tau - \int_0^1 (e^{-\xi G_\tau v_{i-1}})_* \dot{G}_\tau v_{i-1} \, d\xi \right) d\tau \circ \overrightarrow{\exp} \int_{\tau_m}^{t} \left((e^{-G_\tau v_m})_* f_\tau \right.$$
$$\left. - \int_0^1 (e^{-\xi G_\tau v_m})_* \dot{G}_\tau v_m \, d\xi \right) d\tau \circ e^{G_t v(t)}, \qquad \text{for } \tau_m \leq t < \tau_{m+1}. \tag{2.7}$$

The following equality is established in [4].

$$\overrightarrow{\exp} \int_\eta^\zeta \left((e^{-G_\tau v})_* f_\tau - \int_0^1 (e^{-\xi G_\tau v})_* \dot{G}_\tau v \, d\xi \right) d\tau = e^{G_\eta v} \circ \overrightarrow{\exp} \int_\eta^\zeta f_\xi \, d\xi \circ e^{-G_\zeta v}.$$

Applying it to the product (2.7) we obtain

$$Q_t = \left(\prod_{i=1}^{N} \left(\overrightarrow{\exp} \int_{\tau_{i-1}}^{\tau_i} f_\tau \, d\tau \circ e^{G_{\tau_i} u_i} \right) \right) \circ \overrightarrow{\exp} \int_{\tau_m}^{t} f_\tau \, d\tau. \tag{2.8}$$

From (2.8) one derives the following facts for the trajectories generated by the impulsive controls: (i) they are piecewise continuous functions; (ii) their continuous parts are pieces of the trajectories of the vector field f_τ; (iii) their jumps occur at the instances τ_i $(i = 1, \ldots, N)$ are along the trajectories of the time-invariant vector field $G_{\tau_i} u_i$ and correspond to the time-duration 1.

If N is infinite and $u = \sum_{i=1}^{\infty} u_i \delta(\tau - \tau_i)$, with $\tau_i < \tau_{i+1}$ $(i = 0, 1, \ldots)$, $\lim_{i \to \infty} \tau_i = \bar{\tau} \leq T$, then for $t < \bar{\tau}$, we proceed as in the previous example (only finite number of impulses occur before t). If $t \geq \bar{\tau}$, then Q_t is defined by (2.8) with $N = \infty$.

We proved in [3], that if $\sum_{i=1}^{\infty} u_i < \infty$ then this infinite product can be computed as a limit of partial finite products $\prod_{i=1}^{m}$, $m \to \infty$.

2.5 Time-Optimality of Generalized Controls

In this subsection we will use the representation of the generalized trajectories for studying optimal control problems with generalized controls, We will formulate first-order optimality condition for these problems in the Hamiltonian form. An alternative approach and many results regarding optimality of generalized controls can be found in the book of Orlov [10].

Let us start with the definition of attainability for generalized controls. As long as the generalized trajectories are measurable functions, their values at a given instant t are not properly defined, and we define the attainability in approximative sense.

DEFINITION 2.5 *Given a system* (2.1), *a point* $\bar{x}$ *is attainable from the point* x_0 *on the interval* $[0, t]$ *by means of a generalized control* $\bar{u} \in \overset{\circ}{W}{}^r_{-1,\infty}$ *if there exists a sequence of controls* $u^m(\cdot) \in L_1^r[0, T]$, *which converges to the control* $\bar{u}$ *in* DL_1*-norm, such that the points* $x^m(t)$ *of the corresponding trajectories* $x^m(\cdot)$, *(starting at* $x(0) = x_0$*) converge to* $\bar{x}$.

According to (2.5) the set $\mathcal{A}_{x_0}(u; [0, t])$ of points attainable from x_0 on the interval $[0, t]$ by means of the control $u \in \overset{\circ}{W}{}^r_{-1,\infty}$ is contained in the integral manifold $\mathcal{O}_{\tilde{x}}(\hat{G}_t)$ of the integrable (by virtue of the commutativity assumption) differential system $\hat{G}_t = \text{span}\{g_t^1, \ldots, g_t^r\}$ (with t fixed). This manifold passes through the point

$$\tilde{x} = x_0 \circ \overrightarrow{\exp} \int_0^t \left((e^{-G_\tau v(\tau)})_* f_\tau - \int_0^1 (e^{-\xi G_\tau v(\tau)})_* \dot{G}_\tau v(\tau)\, d\xi \right) d\tau.$$

Here again $v(\cdot)$ is the generalized primitive of the control u.

In [11] we proved that the attainable set $\mathcal{A}_{x_0}(u; [0, t])$ coincides with this integral manifold. Therefore there is an r-dimensional manifold of points attainable from given x_0 on a given time interval $[0, t]$ by means of a given (1) generalized control $u \in \overset{\circ}{W}{}^r_{-1,\infty}$.

If $\mathcal{U}$ is a set of generalized controls, then the set of points attainable from x_0 on the time interval $[0, t]$ by means of some control from $\mathcal{U}$ will be denoted by $\mathcal{A}_{x_0}(\mathcal{U}; [0, t])$.

Let us consider time-optimal problem for the system (2.1) with generalized controls $u \in \overset{\circ}{W}{}^r_{-1,\infty}$:

$$t \to \min, \tag{2.9}$$

$$\dot{x}(\tau) = f_\tau(x(\tau)) + G_\tau(x(\tau))u, \quad x(0) = x_0, \quad x \in R^n, \quad u \in \overset{\circ}{W}{}^r_{-1,\infty}, \tag{2.10}$$

$$x_1 \in \mathcal{A}_{x_0}(u, [0, t]). \tag{2.11}$$

Note that the condition (2.11) corresponds to the fixed end-point condition in the classical problem of time optimality.

DEFINITION 2.6 *The generalized control* $\tilde{u} \in \overset{\circ}{W}{}^r_{-1,\infty}$ *is locally optimal for the problem* (2.9)–(2.11), *if for some* δ*-neighborhood* $\mathcal{U}_\delta$ *of* $\tilde{u}$ *in* DL_1*-metric*

$$\forall \tau < t \colon \mathcal{A}_{x_0}(\tilde{u}, [0, t]) \cap \mathcal{A}_{x_0}(\mathcal{U}_\delta, [0, \tau]) = \emptyset.$$

From the representation formula (2.5) it is easy to conclude that the generalized time-optimal control problem (2.9)–(2.11) can be reduced to the following classical time-optimal control problem with variable end-point condition

$$x(t) \in \mathcal{O}_{x_1}(\hat{G}_t). \tag{2.12}$$

for the system

$$\dot{x}(\tau) = (e^{-G_\tau v(\tau)})_* f_\tau - \int_0^1 (e^{-\xi G_\tau v(\tau)})_* \dot{G}_\tau v(\tau)\, d\xi)(x(\tau)), \quad x(0) = x_0, \tag{2.13}$$

with admissible controls $v(\cdot) \in L^{\tau}_{\infty}[0, T]$. Here $\mathcal{O}_{x_1}(\hat{G}_t)$ is the integral manifold of the differential system $\hat{G}_t$ passing through the point x_1.

PROPOSITION 2.7 *A pair* $(\tilde{t}, \tilde{u}) \in R_+ \times \overset{\circ}{W}{}^r_{-1,\infty}$ *is locally optimal for the problem* (2.9)–(2.11) *if and only if for* $\tilde{v}(\cdot)$ *being the generalized primitive of the control* $\tilde{u}$ *the corresponding pair* $(\tilde{t}, \tilde{v}(\cdot)) \in L^{\tau}_{\infty}[0, T]$ *is* L_1*-locally optimal for the time-optimal problem* (2.9), (2.13), (2.12).

By virtue of the Proposition 2.7 a first-order necessary optimality conditions for the problem (2.9)–(2.11) can be derived from (in fact is equivalent to) the corresponding necessary condition for the reduced problem (2.9), (2.13), (2.12).

If the end-point condition (2.12) were admitting an explicit form $\Omega_t(x(t)) = 0$, then the first-order optimality condition is well known and looks as follows.

