Ostrowski type inequalities for functions whose derivatives are h-convex in absolute value

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Abstract

Some new inequalities of Ostrowski type for functions whose derivatives are h-convex in modulus are given. Applications for midpoint inequalities are provided as well.

1 Introduction

1.1 Ostrowski Type Inequalities

Comparison between functions and integral means are incorporated in Ostrowski type inequalities as follows.

The first result in this direction is due to Ostrowski [38].

Theorem 1.1. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) with the property that $|f'(t)| \leq M$ for all $t \in (a, b)$. Then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a)\, M \qquad (1.1)$$

for all $x \in [a, b]$.

The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller quantity.

The following results for absolutely continuous functions hold (see [29] – [31]).

Theorem 1.2. Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous on $[a, b]$. Then, for all $x \in [a, b]$, we have:

$$\left| f\left(x\right) - \frac{1}{b-a} \int_a^b f\left(t\right) dt \right| \qquad (1.2)$$

$$\leq \begin{cases} \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right] (b-a) \|f'\|_\infty & \text{if} \quad f' \in L_\infty [a,b]; \\[3mm] \frac{1}{(\alpha+1)^{\frac{1}{\alpha}}} \left[\left(\frac{x-a}{b-a}\right)^{\alpha+1} + \left(\frac{b-x}{b-a}\right)^{\alpha+1}\right]^{\frac{1}{\alpha}} \times (b-a)^{\frac{1}{\alpha}} \|f'\|_\beta & \text{if} \quad f' \in L_\beta [a,b], \\ & \qquad \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ & \qquad \alpha > 1; \\[3mm] \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right] \|f'\|_1; \end{cases}$$

where $\|\cdot\|_{[a,b],r}$ $(r \in [1,\infty])$ are the usual Lebesgue norms on $L_r[a,b]$, i.e., we recall that

$$\|g\|_{[a,b],\infty} := ess \sup_{t\in[a,b]} |g\left(t\right)|$$

and

$$\|g\|_{[a,b],r} := \left(\int_a^b |g\left(t\right)|^r dt\right)^{\frac{1}{r}}, \ r \in [1,\infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ respectively are sharp in the sense presented in Theorem 1.1.

The above inequalities can also be obtained from the Fink result in [33] on choosing $n = 1$ and performing some appropriate computations.

If one drops the condition of absolute continuity and assumes that f is Hölder continuous, then one may state the result (see for instance [21] and the references therein for earlier contributions):

Theorem 1.3. Let $f : [a,b] \to \mathbb{R}$ be of $r - H-$Hölder type, i.e.,

$$|f\left(x\right) - f\left(y\right)| \leq H |x-y|^r, \text{ for all } \ x,y \in [a,b], \qquad (1.3)$$

where $r \in (0,1]$ and $H > 0$ are fixed. Then, for all $x \in [a,b]$, we have the inequality:

$$\left| f\left(x\right) - \frac{1}{b-a} \int_a^b f\left(t\right) dt \right| \leq \frac{H}{r+1} \left[\left(\frac{b-x}{b-a}\right)^{r+1} + \left(\frac{x-a}{b-a}\right)^{r+1}\right] (b-a)^r. \qquad (1.4)$$

The constant $\frac{1}{r+1}$ is also sharp in the above sense.

Note that if $r = 1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H) (see for instance [13])

$$\left| f\left(x\right) - \frac{1}{b-a} \int_a^b f\left(t\right) dt \right| \leq \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right] (b-a) L, \qquad (1.5)$$

where $x \in [a, b]$. Here the constant $\frac{1}{4}$ is also best.

Moreover, if one drops the condition of the continuity of the function, and assumes that it is of bounded variation, then the following result may be stated (see [15]).

Theorem 1.4. Assume that $f : [a, b] \to \mathbb{R}$ is of bounded variation and denote by $\bigvee_a^b (f)$ its total variation. Then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f) \tag{1.6}$$

for all $x \in [a, b]$. The constant $\frac{1}{2}$ is the best possible.

If we assume more about f, i.e., f is monotonically increasing, then the inequality (1.6) may be improved in the following manner [12] (see also the monograph [28]).

Theorem 1.5. Let $f : [a, b] \to \mathbb{R}$ be monotonic nondecreasing. Then for all $x \in [a, b]$, we have the inequality:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \tag{1.7}$$

$$\leq \frac{1}{b-a} \left\{ [2x - (a+b)] f(x) + \int_a^b sgn\,(t - x) f(t)\, dt \right\}$$

$$\leq \frac{1}{b-a} \left\{ (x-a)[f(x) - f(a)] + (b-x)[f(b) - f(x)] \right\}$$

$$\leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] [f(b) - f(a)].$$

All the inequalities in (1.7) are sharp and the constant $\frac{1}{2}$ is the best possible.

The case for the convex functions is as follows [18]:

Theorem 1.6. Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on $[a, b]$. Then for any $x \in (a, b)$ one has the inequality

$$\frac{1}{2} \left[(b-x)^2 f'_+(x) - (x-a)^2 f'_-(x) \right] \tag{1.8}$$

$$\leq \int_a^b f(t)\, dt - (b-a) f(x)$$

$$\leq \frac{1}{2} \left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a) \right].$$

The constant $\frac{1}{2}$ is sharp in both inequalities. The second inequality also holds for $x = a$ or $x = b$.

For other Ostrowski's type inequalities for the Lebesgue integral, see [3]-[13] and [19].

Inequalities for the Riemann-Stieltjes integral may be found in [14], [16] while the generalization for isotonic functionals was provided in [17].

For the case of functions of self-adjoint operators on complex Hilbert spaces, see the recent monograph [20]

1.2 The Case of Derivatives that are Convex in Modulus

In [17], the author pointed out the following identity in representing an absolutely continuous function. Due to the fact that we use it throughout the paper we give here a short proof.

Lemma 1.7. Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then for any $x \in [a, b]$, one has the equality:

$$f(x) = \frac{1}{b-a} \int_a^b f(t) \, dt + \frac{1}{b-a} \int_a^b (x-t) \left(\int_0^1 f'[(1-\lambda)x + \lambda t] \, d\lambda \right) dt. \tag{1.9}$$

Proof. For any $t, x \in [a, b]$, $x \neq t$, one has

$$\frac{f(x) - f(t)}{x - t} = \frac{1}{x-t} \int_t^x f'(u) \, du = \int_0^1 f'[(1-\lambda)x + \lambda t] \, d\lambda,$$

showing that

$$f(x) = f(t) + (x-t) \int_0^1 f'[(1-\lambda)x + \lambda t] \, d\lambda \tag{1.10}$$

for any $t, x \in [a, b]$.

If we integrate (1.10) over t on $[a, b]$ and divide by $(b-a)$, we deduce the desired identity (1.9).   Q.E.D.

Using the above lemma the following result can be pointed out improving Ostrowski's inequality [4].

Theorem 1.8. Let $f : [a, b] \to \mathbb{C}$ be an absolutely continuous function on $[a, b]$ so that $|f'|$ is convex on (a, b).

(i) If $f' \in L_\infty[a, b]$, then for any $x \in [a, b]$,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \tag{1.11}$$

$$\leq \frac{1}{2} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \left[|f'(x)| + \|f'\|_\infty \right].$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

(ii) If $f' \in L_p[a, b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then for any $x \in [a, b]$,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \tag{1.12}$$

$$\leq \frac{1}{2(q+1)^{\frac{1}{q}}} \left[\left(\frac{b-x}{b-a} \right)^{q+1} + \left(\frac{x-a}{b-a} \right)^{q+1} \right]^{\frac{1}{q}} (b-a)^{\frac{1}{q}} \left\| |f'(x)| + |f'| \right\|_p.$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

(iii) If $f' \in L_1[a, b]$, then for any $x \in [a, b]$,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \tag{1.13}$$

$$\leq \frac{1}{2} \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \left[(b-a) |f'(x)| + \|f'\|_1 \right].$$

In order to extend this result for other classes of functions, we need the following preparatory section.

2 h-Convex Functions

2.1   Some Definitions

We recall here some concepts of convexities that are well known in the literature. Let I be an interval in $\mathbb{R}$.

Definition 2.1 ([32])**.** We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0, 1)$ we have

$$f(tx + (1-t)y) \leq \frac{1}{t} f(x) + \frac{1}{1-t} f(y).$$

Some further properties of this class of functions can be found in [24], [25], [27], [37], [40] and [41]. Among others, its has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

Definition 2.2 ([27])**.** We say that a function $f : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0, 1]$ we have

$$f(tx + (1-t)y) \leq f(x) + f(y).$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions,* i. e. nonnegative functions satisfying

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}$$

for all $x, y \in I$ and $t \in [0, 1]$.

For some results on P-functions see [27] and [39] while for quasi convex functions, the reader can consult [26].

Definition 2.3 ([6])**.** Let s be a real number, $s \in (0, 1]$. A function $f : [0, \infty) \to [0, \infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0, \infty)$ and $t \in [0, 1]$.

For some properties of this class of functions see [1], [2], [6], [7], [22], [23], [34], [35] and [43].