PROPOSITION 2.8 [12] *If the control* $\tilde{v}(\cdot)$ *is a* L_1*-local minimizer for the problem* (2.9), (2.13) *with the variable end-point condition* $\Omega_t(x(t)) = 0$, *then there exists an absolutely continuous covector-function* $\tilde{\psi}(\cdot)$ *and a covector* $\tilde{\nu} \in R^{d^*}$, $(\tilde{\psi}(\cdot), \tilde{\nu}) \neq 0$, *such that the quadruple* $(\tilde{x}(\cdot), \tilde{v}(\cdot), \tilde{\psi}(\cdot), \tilde{\nu})$ *satisfies*:

(i) *(pseudo)-Hamiltonian system*

$$\dot{x} = \frac{\partial H}{\partial \psi}(x, \psi, v, \tau), \quad \dot{\psi} = -\frac{\partial H}{\partial x}(x, \psi, v, \tau), \tag{2.14}$$

with the Hamiltonian

$$H(x, \psi, v, \tau) = \left\langle \psi, (e^{-G_\tau v})_* f_\tau - \int_0^1 (e^{-\xi G_\tau v})_* \dot{G}_\tau v \mathrm{d}\xi)(x) \right\rangle, \tag{2.15}$$

(ii) *the maximality condition*

$$H(\tilde{x}(\tau), \tilde{\psi}(\tau), \tilde{v}(\tau), \tau) = M(\tilde{x}(\tau), \tilde{\psi}(\tau), \tau) = \sup_v H(\tilde{x}(\tau), \tilde{\psi}(\tau), v, \tau), a.e.; \tag{2.16}$$

(iii) *the transversality condition*

$$\tilde{\psi}(t) = \frac{\partial \langle \tilde{\nu}, \Omega_t \rangle}{\partial x} \bigwedge M(\tilde{x}(t), \tilde{\psi}(t), t) + \frac{\partial \langle \tilde{\nu}, \Omega_t \rangle}{\partial t} \geq 0. \tag{2.17}$$

In general it is not a feasible option to put the condition (2.12) into an explicit form, because it means 'integrating' the differential system $\hat{G}_t$. We choose another way which allows us to avoid such an integration. This is done by introducing an auxiliary Hamiltonian $F = \langle \lambda, G_t(x)\tilde{V} \rangle$ and corresponding Hamiltonian system with boundary conditions:

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} = \frac{\partial F}{\partial \lambda} = G_t \tilde{V}, \quad \frac{\mathrm{d}\lambda}{\mathrm{d}\theta} = -\frac{\partial F}{\partial z} = -\left\langle \lambda, \frac{\partial (G_t \tilde{V})}{\partial z} \right\rangle, \tag{2.18}$$

$$z(0) = e^{-G_t \tilde{V}}(x_1), \quad z(1) = x_1, \quad \lambda(0) = \tilde{\psi}(t). \tag{2.19}$$

Then (see [4]) the transversality conditions (2.17) for the boundary condition (2.12) can be written as

$$\langle \lambda(1), g_t^i(x_1)\rangle = 0, \quad l = 1, \ldots, r, \quad \bigwedge \int_0^1 \langle \lambda(\eta), \dot{G}_t(z(\eta))\tilde{V}\, \mathrm{d}\eta \geq M(\tilde{x}(t), \tilde{\psi}(t), t). \tag{2.20}$$

THEOREM 2.9 *If a pair* $(\tilde{t}, \tilde{u}) \in R \times \overset{\circ}{W}{}^r_{-1,\infty}$ *is local minimizer for the generalized problem* (2.9)–(2.11) *and* $\tilde{v}(\cdot) \in L^r_\infty[0, \tilde{t}]$ *is the generalized primitive of the control* $\tilde{u}$, *then there exists a quadruple of absolutely continuous functions* $(\tilde{x}(t), \tilde{\psi}(t), \tilde{z}(t), \tilde{\lambda}(t))$ *such that the triple* $(\tilde{x}(\cdot), \tilde{\psi}(\cdot), \tilde{v}(\cdot)$ *satisfies the (pseudo) Hamiltonian system* (2.14) *with the Hamiltonian* (2.15), *the initial condition* $x(0) = x_0$ *and the maximality condition* (2.16), *while the solution* $(\tilde{z}(t), \tilde{\lambda}(t))$ *of the auxiliary Hamiltonian system* (2.18) *satisfies the boundary conditions* (2.19) *and the transversality conditions* (2.20).

Remark Note that $\tilde{x}(\cdot)$ is not a generalized trajectory of the system (2.10).

2.6 Generalized Minimizers in Highly-Singular Linear-Quadratic Optimal Control Problem

We provide here a brief description of the results contained in Ph.D. thesis of Guerra (University of Aveiro, Portugal, 2001).

One considers classical linear quadratic problem of optimal control

$$\begin{aligned} J(x(\cdot), u(\cdot)) &= \int_0^T (x'Px + 2u'Qx + u'Ru)(t)\, \mathrm{d}t \to \min, \quad \dot{x} = Ax + Bu, \\ x(0) &= \bar{x}, \;\; x(T) = \tilde{x}. \end{aligned} \tag{2.21}$$

Here $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$, A, B, P, Q, R are constant matrices of suitable dimensions.

It is well known that for this problem to have finite infimum (one presumes then that $\tilde{x}$ is attainable from $\bar{x}$ for the system $\dot{x} = Ax + Bu$) the positive semidefiniteness of the matrix R is necessary. If, in particular, R is positive definite then we get a regular LQ-problem and the existence of minimizing control in L_2 is guaranteed. The extremal controls, which are candidates for minimizers, are determined by the Pontryagin maximum principle, its optimality is studied by the theory of second variation (conjugate points, Jacobi condition, Riccati differential equation, Hamilton–Jacobi equation etc.)

Much more difficult and challenging is the singular case where R is singular, *i.e.* has nontrivial kernel. Here minimizer may lack to exist in $L_2[0, T]$ due to noncoerciveness of the functional. The singular L-Q problems have been extensively studied over the last 30–40 years. Still the following questions remained unanswered for the LQ problem with an arbitrary singularity: (i)if minimizers lack to exist in $L_2[0, T]$ can the problem be transferred to a bigger space, where generalized minimizers exist? what space could it be? (ii) how to compute (describe) the generalized minimizers? (iii) do these generalized minimizers admit approximation by minimizing sequences of ordinary (square integrable) controls?

These questions have been answered in the thesis of Guerra. The problem has additional difficulties for the vector-valued u. The detailed description of the results can be found in [13–15]; here they are just listed.

- a series of necessary-generalized Goh and generalized Legendre–Clebsch conditions have been obtained; they guarantee that the functional J has finite infimum for some boundary data; order r of singularity for an LQ problem has been introduced;

- if the previous conditions are satisfied and the order of singularity equals r, then it is proved that the singular LQ-problem can be extended onto a set of generalized controls, which is a subspace of Sobolev space $H_{-r}[0, T]$; whenever the infimum of the problem is finite the problem possesses a generalized minimizer in this space of distributions;
- the generalized minimizer is a sum of an analytic function and of a distribution which is a linear combination of the Dirac measures $\delta^{(j)}(t), \delta^{(j)}(t-T)$, $j = 1, \ldots, r-1$ located at the boundary of the interval $[0, T]$; the corresponding generalized trajectory is a distribution (!) belonging to the Sobolev space $H_{-(r-1)}[0, T]$.
- following approximation result is established: the elements of minimizing sequence must converge to the minimizer in corresponding topology of the Sobolev space H_{-r}; this implies (for $r > 2$) high-gain oscillation behaviour of ordinary trajectories, which approximate the minimizing generalized trajectory;
- the notion of conjugate point is introduced and Jacobi-type optimality condition for the generalized control is established;
- the optimal generalized solution in a feedback form is constructed; the feedback can be computed via solution of a couple of Riccati type and linear-matrix differential equations.
- Hamiltonian formalism for the generalized extremals is developed and its relation to Dirac's theory of constrained Hamiltonian mechanics is established.

The most interesting aspects of this work are: (i) a complete theory of existence, uniqueness, optimality, and approximative properties for minimizers of LQ-problem suffering arbitrary singularity; (ii) appearance of distributions of order > 1 as generalized inputs and generalized trajectories; (iii) reformulation of the Dirac theory of constrained Hamiltonian mechanics for the LQ-problem.

3 SECOND VARIATION AND OPTIMALITY OF BANG–BANG CONTROLS

3.1 Introduction

In this section we deal with a nonlinear time-optimal control problem:

$$t \to \min, \tag{3.1}$$

$$\dot{q} = f(q) + G(q)u(\tau), \quad q(0) = q_0, \quad q \in M, \quad u \in U, \tag{3.2}$$

$$q(t_1) = q_1, \tag{3.3}$$

for an affine control system (3.2) with end-point condition (3.3) on a C^∞-smooth n-dimensional manifold M. Here $G(q) = (g^1(q), \ldots, g^r(q))$ and $f(q)$, $g^1(q), \ldots, g^r(q)$ are C^∞-smooth vector fields on M; admissible controls $u(\tau) = (u_1(\tau), \ldots, u_r(\tau))$ are measurable and take their values in a convex compact polyhedron $U \subset R^r$.

We set the problem of *L_1-local optimality* according to the following definition.

DEFINITION 3.1 *A pair $(\tilde{u}(\cdot), \tilde{q}(\cdot))$ meeting (3.2)–(3.3) for $t = T$ is called L_1-locally optimal, if there exist $\Delta > 0$ and a ball $\mathcal{U} \supset \tilde{u}(\cdot)$ in $L_1^r[0, T]$ such that no admissible control from $\mathcal{U}$ can steer the system (3.2) from q_0 to q_1 in time $T' \in [T - \Delta, T)$.*

We call the vector field $\tilde{f}_t = f(q) + G(q)\tilde{u}(t)$ the reference vector field and the corresponding flow $\tilde{P}_t = \overrightarrow{\exp} \int_0^t \tilde{f}_\tau \, d\tau$ the reference flow.