In order to unify the above concepts, S. Varošanec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}$, $(0,1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 2.4 ([46]). Let $h : J \to [0, \infty)$ with h not identical to 0. We say that $f : I \to [0, \infty)$ is an h-convex function if for all $x, y \in I$ we have

$$f (tx + (1 - t) y) \le h (t) f (x) + h (1 - t) f (y)$$

for all $t \in (0, 1)$.

For some results concerning this class of functions see [46], [5], [36], [44], [42] and [45].

2.2 Inequalities of Hermite-Hadamard Type

In [42] the authors proved the following Hermite-Hadamard type inequality for integrable h-convex functions.

Theorem 2.5. Assume that $f : I \to [0, \infty)$ is an h-convex function, $h \in L [0, 1]$ and $f \in L [a, b]$ where $a, b \in I$ with $a < b$. Then

$$\frac{1}{2h \left(\frac{1}{2} \right)} f \left(\frac{a + b}{2} \right) \le \frac{1}{b - a} \int_a^b f (t) \, dt \le [f (a) + f (b)] \int_0^1 h (t) \, dt. \tag{HH}$$

If we write (HH) for $h (t) = t$, then we get the classical Hermite-Hadamard inequality for convex functions.

If we write it for the case of P-type functions, i.e., $h (t) = 1$, then we get the inequality

$$\frac{1}{2} f \left(\frac{a + b}{2} \right) \le \frac{1}{b - a} \int_a^b f (t) \, dt \le f (a) + f (b), \tag{2.1}$$

provided $f \in L [a, b]$, that has been obtained in [27].

If f is integrable on $[a, b]$ and Breckner s-convex on $[a, b]$, for $s \in (0, 1)$, then by taking $h (t) = t^s$ in (HH) we get

$$2^{s-1} f \left(\frac{a + b}{2} \right) \le \frac{1}{b - a} \int_a^b f (t) \, dt \le \frac{f (a) + f (b)}{s + 1} \tag{2.2}$$

that was obtained in [22].

Since for the case of Godunova-Levin class of function we have $h (t) = \frac{1}{t}$, which is not Lebesgue integrable on $(0, 1)$, we cannot apply the left inequality in (HH).

We can introduce now another class of functions.

Definition 2.6. We say that the function $f : I \to [0, \infty)$ is of s-Godunova-Levin type, with $s \in [0, 1]$, if

$$f (tx + (1 - t) y) \le \frac{1}{t^s} f (x) + \frac{1}{(1 - t)^s} f (y), \tag{2.3}$$

for all $t \in (0, 1)$ and $x, y \in I$.

We observe that for $s = 0$ we obtain the class of P-functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(I)$ the class of s-Godunova-Levin functions defined on I, then we obviously have

$$P(I) = Q_0(I) \subseteq Q_{s_1}(I) \subseteq Q_{s_2}(I) \subseteq Q_1(I) = Q(I)$$

for $0 \leq s_1 \leq s_2 \leq 1$.

We have the following Hermite-Hadamard type inequality.

Theorem 2.7. Assume that the function $f : I \to [0, \infty)$ is of s-Godunova-Levin type, with $s \in [0, 1)$. If $f \in L[a, b]$ where $a, b \in I$ and $a < b$, then

$$\frac{1}{2^{s+1}} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)\,dt \leq \frac{f(a) + f(b)}{1-s}. \tag{2.4}$$

We notice that for $s = 1$ the first inequality in (2.4) still holds and was obtained for the first time in [27].

3 Inequalities for Functions Whose Derivatives are h-Convex in Modulus

3.1 The Case of $|f'|$ is h-Convex

The following result holds:

Theorem 3.1. Let $f : [a, b] \to \mathbb{C}$ be an absolutely continuous function on $[a, b]$ so that $|f'|$ is h-convex on (a, b) with $h \in L[0, 1]$.

(i) If $f' \in L_\infty[a, b]$, then for any $x \in [a, b]$,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \tag{3.1}$$

$$\leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \left[|f'(x)| + \|f'\|_\infty \right] \int_0^1 h(t)\,dt.$$

(ii) If $f' \in L_p[a, b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then for any $x \in [a, b]$,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \tag{3.2}$$

$$\leq \frac{1}{(q+1)^{\frac{1}{q}}} \left[\left(\frac{b-x}{b-a} \right)^{q+1} + \left(\frac{x-a}{b-a} \right)^{q+1} \right]^{\frac{1}{q}} \times (b-a)^{\frac{1}{q}} \||f'(x)| + |f'|\|_p \int_0^1 h(t)\,dt.$$

(iii) If $f' \in L_1[a, b]$, then for any $x \in [a, b]$,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \times \left[(b-a) |f'(x)| + \|f'\|_1 \right] \int_0^1 h(t)\,dt. \tag{3.3}$$