A first-order necessary optimality condition for the problem (3.1)–(3.3) is provided by *Pontryagin Maximum Principle* (see [12]). If a pair $(\tilde{u}(\cdot), \tilde{q}(\cdot))$ meets this principle for some covector function (Hamiltonian multiplier) $\tilde{\psi}(\cdot)$, then the triple $(\tilde{u}(\cdot), \tilde{x}(\cdot), \tilde{\psi}(\cdot))$ is called *Pontryagin extremal*, and $\tilde{u}(\cdot)$ is called *extremal control*; there can exist different Pontryagin extremals with different $\tilde{\psi}(\cdot)$ corresponding to the same extremal control $\tilde{u}(\cdot)$.

In what follows we assume that the extremal control $\tilde{u}(\cdot)$ is piecewise C^1-smooth function of τ. Then due to the Maximum Principle the domain $[0, T]$ of $\tilde{u}(\cdot)$ can be subdivided into subintervals $0 = \tau_0 < \tau_1 < \cdots < \tau_m < \tau_{m+1} = T$ in such way that for each $\tau \in (\tau_i, \tau_{i+1})$ the maximality condition of the Maximum Principle is fullfilled on a k_i-dimensional ($k_i \geq 0$) face W_i of the polyhedron U. The subinterval (τ_i, τ_{i+1}) is called *bang–bang*, if $k_i = 0$, *i.e.* the maximum is achieved at a vertex of U, and is called *singular*, if k_i is positive. The points τ_i $(i = 1, \ldots, m)$ are called *switching points*.

As it is well known, the extremality of control $\tilde{u}(\cdot)$ does not imply its optimality. To ascertain optimality one should at least investigate the *second variation* of the control system (3.2) on singular subintervals. But even *bang–bang Pontryagin extremals*, which have no singular arcs, may happen to be nonoptimal. 'Often' a bang–bang extremal ceases to be optimal after certain number of switchings. Corresponding examples, as well as some high-order necessary optimality conditions for bang–bang extremals, can be found in [16–19].

One should note a characteristic feature of bang–bang Pontryagin extremals: the first variation of the system (3.2) along these extremals do not vanish. Due to this fact the traditional approach of the Calculus of Variations and Optimal Control Theory is no longer applicable in this case. Indeed the first-order conditions do not guarantee optimality, while the high-order variations, which are to be defined on the kernel of the first variation, simply do not exist, because this kernel is trivial. To deal with this situation ant to derive high-order necessary optimality conditions Agrachev and Gamkrelidze constructed in [20] first and second variations for bang–bang extremals based on the variation of the instants of switching.

Our approach is close to the one of [20]. We introduce below an extension of the first variation by adding to the space of admissible variations of the extremal control $\tilde{u}(\cdot)$ some finite-dimensional space of Dirac measures, located at the switching points of the extremal.

This extension results in new addends for the first and the second variations, which are called *first* and *second variations of the system at switching points of extremal.*

Studying the first variation at switching points we derive a *first-order sufficient condition of L_1-local optimality for bang–bang Pontryagin extremals* (Theorem 3.9). When this condition is not met for a bang–bang extremal, we bring into consideration a corresponding *second variation at switching points*. It is finite-dimensional quadratic form and its *negative definiteness* is the crucial point for setting *second-order sufficient conditions of optimality for bang–bang Pontryagin extremals* (Theorems 3.10 and 3.11). Most of these results has been published in the preprint [21]; part of the formulations have been presented in [22–24].

3.2 First and Second Variations of Control System. Pontryagin Maximum Principle

In this subsection we define first and second variations of the control system (3.2). We also formulate the *Pontryagin Maximum Principle*, which provides *first-order necessary optimality condition* for $\tilde{u}(\cdot)$.

Consider the linear operator

$$\Phi_T^1 u(\cdot) = \int_0^T X_\tau(q_0) u(\tau)\, d\tau, \tag{3.4}$$

defined on $L^r_\infty[0, T]$; where

$$X_\tau = (X^1_\tau, \ldots, X^r_\tau), \qquad X^i_\tau = (P^{-1}_\tau) * g^i \quad (i = 1, \ldots, r). \tag{3.5}$$

are the pull-backs of the vector fields g_i by the reference flow $\tilde{P}_\tau$ to the point q_0. Obviously Φ^1_T is linear operator from $L^r_\infty[0, T]$ to the tangent space $\mathcal{T}_{q_0}M$.

Let $\tilde{u}(\cdot) \in \mathcal{U}$, where $\mathcal{U}$ is the set of admissible controls of the system (3.2). A *cone of admissible variations* of $\tilde{u}(\cdot)$ is by definition the (convex) conic hull of the set $\mathcal{U} - \tilde{u}(\cdot)$. We denote this cone by $K_{\tilde{u}}\mathcal{U}$.

DEFINITION 3.1 *The restriction of* Φ^1_T *to the cone* $K_{\tilde{u}}\mathcal{U}$ *is called the first variation of the system* (3.2) *on* $[0, T]$ *along the control* $\tilde{u}(\cdot)$. *The image* $\Phi^1_T(K_{\tilde{u}}\mathcal{U})$ *is called the first variational cone along* $\tilde{u}(\cdot)$.

The pull-forward of the first variational cone by $\tilde{P}_T$ is a subset of the set of tangent vectors at $\tilde{x}(T)$ to the (time-T)-attainable set of the system.

DEFINITION 3.2 *A control* $\tilde{u}(\cdot)$ *is called an extremal control for the problem* (3.1)–(3.3) *if the vector* $Y_T = (\tilde{P}^{-1}_T) * \tilde{f}_T$ *can be separated from the first variational cone along* $\tilde{u}(\cdot)$.

The Pontryagin Maximum Principle claims (see [12]) that separability of the vector $\tilde{f}_T$ from this subset is *necessary* for optimality of $\tilde{u}(\cdot)$, or equivalently any optimal control for the problem (3.1)–(3.3) must be an extremal one.

By the definition for an extremal control $\tilde{u}(\cdot)$ there exists a covector $\zeta_0 \in \mathcal{T}^*_{q_0}M$ such that $\forall u(\cdot) \in K_{\tilde{u}}\mathcal{U}$ there holds:

$$\int_0^T \langle \zeta_0, X_\tau(q_0)u(\tau)\rangle \, d\tau \leq 0 \bigwedge \langle \zeta_0, Y_T\rangle \geq 0. \tag{3.6}$$

The last inequality is so called *Transversality Condition*. We call its strengthened form $\langle \zeta_0, Y_T\rangle > 0$ the Strong Transversality Condition.

The inequality (3.6) implies the inequalities

$$\langle \zeta_0, q_0 \circ X_\tau(u - \tilde{u}(\tau))\rangle \leq 0, \quad \forall u \in U, \quad \text{a.e. on } [0, T], \quad \Leftrightarrow \tilde{u}(\tau) \in \arg\max_{u \in U} \langle \zeta_0, q_0 \circ X_\tau u\rangle. \tag{3.7}$$

The conditions (3.6) and (3.7) can be transformed into a standard form of Maximum Principle.

THEOREM 3.3 (Pontryagin Maximum Principle; [12]) *If a triple* $(\tilde{q}(\cdot), \tilde{u}(\cdot), T)$ *is a solution of the optimal control problem* (3.1)–(3.3), *then there exists a non-zero absolutely continuous covector-function* $\tilde{\psi}: R \to \mathcal{T} * M$ *(with* $\tilde{\psi}(l) \in \mathcal{T}_{\tilde{q}(t)}M$*) such that* $(\tilde{q}(\cdot), \tilde{\psi}(\cdot))$ *satisfy a (pseudo)Hamiltonian system with the Hamiltonian*

$$H(q, \psi, u) = \langle \psi, f(q) + G(q)u\rangle, \tag{3.8}$$

and for $(\tilde{q}(\cdot), \tilde{u}(\cdot), \tilde{\psi}(\cdot), T)$ *the following conditions hold*:

(i) *Maximality Condition*:

$$H(\tilde{q}(t), \tilde{\psi}(t), \tilde{u}(t)) = \max\{H(\tilde{q}(t), \tilde{\psi}(t), u) : u \in U\} \quad a.e.\ on\ [0, T]; \tag{3.9}$$

(ii) *Transversality Condition* $H(\tilde{q}(T), \tilde{\psi}(T), \tilde{u}(T)) \geq 0$.

We call *Strong Transversality Condition* the inequality

$$H(\tilde{q}(T), \tilde{\psi}(T), \tilde{u}(T)) > 0. \tag{3.10}$$

We denote $\tilde{\psi}(t)G(\tilde{q}(t))$ by χ_t and call it *switching function*. The maximality condition (3.9) is equivalent to $\langle \chi_t, \tilde{u}(t)\rangle = \max\{\langle \chi_t, u\rangle | u \in U\}$.

Obviously for any $\tau \in [0, T]$ the latter maximum is attained at some (perhaps 0-dimensional) face W_r of the polyhedron U; we will denote by V_τ a directing subspace of this face; $V_\tau = \text{span}\{W_\tau - w\}$, where $w \in W$ can be chosen arbitrarily. Evidently $\forall_v \in V_\tau$, $\tau \in [0, T] : \langle \chi_\tau, v\rangle = 0$.

In what follows we assume the mapping $\tau \to V_\tau$ to be piecewise-constant with $0 = \tau_0 < \tau_1 < \cdots < \tau_m < \tau_{m+1} = T$ determining the intervals of constancy. We call $(\tau_i, \tau_{i+1}]$ a *bang–bang interval of the extremal control* $\tilde{u}(\cdot)$ if $\dim V_\tau = 0$, or equivalently W_τ is a vertex of U for $\tau \in (\tau_i, \tau_{i+1})$ and a *singular interval* if $\dim V_\tau > 0$ on (τ_i, τ_{i+1}). The points τ_i are called *switching points*; if τ_i separates two bang–bang intervals, then we shall call it a *bang–bang switching point*. It is obvious that $W_{\tau_i \pm 0} \subseteq W_{\tau_i}$ for any τ_i and $\dim W_{\tau_i} \geq 1$. The face W_{τ_i} is called *face of switching at* τ_i.

Below we put $X_i, W_i, V_i, X_{i\pm 0}, W_{i\pm 0}, V_{i\pm 0}$ instead of $X_{\tau_i}, W_{\tau_i}, V_{\tau_i}, X_{\tau_i \pm 0}, W_{\tau_i \pm 0}, V_{\tau_i \pm 0}$.

Let us consider two faces $W \subseteq W'$ of the polyhedron $U \subset R^r$. For all points $w \in \text{relint}\, W$ the conic hulls of the sets $W' - w$ coincide. We shall call any of them the *tangent cone* to W' at W and denote this cone by $\mathcal{K}_W W'$. These tangent cones are closed.

Any of these cones is naturally imbedded into the directing linear space V' of the face W'. The cone $\mathcal{K}_W W'$ is pointed if and only if W is a vertex of U; otherwise the directing space V of the face W is the maximal linear subspace of $\mathcal{K}_W W'$: $V = \mathcal{K}_W W' \cap (-\mathcal{K}_W W')$.

The following two *Genericity Assumptions* will be involved in the optimality conditions.

STRONG GENERICITY ASSUMPTION FOR BANG–BANG SWITCHINGS *For every bang–bang switching point* τ_i *the maximum* $\max\{H(\tilde{q}(t), \tilde{\psi}t, u) : u \in U\}$ *is achieved on an edge (*$=1$*-dimensional face) of U; the left and right derivatives* $\dot{\chi}_{\tau_j \pm 0}$ *of the switching function* χ_τ *are not orthogonal to this edge of switching.*

The nonorthogonality of $\dot{\chi}_{\tau_i \pm 0}$ to the edge of switching can be better visualized in the case of scalar u's, when the segment U (for example $U = [-1, 1]$) is edge itself. By virtue of the Maximum Principle the sign of extremal control $\tilde{u}(\tau)$ must coincide with the sign of the (scalar valued) function χ_τ and the 'non-orthogonality' means $\dot{\chi}_{\tau_i \pm 0} \neq 0$, that implies the transversality of the graph $\tau \mapsto \chi_\tau$ to the τ-axis at the point $(\tau_i, 0)$.

WEAK GENERICITY ASSUMPTION FOR BANG–BANG SWITCHINGS *For every switching point* τ_i *the maximum* $\max\{H(\tilde{q}(t), \tilde{\psi}(t), u) : u \in U\}$ *is achieved on a face* W_i *of U. For the cones* $K_i^- = \mathcal{K}_{W_{i-0}} W_i$, $K_i^+ = \mathcal{K}_{W_{i+0}} W_i$ *and the switching function* χ_τ *we have*

$$\dot{\chi}_{\tau_i - 0}\xi \neq 0, \quad \forall \xi \in K_i^-; \qquad \dot{\chi}_{\tau_i + 0}\xi \neq 0, \quad \forall \xi \in K_i^+. \tag{3.11}$$

Actually the inequalities (3.11) together with the maximality condition (3.9) imply: $\dot{\chi}_{\tau_i-0}\xi > 0, \forall\xi \in K_i^-$; $\dot{\chi}_{\tau_i+0}\xi < 0, \forall\xi \in K_i^+$. For a bang–bang switching point τ_i the cones K_i^-, K_i^+ are the tangent cones to the face W_j at the vertices $\tilde{u}(\tau_i - 0)$ and $\tilde{u}(\tau_i + 0)$ correspondingly. It is clear that for bang–bang switchings the Strong Genericity Assumption is a particular case of the Weak one.

In what follows we assume the Weak Genericity Assumption and the Strong Transversality Condition to hold for any extremal under consideration.

Now we introduce *second variation of the system* (3.2) along an extremal $(\tilde{u}(\cdot), \tilde{x}(\cdot), \tilde{\psi}(\cdot))$ on $[0, T]$. It is defined on the kernel of the first variation Φ_T^1:

$$\ker \Phi_T^1 = \left\{ u(\cdot) \in K_{\tilde{u}}\mathcal{U}\colon \int_0^T X_\tau(q_0)u(\tau)\,\mathrm{d}\tau = 0 \right\}. \tag{3.12}$$

This equality $\int_0^T X_\tau(q_0)u(\tau)\,\mathrm{d}\tau = 0$ together with (3.6) imply an inclusion $u(\tau) \in V_\tau$ a.e. on $[0, T]$; in particular, $u(\tau)$ must vanish on the bang–bang intervals of $\tilde{u}(\cdot)$. If $\tilde{u}(\cdot)$ is a bang–bang extremal control, then $\ker\Phi_T^1$ is trivial linear subspace.

DEFINITION 3.4 *The second variation of the system* (3.2) *along the extremal* $(\tilde{u}(\cdot), \tilde{x}(\cdot), \tilde{\psi}(\cdot))$ *on* $[0, T]$ *is the quadratic form*:

$$\zeta_0\Phi_T''(u(\cdot)) = \int_0^T \left\langle \zeta_0, \left[\int_0^\tau X_\theta u(\theta)\,\mathrm{d}\theta, X_\tau u(\tau)\right](q_0)\right\rangle \mathrm{d}\tau, \quad u(\cdot) \in \ker \Phi_T^1. \tag{3.13}$$

3.3 Extended First and Second Variations

From now on we deal with a fixed Pontryagin extremal $(\tilde{u}(\cdot), \tilde{x}(\cdot), \tilde{\psi}(\cdot))$. Our idea is to define first variation with extended domain in such a way that the triviality of the kernel of the *extended first variation* along the bang–bang extremal control $\tilde{u}(\cdot)$ implies L_1-local optimality of $\tilde{u}(\cdot)$. If this extended first variation can be nullified, then we define an *extended second variation* on its kernel and formulate second-order sufficient conditions of L_1-local optimality for the bang–bang extremal control $\tilde{u}(\cdot)$.

We extend the domain of Φ_T^1 by adding some Dirac measures located at the switching points of the bang–bang extremal. To introduce them let us consider the directing subspaces $V_j = V_{\tau_j}$ of the switching faces $W_j = W_{\tau_j} (j = 1, \ldots, m)$ and introduce the generalized functions of the form:

$$\omega = \sum_{j=1}^m \omega_j\delta(\tau - \tau_j), \quad \omega_j \in V_j (j = 1, \ldots, m); \tag{3.14}$$

where τ_j $(j = 1, \ldots, m)$ are the switching points of the extremal. We will denote by $\Delta\{\tau_j, V_j\}$ the set of the generalized controls (3.14).

A natural way to construct extended first and second variations would be extending the input/trajectory map Φ_T^1 onto the extended domain (see the previous section), which includes the Dirac measures (3.14), and then defining extended first and second variations as first and second differentials of this extension.

DEFINITION 3.1 *Let $\Delta\{\tau_j, V_j\}$ be the set of Dirac measures* (3.14). *The linear operator $\Phi_T^{l\varepsilon}$: $\mathcal{K}_{\tilde{u}}\mathcal{U} \oplus \Delta\{\tau_j, V_j\} \to \mathcal{T}_{q_0}M$, defined by*

$$\Phi_T^{e'}(u(\cdot) \oplus \omega) = \left(\int_0^T X_\tau(q_0)u(\tau)\,\mathrm{d}\tau + \sum_{j=1}^m X_j(q_0)\omega_j\right) \tag{3.15}$$

is called an extended first variation along the extremal. The summand

$$\sum_{j=1}^m X_j(q_0)\omega_j \tag{3.16}$$

is called the first variation at the switching points of the extremal.