Proof. (i). Using (1.9) and taking the modulus, we have

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| = \frac{1}{b-a} \left| \int_a^b \int_0^1 (x-t) f' \left[(1-\lambda) x + \lambda t \right] d\lambda dt \right|$$

$$\leq \frac{1}{b-a} \int_a^b \int_0^1 |x-t| \left| f' \left[(1-\lambda) x + \lambda t \right] \right| d\lambda dt := K$$

Utilizing the h-convexity of $|f'|$ we have

$$K \leq \frac{1}{b-a} \int_a^b \int_0^1 |x-t| \left[h (1-\lambda) |f'(x)| + h (\lambda) |f'(t)| \right] d\lambda dt$$

$$= \frac{1}{b-a} \int_a^b |x-t| \left[|f'(x)| \int_0^1 h (1-\lambda) d\lambda + |f'(t)| \int_0^1 h (\lambda) d\lambda \right] dt$$

$$= \frac{1}{b-a} \int_0^1 h (\lambda) d\lambda \int_a^b |x-t| \left[|f'(x)| + |f'(t)| \right] dt := M (x) \int_0^1 h (\lambda) d\lambda$$

$$\leq \frac{1}{b-a} \int_0^1 h (\lambda) d\lambda \ ess \sup_{t \in [a,b]} \left[|f'(x)| + |f'(t)| \right] \int_a^b |x-t| \, dt$$

$$= \left[\frac{(x-a)^2 + (b-x)^2}{2 (b-a)} \right] \left[|f'(x)| + \|f'\|_\infty \right] \int_0^1 h (\lambda) d\lambda$$

$$= \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \left[|f'(x)| + \|f'\|_\infty \right] \int_0^1 h (\lambda) d\lambda,$$

for any $x \in [a, b]$, and the inequality (3.1) is proved.

(ii). As above, we have

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{1}{b-a} \int_a^b |x-t| \left[|f'(x)| + |f'(t)| \right] dt := M (x) \int_0^1 h (\lambda) d\lambda.$$

Using Hölder's integral inequality for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, we get that

$$M(x) \leq \frac{1}{b-a} \left(\int_a^b |x-t|^q \, dt \right)^{\frac{1}{q}} \left(\int_a^b \left(|f'(x)| + |f'(t)| \right)^p dt \right)^{\frac{1}{p}}$$

$$= \frac{1}{b-a} \left[\frac{(b-x)^{q+1} + (x-a)^{q+1}}{q+1} \right]^{\frac{1}{q}} \| |f'(x)| + |f'| \|_p$$

and the inequality (3.2) is proved.

(iii). We also have that

$$M(x) \le \sup_{t \in [a,b]} |x - t| \frac{1}{b-a} \int_a^b \left[|f'(x)| + |f'(t)| \right] dt$$

$$= \frac{1}{b-a)} \max(x-a, b-x) \left[(b-a) |f'(x)| + \int_a^b |f'(t)| \, dt \right]$$

$$= \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \left[(b-a) |f'(x)| + \|f'\|_1 \right]$$

and the inequality (3.3) is proved.   Q.E.D.

The following particular case is interesting.

Corollary 3.2. With the assumptions of Theorem 3.1, we have the midpoint inequality

$$\left| f\left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \tag{3.4}$$

$$\le \frac{1}{4} (b-a) \left[\left| f'\left(\frac{a+b}{2} \right) \right| + \|f'\|_\infty \right] \int_0^1 h(t) \, dt,$$

provided $f' \in L_\infty[a,b]$.

If $f' \in L_p[a,b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then, we have,

$$\left| f\left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \tag{3.5}$$

$$\le \frac{1}{2} (b-a)^{\frac{1}{q}} \left(\int_a^b \left[\left| f'\left(\frac{a+b}{2} \right) \right| + |f'(t)| \right]^p dt \right)^{\frac{1}{p}} \int_0^1 h(t) \, dt.$$

If $f' \in L_1[a,b]$, then

$$\left| f\left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \tag{3.6}$$

$$\le \frac{1}{2} \left[(b-a) \left| f'\left(\frac{a+b}{2} \right) \right| + \int_a^b |f'(t)| \, dt \right] \int_0^1 h(t) \, dt.$$

Remark 3.3. We observe that if $|f'|$ is convex on (a,b), then Theorem 3.1 reduces to Theorem 1.8.

Assume that $|f'|$ is Breckner s-convex on $[a,b]$, for $s \in (0,1)$.