This definition is justified by the following proposition.

LEMMA 3.5 *For every $j = 1, \dots, m$ let a δ-sequence of controls $w_k^j(\cdot)(k = 1, \dots)$ tend weakly to the Dirac measure $\omega_j\delta(\tau - \tau_j)$. Put $w_k(\cdot) = \sum_{j=1}^m w_k^j(\cdot)$. Then as $k \to \infty$ the values of the first variation $\Phi'_T(u(\cdot) + w_k(\cdot)) = q_0 \circ (\int_0^T X_\tau(u(\tau) + w_k(\tau))\,\mathrm{d}\tau$ tend to the value of the extended first variation* (3.15).

(Note that we are not able to approximate Dirac measures by *admissible* controls, since the latter are bounded.)

If the extended first variation along $\tilde{u}(\cdot)$ can be nullified, then we define on its kernel an *extended second variation along the extremal.*

DEFINITION 3.6 *The extended second variation of the system* (3.2) *along the Pontryagin extremal $(\tilde{u}(\cdot), \tilde{x}(\cdot), \tilde{\psi}(\cdot))$ is the quadratic form*:

$$\frac{1}{2}\Bigg\langle \zeta_0, \int_0^T \left[\int_0^\tau X_\xi u(\xi)\,\mathrm{d}\xi + \sum_{\tau_j \le \tau} X_j\omega_j, X_\tau u(\tau)\right](q_0)\,\mathrm{d}\tau$$
$$+ \sum_{i=1}^m q_0 \circ \left[\int_0^{\tau_i} X_\xi u(\xi)\,\mathrm{d}\xi + \sum_{j=1}^i X_j\omega_j, X_i\omega_i\right](q_0)\Bigg\rangle, \tag{3.17}$$

where $u(\cdot) \oplus \omega \in \ker \Phi_T^{e'}$, i.e. $u(\cdot) \oplus \omega \in \mathcal{K}_{\tilde{u}}\mathcal{U} \oplus \Delta\{\tau_j, V_j\}$ and

$$\int_0^T X_\tau(q_0)u(\tau)\,\mathrm{d}\tau + \sum_{j=1}^m X_j(q_0)\omega_j = 0. \tag{3.18}$$

Putting $u(\cdot) \equiv 0$ in (3.17) and (3.18), one obtains the quadratic form

$$\frac{1}{2}\left\langle \zeta_0, q_0 \circ \left[\sum_{j=1}^i X_j\omega_j, X_i\omega_i\right]\right\rangle \tag{3.19}$$

defined on the kernel of the first variation at the switching points, which is a finite-dimensional subspace:

$$\left\{\omega = (\omega_1, \dots, \omega_m) \in \bigoplus_{j=1}^{m} V_j \colon q_0 \circ \sum_{j=1}^{m} X_j \omega_j = 0\right\}. \tag{3.20}$$

DEFINITION 3.7 *The quadratic form* (3.19) *defined on the subspace* (3.20) *is called the second variation of the system* (3.2) *at the switching points of the extremal* $(\tilde{u}(\cdot), \tilde{x}(\cdot), \tilde{\psi}(\cdot))$.

3.4 First- and Second-Order Sufficient Optimality Condition for Bang–Bang Pontryagin Extremals

We have introduced above the extended first and second variations along a Pontryagin extremal. It turns out that if the kernel of the extended first variation is trivial, or, equivalently, *if the kernel of the first variation at the switching points is trivial, then the bang–bang Pontryagin extremal is optimal* (under the following additional assumption).

DEFINITION 3.8 *We say, that second-order Horizontality Conditions for Switchings of the bang–bang Pontryagin extremal* $(\tilde{u}(\cdot), \tilde{x}(\cdot), \tilde{\psi}(\cdot))$ *hold, if:*

$$\langle \tilde{\psi}(\tau_j, \tilde{q}(\tau_j) \circ [Gv, Gv']\rangle = 0, \quad \forall v, v' \in V_j, \quad \forall j = 1, \dots, m, \tag{3.21}$$

for all its switching points $\tau_j (j = 1, \dots, m)$.

Goh has established that the fulfillment of the second-order Horizontality Condition along a singular subarc of extremal is necessary for optimality of this subarc. We require this condition to hold 'along' the degenerate singular 'subarcs', which are the switching points.

If the *Strong Genericity Assumption* holds, then $\dim V_j = 1$ $(j = 1, \dots, m)$, and the second-order Horizontality Condition (3.21) for Switchings holds automatically.

THEOREM 3.9 (first-order sufficient optimality condition for bang–bang extremals) *Let a bang–bang Pontryagin extremal* $(\tilde{u}(\cdot), \tilde{x}(\cdot), \tilde{\psi}(\cdot))$ *meet the Strong Transversality Condition* (3.10), *the Weak Genericity Assumption* (4.12) *and the second-order Horizontality Condition for Switchings. If the first variation at the switching points of the bang–bang extremal has trivial kernel, i.e.,*

$$\sum_{i=1}^{m} X_i(q_0) v_i = 0, \quad v_i \in V_i \ (i = 1, \dots, m) \Rightarrow v_i = 0 \ (i = 1, \dots, m), \tag{3.22}$$

then the bang–bang extremal control $\tilde{u}(\cdot)$ *is optimal for the problem* (1.1)–(1.3).

If the first variation at the switching points possesses a nontrivial kernel we establish in this case the second-order sufficient optimality condition (Theorem 3.10). Regretably this condition is not as sharp as one could wish. Below we shall provide some comments and also a *sharper form of the sufficient condition* for the case where *Strong Genericity Assumption* holds (Theorem 3.11).

THEOREM 3.10 (second-order sufficient optimality condition for bang–bang extremals) *Let the Strong Transversality Condition* (3.10), *the Weak Genericity Assumption and the* *second-order Horizontality Condition for Switchings* (3.21) *hold for the bang–bang Pontryagin extremal* $(\tilde{u}(\cdot), \tilde{x}(\cdot), \tilde{\psi}(\cdot))$. *If the second variation at the switching points* $\tau_j\,(j = 1, \ldots, m)$ *of the extremal, i.e. the quadratic form*

$$Q(u) = \sum_{i=1}^{m} \left\langle \zeta_0, q_0 \circ \left[\sum_{j<i} X_j v_j, X_i v_i \right] \right\rangle, \tag{3.23}$$

is nonpositive (negative semidefinite) on the kernel of the first variation at switching points

$$\left\{ v = (v_1, \ldots, v_m) \in \bigoplus_{i=1}^{m} V_i \middle| \sum_{i=1}^{m} q_0 \circ X_i v_i = 0 \right\}, \tag{3.24}$$

then $\tilde{u}(\cdot)$ *is optimal for the problem* (3.1)–(3.3).

As one can see the formulation of the *sufficient* condition is somewhat unusual, in the sense that it involves negative semidefiniteness of the second variation, while usually second order sufficient optimality conditions involve strict definiteness of the second variation. We should also emphasize, that negative semidefiniteness of the second variation at the switching points *is not necessary* for the optimality of $\tilde{u}(\cdot)$. The reason is that the first and the second variations at switching points have been constructed on the basis of Dirac measures (located at switching points) used as generalized variations of $\tilde{u}(\cdot)$. If it were possible to approximate these Dirac measures by 'ordinary' controls, this construction would be quite adequate. Since the *admissible controls are bounded* we cannot arrange such approximations.

If Strong Genericity Assumption holds let us consider the unit directing vectors $\ell_i\,(i = 1, \ldots, m)$ of the switching edges $V_i\,(i = 1, \ldots, m)$; $\ell_i = (\eta_i^+ - \eta_i^-)/|\eta_i^+ - \eta_i^-|$, where $\eta_i^- = \tilde{u}\,(\tau_i - 0)$, $\eta_i^+ = \tilde{u}(\tau_i + 0)$. Let $M_i = |\eta_i^+ - \eta_i^-|\ (i = 1, \ldots, m)$ be the lengths of the switching edges. The variables $v_i\,(i = 1, \ldots, m)$ involved into the expression for the second variation are scalar. Let us introduce the quadratic form

$$-\frac{1}{2}\left(\sum_{i=1}^{m} \langle \zeta_0, \dot{X}_i(q_0)\ell_i \rangle \frac{v_i^2}{2M_i} + \sum_{i=1}^{m} \left\langle \zeta_0 \left[\sum_{j<i} X_j v_j, X_i v_i \right] (q_0) \right\rangle \right) \tag{3.25}$$

with the domain (3.24); obviously its values are less than the values of the quadratic form (3.23)–(3.24).

THEOREM 3.11 (second-order sufficient optimality condition under Strong Genericity Assumption) *Let the Strong Transversality Condition* (3.10) *and the Strong Genericity Assumption hold for the bang–bang Pontryagin extremal* $(\tilde{u}(\cdot), \tilde{x}(\cdot), \tilde{\psi}(\cdot))$. *If the quadratic form* (3.25) *with the domain* (3.24) *is negative definite, then* $\tilde{u}(\cdot)$ *is optimal for the problem* (3.1)–(3.3).