(a) If $f' \in L_\infty[a, b]$, then for any $x \in [a, b]$,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \tag{3.7}$$

$$\leq \frac{1}{s+1} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \left[|f'(x)| + \|f'\|_\infty \right].$$

(aa) If $f' \in L_p[a, b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then for any $x \in [a, b]$,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \tag{3.8}$$

$$\leq \frac{1}{(s+1)(q+1)^{\frac{1}{q}}} \left[\left(\frac{b-x}{b-a} \right)^{q+1} + \left(\frac{x-a}{b-a} \right)^{q+1} \right]^{\frac{1}{q}} \times (b-a)^{\frac{1}{q}} \||f'(x)| + |f'|\|_p.$$

(aaa) If $f' \in L_1[a, b]$, then for any $x \in [a, b]$,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{1}{s+1} \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \times \left[(b-a) |f'(x)| + \|f'\|_1 \right]. \tag{3.9}$$

Assume that $|f'|$ is of s-Godunova-Levin type, with $s \in [0, 1)$.

(b) If $f' \in L_\infty[a, b]$, then for any $x \in [a, b]$,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \tag{3.10}$$

$$\leq \frac{1}{1-s} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \left[|f'(x)| + \|f'\|_\infty \right].$$

(bb) If $f' \in L_p[a, b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then for any $x \in [a, b]$,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \tag{3.11}$$

$$\leq \frac{1}{(1-s)(q+1)^{\frac{1}{q}}} \left[\left(\frac{b-x}{b-a} \right)^{q+1} + \left(\frac{x-a}{b-a} \right)^{q+1} \right]^{\frac{1}{q}} \times (b-a)^{\frac{1}{q}} \||f'(x)| + |f'|\|_p.$$

(bbb) If $f' \in L_1[a, b]$, then for any $x \in [a, b]$,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{1}{1-s} \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \times \left[(b-a) |f'(x)| + \|f'\|_1 \right]. \tag{3.12}$$

3.2   The Case of $|f'|^p$ is h-Convex

The following result also holds:

Theorem 3.4. Let $f : [a, b] \to \mathbb{C}$ be an absolutely continuous function on $[a, b]$ so that $|f'|^p$ with $p > 1$ is h-convex on (a, b) and $h \in L[0, 1]$.

(i) If $f' \in L_\infty[a, b]$, then for any $x \in [a, b]$,

$$\left| f(x) - \frac{1}{b - a} \int_a^b f(t)dt \right| \tag{3.13}$$

$$\leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b - a} \right)^2 \right] (b - a) \times [|f'(x)|^p + \|f'\|_\infty^p]^{1/p} \left(\int_0^1 h(t)\, dt \right)^{1/p}.$$

(ii) If $f' \in L_p[a, b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then for any $x \in [a, b]$,

$$\left| f(x) - \frac{1}{b - a} \int_a^b f(t)dt \right| \tag{3.14}$$

$$\leq \frac{(b - a)^{\frac{1}{q}}}{(q + 1)^{1/q}} \left[\left(\frac{b - x}{b - a} \right)^{q+1} + \left(\frac{x - a}{b - a} \right)^{q+1} \right]^{1/q} \times \left[(b - a)|f'(x)|^p + \|f'\|_p^p \right]^{1/p} \left(\int_0^1 h(t)\, dt \right)^{1/p}.$$

(iii) If $f' \in L_p[a, b]$, then for any $x \in [a, b]$,

$$\left| f(x) - \frac{1}{b - a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b - a} \right| \right] \times \left\| |f'(x)|^p + |f'|^p \right\|^p \left(\int_0^1 h(t)\, dt \right)^{1/p} \tag{3.15}$$

$$\leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b - a} \right| \right] \times \left((b - a)|f'(x)|^p + \|f'\|_p^p \right)^{1/p} \left(\int_0^1 h(t)\, dt \right)^{1/p}.$$

Proof. As in the proof of Theorem 3.1 we have

$$\left| f(x) - \frac{1}{b - a} \int_a^b f(t)dt \right| = \frac{1}{b - a} \left| \int_a^b \int_0^1 (x - t) f'[(1 - \lambda)x + \lambda t]\, d\lambda dt \right|$$

$$\leq \frac{1}{b - a} \int_a^b |x - t| \left(\int_0^1 |f'[(1 - \lambda)x + \lambda t]|\, d\lambda \right) dt := K$$

for any $x \in [a, b]$.

By Hölder's integral inequality we have

$$\int_0^1 |f'[(1 - \lambda)x + \lambda t]|\, d\lambda \leq \left(\int_0^1 1^q d\lambda \right)^{1/q} \left(\int_0^1 |f'[(1 - \lambda)x + \lambda t]|^p\, d\lambda \right)^{1/p}$$

$$= \left(\int_0^1 |f'[(1 - \lambda)x + \lambda t]|^p\, d\lambda \right)^{1/p}$$

for any $x \in [a, b]$, where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$.