There is a recent result by Agrachev *et al.* [25] which establishes a stronger sufficient condition under strong genericity assumption. It is a development of the previous work of Agrachev and Gamkrelidze [20] on necessary conditions for optimality of bang–bang extremals. The key tool are first and second variation corresponding to the variation of the moments of switchings. The sufficiency is proved by the field theory techniques.

4 HIGH-ORDER AVERAGING AND STABILITY

4.1 Introduction and General Problem Setting

In this section we study stability and asymptotic stability properties of periodically time-varying systems. In particular we will deal with linear systems with fast oscillating coefficients $\dot{x} = A(t/\varepsilon)x$ and with nonlinear systems $\dot{x} = f(t, x)$ whose stability can not be concluded by the use of linearization principle.

The systems of this type draw much attention for a number of reasons. Differential equations with periodic terms and the corresponding averaging techniques were studied since the 18th century. Stability issues for periodic differential equations have been addressed since long ago. Substantial contributions to the theory were made by Lyapunov; further results have been obtained by Perron, Chetaev, Malkin, Erugin, Barbashin, Bellman, Krasovsky, LaSalle, Massera, Cesari among others.

In control theory and mechanics the discovery of stabilizing effect of vibration in the reverse pendulum example inspired the study of time varying feedback laws. A regular approach to the issues of 'vibrational controllability' and 'vibrational stabilizability' for control systems has been developed by Bellman *et al.* [26]. This topic got a new impulse after the discovery of obstructions (*e.g.* Brockett's criterion) to time-invariant stabilizability and after Coron [27] obtained a general result on time-variant stabilizability for controllable nonlinear control systems. Since then many important contributions to the problem of design of time-varying feedbacks for the control systems have been done.

The most famous tool for studying stability – the Lyapunov direct method – is much more difficult to apply to time-variant systems. It is much harder to construct a corresponding Lyapunov function, which decreases along trajectories of time-variant (in particular of fast-oscillating) systems. It has been observed (see [28] and references therein) that weaker condition of partial decrease may guarantee asymptotic stability, but this weaker condition is difficult to verify. One can find many interesting results of this kind in publications by Aeyels, M'Closkey, Moreau, Murray, Peuteman (see for example [28–30] and references therein).

What we suggest is kind of high-order averaging procedure for studying stability/asymptotic stability properties of time-variant systems. The result of this procedure is a time-invariant vector field represented as a (formal) series. The first-term of the series (first-order averaging) corresponds to standard averaging defined in Bogoliubov's theorem. Obviously we only need high-order averagings if the standard averaging leads to a system with critical equilibria. This for example always will be the case in a problem of time-varying stabilization of a nonholonomic system.

When computing the high-order averagings we use tools of chronological calculus – a technique of asymptotic expansions for the flows generated by time-varying nonlinear vector fields; this calculus has been developed by Agrachev and Gamkrelidze [9] in 70's. This approach is Lie algebraic in its nature; also the high-order averagings are computed via Lie-Poisson brackets of the vector fields appearing in the right-hand side of the differential equation. The computation of the high-order averagings is technically rather involved. Below we just provide formulae for the first three terms of the expansion of the logarithm.

One can find in [31–33] a description of the technique of high-order averaging and some of its applications to studying stability, in particular: (i) stability results for time-periodic linear systems, and in particular for the second- and third-order linear differential equations, with fast oscillating coefficients, (ii) results on time variant stabilization of bilinear systems and of nonholonomic systems without drift term; (iii) stability of equilibrium for time-periodic nonlinear system. In this presentation we illustrate the techniques by an example of reversed pendulum and by study of asymptotic stability of a nonlinear time-periodic system.

4.2 High-Order Averaging for Linear Systems

It is much easier to introduce the notions of high-order averaging for linear time-variant systems.

Consider the system

$$\left(\frac{dx}{dt} = \right)\dot{x}(t) = A\left(\frac{t}{\varepsilon}\right)x(t), \tag{4.1}$$

where $x \in \mathbb{R}^n$ or $\mathbb{C}^n$, $A(\tau)$ is $\mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$-valued function continuous and 1-periodic with respect to τ, $\varepsilon > 0$ is a small parameter. What one can say about stability/asymptotic stability of such a system? Clearly spectra of the matrices $A(\tau)$ for 'frozen' τ do not define the stability properties of this system.

Averaging is another option. The classical averaging result says that, if all the eigenvalues of the corresponding averaged matrix $\int_0^1 A(\tau)\,d\tau$ are located in the open left complex half-plane, then one can assert asymptotic stability of the system (4.1) for all sufficiently small $\varepsilon > 0$.

We want to find out what happens in the critical cases where some of the eigenvalues of this averaged matrix are located on the imaginary axis (for example vanish). This question can be reduced to a more explicit one. Proceeding with the time substitution $\tau = t/\varepsilon$ and denoting the derivative $dx/d\tau$ again by $\dot{x}$ we arrive to the equation

$$\frac{dx}{d\tau} = \varepsilon A(\tau)x(\tau) \tag{4.2}$$

with 1-periodic coefficients. Consider the corresponding matrix differential equation

$$\frac{d}{d\tau}X = \varepsilon A(\tau)X(\tau), \quad X \in \mathbb{R}^{n\times n} \text{ or,} \quad X \in \mathbb{C}^{n\times n} \tag{4.3}$$

and introduce the monodromy matrix $M_\varepsilon = X(1)$. This monodromy matrix determines stability or instability of the system (4.2): if the eigenvalues of M_ε (the multiplicators) all belong to the interior of the unit circle, then the system is asymptotically stable. If at least one of these eigenvalues lies outside the unit circle, then the system is unstable.

Obviously the principal difficulty in applying this result is impossibility to compute the spectra of the matrix M_ε. The only feasible idea is to deal with an expansion of M_ε with respect to ε; as it is known for small $\varepsilon > 0$ the matrix M_ε can be represented as a convergent power series in ε.

Instead of dealing with M_ε we will deal with its logarithm. It is more suitable object in a nonlinear case, where the monodromy map is a nonlinear diffeomorphism. In this latter case the 'logarithm' of this map, (if exists!) is a vector field – an element of a Lie algebra whose linear structure is suitable for expansions into series.

We denote by Λ_ε the logarithm of the matrix M_ε. The corresponding stability criterion becomes: the equilibrium of the system (4.1) (with fixed $\varepsilon > 0$) is asymptotically stable if all the eigenvalues of the logarithm Λ_ε are located in the open left half-plane and unstable if any of them belongs to the open right half-plane.

The matrix Λ_ε admits a power series representation:

$$\Lambda_\varepsilon = \varepsilon\Lambda^{(1)} + \varepsilon^2\Lambda^{(2)} + \cdots.$$

(It is easy to see that $\Lambda^{(1)} = A^1 = \int_0^1 A(\tau)\,d\tau$.) One can come to a conclusion about the stability properties by analyzing the truncations of the latter series. It has been said already that asymptotic stability of $\dot{x} = \Lambda^{(1)}x$ implies asymptotic stability for the system (4.1) if $\varepsilon > 0$ is sufficiently small.

Obviously the logarithm Λ_ε of the monodromy matrix M_ε provides a complete information concerning turnpike or averaged behaviour of the solution of the time-variant system (4.1). According to Floquet theorem this solution 'oscillates around' the trajectory $e^{t\Lambda_\varepsilon}$ of the time-invariant system $dX/d\tau = \Lambda_\varepsilon X$, $X(0) = I$. Therefore Λ_ε can be seen as kind of 'complete averaging' of the system (4.2), while the truncations of its expansion can be seen as partial averagings of the system.

A major problem is the computation of the expansion for Λ_ε. Below we provide the expressions for the first terms of the series in nonlinear setting.

4.3 High-Order Averaging for Nonlinear Systems

Let us consider the nonlinear system

$$\dot{x} = f(\varepsilon^{-1}t, x),$$

where $f(\tau, x)$, denoted below as $f_\tau(x)$, is 1-periodic with respect to τ. Assume $f_\tau(0) = 0$, $\forall\tau$. Introducing the fast time variable $\tau = t/\varepsilon$ we come to the equation

$$\frac{dx}{d\tau} = \varepsilon f_\tau(x). \tag{4.4}$$

Let P_ε^τ be the flow generated by $f_\tau(x)$. This flow is denoted by $P_\varepsilon^\tau = \overrightarrow{\exp} \int_0^\tau \varepsilon f_\theta(x)\,d\theta$. The diffeomorphism $M_\varepsilon = P_\varepsilon^1$ is a *time-1-map* or a *monodromy map*. Obviously $P_\varepsilon^{\tau+1} = M_\varepsilon \circ P_\varepsilon^\tau$ and the origin is a fixed point of M_ε.

It is known that the origin O is asymptotically stable for the system if it is an asymptotically stable fixed point for the map M_ε: $\forall\Delta > 0\ \exists\ \delta > 0$ such that $\rho(x, O) < \delta \Rightarrow \rho(M_\varepsilon^n(x), 0) < \Delta$ and $\rho(M_\varepsilon^n(x), 0) \to 0$ as $n \to +\infty$.

As we said it is more proper to deal with a linear object – the *logarithm of the diffeomorphism* M_ε. Time-invariant vector field Λ_ε being a logarithm of the diffeomorphism M_ε means that M_ε is included in the flow $e^{\Lambda_\varepsilon t}$ generated by the vector field Λ_ε: $M_\varepsilon = e^{\Lambda_\varepsilon t}|_{t=1} = e^{\Lambda_\varepsilon}$. We may consider the expansion of Λ_ε

$$\Lambda_\varepsilon = \sum_{i=1}^{\infty} \varepsilon^i \Lambda^{(i)} \tag{4.5}$$

and try to proceed in a similar way as in the linear case.

There are additional difficulties on this way. In contrast to the linear case it is not always that the logarithm Λ_ε exists. It is known that not any diffeomorphism can be included into a flow generated by time-invariant vector field. Still if one manages to construct a formal power expansion for Λ_ε, then the exponentials of its truncations provide 'nice asymptotics' for the diffeomorphism M_ε and we are often able to conclude asymptotic stability or instability.

If the logarithm Λ_ε exists, then the origin is its equilibrium point and we can state the following two results, the second one being a nonlinear version of Floquet theorem for time-periodic systems.

THEOREM 4.1 *If the origin is an asymptotically stable equilibrium point for* Λ_ε *then it is an asymptotically stable equilibrium point for the system* $\dot{x} = f(\varepsilon^{-1}t, x)$.

THEOREM 4.2 (nonlinear Floquet theorem) *Assume that* P^τ *is the flow generated by a time-periodic (of period 1) differential equation* $\mathrm{d}x/\mathrm{d}\tau = f_\tau(x)$. *If the vector field* Λ *is the logarithm of the diffeomorphism* P^1, *then the flow* P^τ *can be represented as a composition* $P^\tau = e^{\Lambda\tau} \circ \Phi^\tau$ *of the flow* $e^{\Lambda\tau}$ *and of the* 1*-periodic flow* Φ^τ (1*-periodicity of a flow* Φ^τ *means* $\Phi^\tau = \Phi^{\tau+1}$).

4.4 Computation of the Logarithm of a Diffeomorphism

For a diffeomorphism included into the flow generated by time-varying ODE one can compute the logarithm via 'formal chronological series'. It has been accomplished in [9]; in [34] another method of its computation has been provided.

Here we provide only the first terms of this series expressed via Lie-Poisson brackets:

$$\Lambda^1(X_\cdot) = \int_0^1 X_t\,\mathrm{d}t, \quad \Lambda^2(X_\cdot) = \frac{1}{2}\int_0^1 \left[\int_0^{t_1} X_{t_2}\,\mathrm{d}t_2, X_{t_1}\right]\mathrm{d}t_1,$$
$$\Lambda^3(X_\cdot) = -\frac{1}{2}[\Lambda^1(X_\cdot), \Lambda^2(X_\cdot)] + \frac{1}{3}\int_0^1 \mathrm{ad}^2\left(\int_0^{t_1} X_{t_2}\,\mathrm{d}t_2\right)X_{t_1}\,\mathrm{d}t_1.$$

Here $\mathrm{ad}\,XY = [X, Y]$, $\mathrm{ad}^i XY = [X, \mathrm{ad}^{i-1} XY]$, $i \geq 2$.

Let us comment on a particular case of linear vector fields $X_t(x) = A_t x$. For linear vector fields Ax and Bx the Lie bracket is again a linear vector field equal to $[Ax, Bx] = BAx - ABx = -[A, B]x$, where $[A, B]$ is the commutator of matrices A and B. Suppressing x we may treat the terms $\Lambda^{(i)}(A_\cdot)$ in the expansion (4.5) as matrices. To compute these matrices one has to substitute in the previous expressions $X_\cdot$ by $A_\cdot$ and multiply each Lie monomial of order k by $(-1)^{k-1}$.

4.5 Asymptotic Stability for Linear Systems: an Example

Here we arrange an example of linear systems, whose stability properties are determined by high-order averaging(s).

Example Consider the linear system

$$\dot{x}_1 = -x_1, \qquad \dot{x}_2 = x_1 + \frac{\varepsilon x_2}{3},$$

for which the origin is an unstable equilibrium. We put the question if one can perturb this system by a fast oscillating term with a zero average and achieve asymptotic stability. For example can the origin be asymptotically stable for the system

$$\dot{x}_1 = -x_1 + \alpha x_2 \sin\frac{2\pi t}{\varepsilon}, \qquad \dot{x}_2 = x_1 + \frac{\varepsilon x_2}{3}. \tag{4.6}$$

To answer this question we have to calculate first- and second-order averagings. The first-order averaging equals to $\varepsilon\Lambda^{(1)} = \mathrm{diag}\{-\varepsilon, \varepsilon^2/3\}$, being unstable. The second order averaging equals $\varepsilon^2\Lambda^{(2)} = (\varepsilon^2/2)\int_0^1[A_{t_1}, \int_0^{t_1} X_{t_2}\,\mathrm{d}t_2]\mathrm{d}t_1$, where $A(t)$ is the coefficient matrix of the system (4.6). Proceeding with the computations we obtain

$$\varepsilon\Lambda^{(1)} + \varepsilon^2\Lambda^{(2)} = \begin{pmatrix} -\varepsilon + \alpha\frac{\varepsilon^2}{\pi} & -\alpha\frac{\varepsilon^2}{\pi}\left(1 + \frac{\varepsilon}{3}\right) \\ \varepsilon & \frac{\varepsilon^2}{3} \end{pmatrix}.$$

Its trace equals $-\varepsilon + o(\varepsilon)$, while its determinant equals $\varepsilon^3(-1/3 + \alpha/\pi)$. It easy to verify that $\Lambda^{(k)}$ with $k > 2$ contribute to the value of determinant $o(\varepsilon^3)$. Therefore we may conclude, that if $\alpha > \pi/3$ (respectively $\alpha < \pi/3$) then there exists ε_0 such that $\forall\varepsilon < \varepsilon_0$ the origin is asymptotically stable (respectively unstable) for the system (4.6).

4.6 Stabilization of Equilibrium of Linearized Reverse Pendulum

As it is well known the upper position of a pendulum – the *reverse pendulum* – can be made stable if the suspension of the pendulum is subject to (sufficiently) fast harmonic oscillation (see [35]). Here we represent the case where the suspension is subject to fast oscillation of arbitrary form $\delta s(kt)$ and establish conditions for it to stabilize the upper position of the pendulum. We assume $\delta > 0$ and $k > 0$ to be a small and a large parameter respectively, and $s(\tau)$ to be 1-periodic C^2-function. Small oscillations of the pendulum in a neighborhood of upper equilibrium point are described by the equation

$$\ddot{x} - (\omega^2 + \delta k^2\ddot{s}(kt))x = 0, \tag{4.7}$$

where ω is proper frequency of the pendulum. The sign ‘ $-$ ’ in this equation corresponds to the upper equilibrium of the pendulum. Without loss of generality we may assume $\dot{s}(0) = 0$. The stability condition for the reverse pendulum is provided by the following theorem (see [32]).

THEOREM 4.3 *For each $\varepsilon > 0$ there exist $\delta_0 > 0, k_0 > 0$ such that the equilibrium of the reverse pendulum is stable provided that $0 < \delta < \delta_0, k > k_0$ and $\delta^2\int_0^1 \dot{s}^2(\tau)\,\mathrm{d}\tau \geq (\omega^2/k^2) + \varepsilon$ and is unstable provided that $0 < \delta < \delta_0$, $k > k_0$ and $\delta^2\int_0^1 \dot{s}^2(\tau)\,\mathrm{d}\tau \leq (\omega^2/k^2) - \varepsilon$.*

Putting $\dot{x} = y$, rewriting Eq. (4.7) as a system of first-order equations and proceeding with the substitution $\tau = kt$ we arrive to the system

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = (k^{-1}A + \delta k B_\tau)z, \tag{4.8}$$

where

$$z = (x, y), \quad A = \begin{pmatrix} 0 & 1 \\ \omega^2 & 0 \end{pmatrix}, \quad B_\tau = \begin{pmatrix} 0 & 0 \\ \ddot{s}(\tau) & 0 \end{pmatrix}.$$