Since $|f'|^p$ is h-convex on (a, b) with $h \in L[0, 1]$, then

$$\int_0^1 |f'[(1 - \lambda) x + \lambda t]|^p \, d\lambda \leq [|f'(x)|^p + |f'(t)|^p] \int_0^1 h(\lambda) \, d\lambda,$$

for any $x \in [a, b]$.

Therefore

$$K \leq \frac{1}{b - a} \left(\int_0^1 h(\lambda) \, d\lambda \right)^{1/p} \int_a^b |x - t| \left[|f'(x)|^p + |f'(t)|^p \right]^{1/p} dt \tag{3.16}$$

for any $x \in [a, b]$.

(i). Now, if $f' \in L_\infty[a, b]$ then

$$\int_a^b |x - t| \left[|f'(x)|^p + |f'(t)|^p \right]^{1/p} dt$$

$$\leq \underset{t \in [a, b]}{ess \ \sup} \left[|f'(x)|^p + |f'(t)|^p \right]^{1/p} \int_a^b |x - t| \, dt$$

$$= \left[|f'(x)|^p + \|f'\|_\infty^p \right]^{1/p} \frac{1}{2} \left[(x - a)^2 + (b - x)^2 \right]$$

for any $x \in [a, b]$, and utilizing (3.16), the inequality (3.13) is proved.

(ii). If $f' \in L_p[a, b]$, $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\int_a^b |x - t| \left[|f'(x)|^p + |f'(t)|^p \right]^{1/p} dt$$

$$\leq \left(\int_a^b |x - t|^q \, dt \right)^{1/q} \left(\int_a^b \left(\left[|f'(x)|^p + |f'(t)|^p \right]^{1/p} \right)^p dt \right)^{1/p}$$

$$= \left[\frac{(b - x)^{q+1} + (x - a)^{q+1}}{q + 1} \right]^{1/q} \left[(b - a) |f'(x)|^p + \|f'\|_p^p \right]^{1/p}$$

$$= \frac{(b - a)^{1 + \frac{1}{q}}}{(q + 1)^{1/q}} \left[\left(\frac{b - x}{b - a} \right)^{q+1} + \left(\frac{x - a}{b - a} \right)^{q+1} \right]^{1/q} \times \left[(b - a) |f'(x)|^p + \|f'\|_1^p \right]^{1/p}$$

for any $x \in [a, b]$, and by (3.16) we deduce the desired inequality (3.14).

(iii). If $f' \in L_p[a,b]$, then by Hölder's inequality we also have

$$\int_a^b |x - t| \left[|f'(x)|^p + |f'(t)|^p \right]^{1/p} dt$$

$$\leq \sup_{t \in [a,b]} |x - t| \int_a^b \left[|f'(x)|^p + |f'(t)|^p \right]^{1/p} dt$$

$$= \max\{x - a, b - x\} \int_a^b \left[|f'(x)|^p + |f'(t)|^p \right]^{1/p} dt$$

$$= (b - a) \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b - a} \right| \right] \left\| |f'(x)|^p + |f'|^p \right\|^p$$

$$\leq (b - a) \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b - a} \right| \right] \left(\int_a^b \left[|f'(x)|^p + |f'(t)|^p \right] dt \right)^{1/p}$$

$$= (b - a) \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b - a} \right| \right] \left((b - a) |f'(x)|^p + \|f'\|_p^p \right)^{1/p}$$

for any $x \in [a,b]$. Q.E.D.

The following midpoint type inequalities are of interest.

Corollary 3.5. With the assumptions of Theorem 3.4, we have the inequality

$$\left| f\left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t)dt \right| \tag{3.17}$$

$$\leq \frac{1}{4} (b - a) \left[\left| f'\left(\frac{a+b}{2} \right) \right|^p + \|f'\|_\infty^p \right]^{1/p} \left(\int_0^1 h(t) \, dt \right)^{1/p},$$

provided $f' \in L_\infty[a,b]$.

If $f' \in L_p[a,b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then we have

$$\left| f\left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t)dt \right| \tag{3.18}$$

$$\leq \frac{1}{2(q+1)^{1/q}} (b - a)^{\frac{1}{q}} \times \left[(b - a) \left| f'\left(\frac{a+b}{2} \right) \right|^p + \|f'\|_p^p \right]^{1/p} \left(\int_0^1 h(t) \, dt \right)^{1/p}.$$

If $f' \in L_p[a, b]$, then

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \tag{3.19}$$

$$\leq \frac{1}{2}\left\| \left| f'\left(\frac{a+b}{2}\right)\right|^p + |f'|^p \right\|^p \left(\int_0^1 h(t)\,dt\right)^{1/p}$$

$$\leq \frac{1}{2}\left((b-a)\left| f'\left(\frac{a+b}{2}\right)\right|^p + \|f'\|_p^p\right)^{1/p} \left(\int_0^1 h(t)\,dt\right)^{1/p}.$$

Remark 3.6. The interested reader can state the corresponding particular inequalities for different h-convex functions. However the details are omitted.

References

[1] M. Alomari and M. Darus, The Hadamard's inequality for s-convex function. *Int. J. Math. Anal.* (Ruse) **2** (2008), no. 13-16, 639–646.

[2] M. Alomari and M. Darus, Hadamard-type inequalities for s-convex functions. *Int. Math. Forum* **3** (2008), no. 37-40, 1965–1975.

[3] G. A. Anastassiou, Univariate Ostrowski inequalities, revisited. *Monatsh. Math.,* **135** (2002), no. 3, 175–189.

[4] N. S. Barnett, P. Cerone, S. S. Dragomir, M. R. Pinheiro,and A. Sofo, Ostrowski type inequalities for functions whose modulus of the derivatives are convex and applications. *Inequality Theory and Applications*, **Vol. 2** (Chinju/Masan, 2001), 19–32, Nova Sci. Publ., Hauppauge, NY, 2003. Preprint: *RGMIA Res. Rep. Coll.* **5** (2002), No. 2, Art. 1 [`Online` `http://rgmia.org/papers/v5n2/Paperwapp2q.pdf`].

[5] M. Bombardelli and S. Varošanec, Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities. *Comput. Math. Appl.* **58** (2009), no. 9, 1869–1877.

[6] W. W. Breckner, Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen. (German) *Publ. Inst. Math.* (Beograd) (N.S.) **23(37)** (1978), 13–20.

[7] W. W. Breckner and G. Orbán, Continuity properties of rationally s-convex mappings with values in an ordered topological linear space. Universitatea "Babeş-Bolyai", Facultatea de Matematica, Cluj-Napoca, 1978. viii+92 pp.

[8] P. Cerone and S. S. Dragomir, Midpoint-type rules from an inequalities point of view, Ed. G. A. Anastassiou, *Handbook of Analytic-Computational Methods in Applied Mathematics*, CRC Press, New York. 135-200.

[9] P. Cerone and S. S. Dragomir, New bounds for the three-point rule involving the Riemann-Stieltjes integrals, in *Advances in Statistics Combinatorics and Related Areas*, C. Gulati, et al. (Eds.), World Science Publishing, 2002, 53-62.

[10] P. Cerone, S. S. Dragomir and J. Roumeliotis, Some Ostrowski type inequalities for n-time differentiable mappings and applications, *Demonstratio Mathematica*, **32**(2) (1999), 697—712.

[11] G. Cristescu, Hadamard type inequalities for convolution of h-convex functions. *Ann. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convexity* **8** (2010), 3–11.

[12] S. S. Dragomir, Ostrowski's inequality for monotonous mappings and applications, *J. KSIAM,* **3**(1) (1999), 127-135.

[13] S. S. Dragomir, The Ostrowski's integral inequality for Lipschitzian mappings and applications, *Comp. Math. Appl.,* **38** (1999), 33-37.

[14] S. S. Dragomir, On the Ostrowski's inequality for Riemann-Stieltjes integral, *Korean J. Appl. Math.,* **7** (2000), 477-485.

[15] S. S. Dragomir, On the Ostrowski's inequality for mappings of bounded variation and applications, *Math. Ineq. & Appl.,* **4**(1) (2001), 33-40.

[16] S. S. Dragomir, On the Ostrowski inequality for Riemann-Stieltjes integral $\int_a^b f(t) \, du(t)$ where f is of Hölder type and u is of bounded variation and applications, *J. KSIAM,* **5**(1) (2001), 35-45.

[17] S. S. Dragomir, Ostrowski type inequalities for isotonic linear functionals, *J. Inequal. Pure & Appl. Math.,* **3**(5) (2002), Art. 68.

[18] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products. *J. Inequal. Pure Appl. Math.* **3** (2002), no. 2, Article 31, 8 pp.

[19] S. S. Dragomir, An Ostrowski like inequality for convex functions and applications, *Revista Math. Complutense*, **16**(2) (2003), 373-382.

[20] S. S. Dragomir, *Operator Inequalities of Ostrowski and Trapezoidal Type*. Springer Briefs in Mathematics. Springer, New York, 2012. x+112 pp. ISBN: 978-1-4614-1778-1

[21] S. S. Dragomir, P. Cerone, J. Roumeliotis and S. Wang, A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis, *Bull. Math. Soc. Sci. Math. Romanie,* **42(90)** (4) (1999), 301-314.