Applying the variational formula of the chronological calculus (see [9, 34]) we derive the following formula for the fundamental matrix of the system (4.8):

$$\overrightarrow{\exp}\int_0^1 (k^{-1}A + \delta k B_\tau)\mathrm{d}\tau = \overrightarrow{\exp}\int_0^1 e^{\delta k\,\mathrm{ad}\int_0^\sigma B_\tau\,\mathrm{d}\tau}k^{-1}A\,\mathrm{d}\sigma \circ e^{\delta k\int_0^1 B_\tau\,\mathrm{d}\tau}.$$

The second factor of the composition equals I because $\int_0^1 \ddot{s}(\tau)\,d\tau = 0$ and hence $\int_0^1 B_\tau\,d\tau = 0$. What for the first factor then the expansion for the exponential in $C_\sigma = e^{\delta k a d \int_0^\sigma B_\tau\,d\tau} k^{-1}A$ terminates by the second term due to nilpotency of B_τ and we get

$$C_\sigma = \begin{pmatrix} -\delta\dot{s}(\sigma) & k^{-1} \\ k^{-1}\omega^2 - \delta^2 k\dot{s}^2(\sigma) & \delta\dot{s}(\sigma) \end{pmatrix}.$$

Matrix C_σ is 1-periodic in σ; taking its (first-order) averaging we obtain

$$\int_0^1 C_\sigma\,d\sigma = \Lambda^{(1)} = \begin{pmatrix} 0 & k^{-1} \\ k^{-1}\omega^2 - \delta^2 k \int_0^1 \dot{s}^2(\sigma)\,d\sigma & 0 \end{pmatrix}. \tag{4.9}$$

The first-order rest term admits an estimate $O(\delta^2 + k^{-2})$, as $\delta \to 0, k \to \infty$. As far as the complete averaging of C_σ is a trace-free matrix then the stability of the system is defined by (the sign of) its determinant. According to the above estimates the determinant of the complete averaging equals

$$-k^{-2}\omega^2 + \delta^2 \int_0^1 \dot{s}^2(\sigma)\,d\sigma + O(\delta^4 + k^{-4}),$$

from where the result of the Theorem 4.3 follows.

For the lack of space we do not present here a more technically involved result on stabilization of reverse double pendulum; it will appear elsewhere.

4.7 High-Order Averaging for Time-Periodic Nonlinear Systems and Stability

Let us show how the technique works in time-periodic nonlinear case. Assume $X_t(x)$ to be nonlinear time-periodic vector field in $\mathbb{R}^n$, continuous together with all its partial derivatives with respect to x_i's; $X_t(0) \equiv 0, \forall t$.

We will need to introduce a notion of homogeneity. To this end let us consider a dilation

$$\delta_\varepsilon^r : \mathbb{R}^n \to \mathbb{R}^n \quad \delta_\varepsilon^r(x_1, \ldots, x_n) = (\varepsilon^{r_1}x_1, \ldots, \varepsilon^{r_n}x_n), \quad \varepsilon > 0, \ r_i \geq 0, \ i = 1, \ldots, n.$$

The dilation establishes weights for (actually defines the filtrations in the algebras of) smooth functions, differential operators and vector fields in $\mathbb{R}^n$ (see [34] for details). For example the weight of a monomial $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ equals $\sum_{k=1}^n \alpha_k$; the weight of a monomial vector field $x_1^{\alpha_1}\cdots x_n^{\alpha_n}\partial/\partial x_\beta$ equals $-r_\beta + \sum_{k=1}^n \alpha_k$. In particular case of $\alpha_1 = \cdots = \alpha_n = 1$ the weight of a monomial is equal to its degree, the weight of a monomial vector field is 1 less then the degree of the monomial.

Let us say that a vector field is of weight $\geq s$ if all the monomials of its Taylor expansion with respect to x are of weights $\geq s$ (see [34] for 'more invariant' definition). For the rest of this section the following assumption holds: $X_t(x)$ is of weight $\geq s$ $(s > 0)$ with respect to x.

Consider the *Fourier expansion* $\sum_{k=-\infty}^{\infty} X^k e^{ikt}$ for $X_t(x)$. Obviously X^0 coincides with the first-order averaging of X: $X^0 = (1/2\pi)\int_0^{2\pi} X_t(x)\,dt$ and all the vector fields X^k are of weight $\geq s$. We assume that arbitrary weight can be assigned to the vanishing vector field. Let us represent X^k as $\tilde{X}^k + \cdots$, where the principal parts $\tilde{X}^k$ are homogeneous of weight s while the rest terms are of weights $> s$ for $k \neq 0$. A condition for X^0 will be specified later on.

We establish a sufficient local asymptotic stability condition for the vector field $X_t(x)$ using its second-order averaging.

THEOREM 4.4 *Let $X_t(x) \sim \sum_{k=-\infty}^{\infty} X^k e^{ikt}$. Let $X^k = \tilde{X}^k + \cdots$ will be of weight $\geq s$ and $X^0 = \tilde{X}^k + \cdots$ of weight $\geq 2s$. If the origin is locally asymptotically stable equilibrium for the time-invariant homogeneous vector field*

$$\tilde{X}^0(x) - \sum_{k=1}^{\infty} ik^{-1}[\tilde{X}^k, \tilde{X}^{-k}] \tag{4.10}$$

then it is locally asymptotically stable for the vector field $X_t(x) = X^0 + \sum_{k=-\infty}^{\infty} X^k e^{ikt}$.

If one considers a real form Fourier series $X_t(x) = X^0(x) + \sum_{i=1}^{\infty}(Z^k \cos kt + Y^k \sin kt)$ then one concludes with the following corollary (we denote again $\tilde{X}^0, \tilde{Z}^k, \tilde{Y}^k$ the principal parts of X^0, Z^k, Y^k).

COROLLARY 4.5 *If the origin is locally asymptotically stable equilibrium for the time-invariant homogeneous vector field*

$$\tilde{X}^0(x) + \frac{1}{4}\sum_{k=1}^{\infty} k^{-1}[\tilde{Z}^k, \tilde{Y}^k] \tag{4.11}$$

then it is locally asymptotically stable for the vector field $X_t(x)$.

Remark The latter result can be compared with the result of M'Closkey and Morin [36] who studied the system

$$\dot{x} = X^0(x) + X^1(x)\cos 2\pi t + Y^1(x)\sin 2\pi t$$

under similar homogeneity assumptions for the vector fields X^0, X^1, Y^1.

We sketch the proof of the Theorem 4.4. Take the time-2π map for $X_t(x)$. Calculating the first two terms of the expansion of its logarithm one obtains: $\Lambda^{(1)} = 2\pi X^0(x), \Lambda^{(2)} = -2\pi\sum_{k=1}^{\infty} ik^{-1}[X^k, X^{-k}]$. Let $\tilde{\Lambda}^{(1)} = 2\pi\tilde{X}^0, \tilde{\Lambda}^{(2)} = -2\pi\sum_{k=1}^{\infty} ik^{-1}[\tilde{X}^k, \tilde{X}^{-k}]$. Then $\Lambda = \tilde{\Lambda}^{(1)} + \tilde{\Lambda}^{(2)} + \cdots$, where the omitted terms are of weights $>2s$. Therefore the homogeneous of weight $2s$ vector field $\tilde{\Lambda}^{(1)} + \tilde{\Lambda}^{(2)}$, coinciding with (4.10), is the homogeneous principal part of Λ. It can be proven that $\overrightarrow{\exp}\int_0^{2\pi} X_t\, dt = \overrightarrow{\exp}\int_0^{2\pi} \tilde{\Lambda}^{(1)} + \tilde{\Lambda}^{(2)} + Z_t(x)\, dt$, where $Z_t(x)$ has weight $>2s$. If the origin is asymptotically stable equilibrium for $\tilde{\Lambda}^{(1)} + \tilde{\Lambda}^{(2)}$ then its asymptotic stability for $X_t(x)$ can be concluded by application of Massera–Hermes theorem [37, 38].

What happens if $X^0 = 0$ and $[X^k, X^{-k}] = 0, \forall k = 1, \ldots$?

It is easy to see that the second-order averaging $\Lambda^{(2)}$ vanishes in this case. We need to invoke higher-order averagings to try to establish asymptotic stability. In the next result the notation is the same as in the Theorem 4.4.

THEOREM 4.6 *If $X_t(x) = \sum_{k=-\infty}^{\infty} X^k e^{ikt}$, and $X^0 = 0 \bigwedge [X^k, X^{-k}] = 0, \forall k = 1, \ldots,$ then the origin is locally asymptotically stable equilibrium for $X_t(x)$ if it is asymptotically stable equilibrium for the time-invariant homogeneous vector field*

$$-\sum_{k,l=-\infty}^{\infty} \frac{1}{kl}\left[\tilde{X}^k, [\tilde{X}^l, \tilde{X}^{-(k+l)}]\right]. \tag{4.12}$$

Acknowledgements

This work has been partially supported by the Foundation for Science and Technology (FCT), Portugal via the program of multi-annual funding of the R&D Unit 'Mathematics and Applications' of the University of Aveiro and via the program SAPIENS, project POCTI/MAT/41683/2001.

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