[22] S.S. Dragomir and S. Fitzpatrick, The Hadamard inequalities for s-convex functions in the second sense. *Demonstratio Math.* **32** (1999), no. 4, 687–696.

[23] S.S. Dragomir and S. Fitzpatrick,The Jensen inequality for s-Breckner convex functions in linear spaces. *Demonstratio Math.* **33** (2000), no. 1, 43–49.

[24] S. S. Dragomir and B. Mond, On Hadamard's inequality for a class of functions of Godunova and Levin. *Indian J. Math.* **39** (1997), no. 1, 1–9.

[25] S. S. Dragomir and C. E. M. Pearce, On Jensen's inequality for a class of functions of Godunova and Levin. *Period. Math. Hungar.* **33** (1996), no. 2, 93–100.

[26] S. S. Dragomir and C. E. M. Pearce, Quasi-convex functions and Hadamard's inequality, *Bull. Austral. Math. Soc.* **57** (1998), 377-385.

[27] S. S. Dragomir, J. Pečarić and L. Persson, Some inequalities of Hadamard type. *Soochow J. Math.* **21** (1995), no. 3, 335–341.

[28] S. S. Dragomir and Th. M. Rassias (Eds), *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publisher, 2002.

[29] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_1-norm and applications to some special means and to some numerical quadrature rules, *Tamkang J. of Math.*, **28** (1997), 239-244.

[30] S. S. Dragomir and S. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Appl. Math. Lett.*, **11** (1998), 105-109.

[31] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_p-norm and applications to some special means and to some numerical quadrature rules, *Indian J. of Math.,* **40**(3) (1998), 245-304.

[32] E. K. Godunova and V. I. Levin, Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions. (Russian) *Numerical mathematics and mathematical physics* (Russian), 138–142, 166, Moskov. Gos. Ped. Inst., Moscow, 1985

[33] A. M. Fink, Bounds on the deviation of a function from its averages, *Czechoslovak Math. J.,* **42**(117) (1992), No. 2, 298-310.

[34] H. Hudzik and L. Maligranda, Some remarks on s-convex functions. *Aequationes Math.* **48** (1994), no. 1, 100–111.

[35] U. S. Kirmaci, M. Klaričić Bakula, M. E Özdemir and J. Pečarić, Hadamard-type inequalities for s-convex functions. *Appl. Math. Comput.* **193** (2007), no. 1, 26–35.

[36] M. A. Latif, On some inequalities for h-convex functions. *Int. J. Math. Anal.* (Ruse) **4** (2010), no. 29-32, 1473–1482.

[37] D. S. Mitrinović and J. E. Pečarić, Note on a class of functions of Godunova and Levin. *C. R. Math. Rep. Acad. Sci. Canada* **12** (1990), no. 1, 33–36.

[38] A. Ostrowski, Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, *Comment. Math. Hel,* **10** (1938), 226-227.

[39] C. E. M. Pearce and A. M. Rubinov, P-functions, quasi-convex functions, and Hadamard-type inequalities. *J. Math. Anal. Appl.* **240** (1999), no. 1, 92–104.

[40] J. E. Pečarić and S. S. Dragomir, On an inequality of Godunova-Levin and some refinements of Jensen integral inequality. *Itinerant Seminar on Functional Equations, Approximation and Convexity* (Cluj-Napoca, 1989), 263–268, Preprint, 89-6, Univ. "Babeş-Bolyai", Cluj-Napoca, 1989.

[41] M. Radulescu, S. Radulescu and P. Alexandrescu, On the Godunova-Levin-Schur class of functions. *Math. Inequal. Appl.* **12** (2009), no. 4, 853–862.

[42] M. Z. Sarikaya, A. Saglam, and H. Yildirim, On some Hadamard-type inequalities for h-convex functions. *J. Math. Inequal.* **2** (2008), no. 3, 335–341.

[43] E. Set, M. E. Özdemir and M. Z. Sarıkaya, New inequalities of Ostrowski's type for s-convex functions in the second sense with applications. *Facta Univ. Ser. Math. Inform.* **27** (2012), no. 1, 67–82.

[44] M. Z. Sarikaya, E. Set and M. E. Özdemir, On some new inequalities of Hadamard type involving h-convex functions. *Acta Math. Univ. Comenian.* (N.S.) **79** (2010), no. 2, 265–272.

[45] M. Tunç, Ostrowski-type inequalities via h-convex functions with applications to special means. *J. Inequal. Appl.* **2013**, 2013:326.

[46] S. Varošanec, On h-convexity. *J. Math. Anal. Appl.* **326** (2007), no. 1, 303–311.