

Holographic Brownian motion in 2 + 1 dimensional hairy black holes

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Received: 17 December 2013 / Accepted: 19 February 2014 / Published online: 4 March 2014

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Abstract In this paper, we investigate the dynamics of a heavy quark for plasmas corresponding to three dimensional hairy black holes. We utilize the AdS/CFT correspondence to study the holographic Brownian motion of this particle with different kinds of hairy black holes. For an uncharged black hole in the low frequency limit we derive analytic expressions for the correlation functions and the response functions and verify that the fluctuation–dissipation theorem holds in the presence of a scalar field against a metric background. In the case of a charged black hole, we think that the results are similar to that derived for an uncharged black hole.

1 Introduction

Heavy-ion collision experiments at RHIC are believed to create a strongly coupled quark gluon plasma (sQGP) [1–3]. The QGP is a phase of QCD that is thought to be very similar to the plasma of $N = 4$ super Yang Mills theory at finite temperature. One of the current challenges in theoretical particle physics is to compute the properties of this strongly coupled plasma. The AdS/CFT correspondence [4–7] has led to many profound insights into the nature of strongly coupled gauge theories. This gauge/gravity duality provides the possibility of computing some properties of QGP [8–11]. QGPs contain quarks and gluons, like hadrons, but unlike hadrons, the mesons and baryons lose their identities and dissolve into a fluid of quarks and gluons. A heavy quark immersed in this fluid undergoes the Brownian motion [12–15] at finite temperature. The AdS/CFT correspondence can be utilized to investigate the Brownian motion of this particle. In the context of this duality, the dual statement of the quark in QGP corresponds to the end point of an open string that extends

from the boundary to the black-hole horizon. The black-hole environment excites the modes of the string by Hawking radiation. It was found that, once these modes are quantized, the end point of the string at the boundary shows a Brownian motion which is described by the Langevin equation [13–15].

In the formulation of the AdS/CFT correspondence, the fields of gravitational theory would be related to the corresponding boundary theory operators [6,7], such as that their boundary value should couple to the operators. In this way, instead of using the boundary field theory to obtain the correlation function of the quantum operators, one can determine these correlators by the thermal physics of black holes and use them to compute the correlation functions. For different theories of gravity one can make an association with various plasmas in the boundary. In this paper we follow different works [16–22] to investigate the Brownian motion of a particle in a two dimensional plasma of which the gravity dual is described by a three dimensional hairy metric background [23–31]. We obtain the solutions to the equation of motion of uncharged hairy black holes by using a matching technique in the low frequency limit. We utilize these solutions in investigating the Brownian motion of the particle. We obtain an expression for the response functions and correlation functions and show that the fluctuation–dissipation theorem holds in the presence of a scalar field against a metric background. For the case of a charged black hole we make some comments.

This paper is structured as follows: in Sect. 2, we review different kinds of three dimensional hairy black holes. Section 3 is assigned to looking for a holographic realization of Brownian motion on the boundary and bulk side of theory. We study the holographic Brownian motion in hairy black holes and the fluctuation–dissipation theorem in the presence of a scalar field in a metric background in Sect. 4. In Sect. 5, we summarize our work in this paper and make

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some comments on our results and close with our conclusions.

2 Hairy black holes in 2 + 1 dimensions

There is already a huge amount of literature on the subject of gravity coupled with a scalar field [23–26, 28–31]. Black-hole solutions in such theories are known as hairy black holes. In this paper, we are interested in the study of the black-hole solution in Einstein–Maxwell–scalar gravity with a nonminimally coupled scalar field in (2 + 1) dimensions [23–25]. The action reads

$$I = \frac{1}{2} \int d^3x \sqrt{-g} \left[R + g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \xi R \phi^2 - 2V(\phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right], \tag{1}$$

where ξ is the coupling constant between gravity and the scalar field, which will be fixed be as $\xi = \frac{1}{8}$ [24].

A static, circularly solution to the above action which represents a charged hairy black hole can be written thus:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\psi^2, \tag{2}$$

where

$$f(r) = \frac{r^2}{l^2} - M + \frac{\left(\frac{Q^2}{6} - 2M\right) B}{3r} - \frac{Q^2}{2} \left(1 + \frac{2B}{3r}\right) \ln(r), \tag{3}$$

Q is the electric charge, l is an integration constant, and B is related to the scalar field by

$$\phi(r) = \pm \sqrt{\frac{8B}{r+B}}. \tag{4}$$

When $B = 0$, the scalar field ϕ vanishes, and the system becomes the Einstein–Maxwell–AdS theory. The solution is known as a static charged BTZ black hole [31]. We have

$$f(r) = \frac{r^2}{l^2} - M - \frac{Q^2}{2} \ln(r), \tag{5}$$

where M is the mass of BTZ black hole. The horizon radius in terms of the black-hole mass and charge are obtained as

$$r_+ = \exp\left(-\frac{1}{2} \text{Lambert } W\left[\frac{-4 \exp\left(\frac{-4M}{Q^2}\right)}{l^2 Q^2}\right] - \frac{2M}{Q^2}\right). \tag{6}$$

In the case of an uncharged black hole we set $Q = 0$ in Eq. (3), and then this equation is reduced to the following:

$$f(r) = \frac{r^2}{l^2} - M - \frac{2BM}{3r} = \frac{(r - r_+)(r - r_1)(r - r_2)}{3rl^2}, \tag{7}$$

where

$$r_+ = \frac{1}{3} Y(M, B) + \frac{Ml^2}{Y(M, B)} \tag{8}$$

is the black-hole horizon with

$$Y(M, B) = \left[9BMl^2 + 3\sqrt{-3M^3l^6 + 9B^2M^2l^4}\right]^{\frac{1}{3}}. \tag{9}$$

In Eq. (7) r_1 and r_2 are given by

$$r_1 = -\frac{1}{2} \left[\frac{1}{3} Y(M, B) + \frac{Ml^2}{Y(M, B)} \right] + i \frac{\sqrt{3}}{2} \left[\frac{1}{3} Y(M, B) - \frac{Ml^2}{Y(M, B)} \right] = r_2^*. \tag{10}$$

We can derive the mass and temperature of the black hole in terms of the entropy s as follows:

$$M = \frac{3s^3}{16\pi^2 l^2 (8\pi B + 3s)}, \quad T = \frac{9s^2 (4\pi B + s)}{8\pi^2 l^2 (8\pi B + 3s)^2}, \tag{11}$$

where $s = 4\pi r_+$. In the special case of $M = \frac{3B^2}{l^2}$, where the self-interacting scalar vanishes, we obtain the conformal black hole, with [26]

$$f(r) = \frac{(r - 2B)(r + B)^2}{rl^2}, \tag{12}$$

which yields the following black-hole mass and temperature:

$$M = \frac{3s^2}{64\pi^2 l^2}, \quad T = \frac{3s}{32\pi^2 l^2}. \tag{13}$$

3 Holographic Brownian motion

3.1 Dictionary of Brownian motion in the boundary

In the field theory or boundary side of the AdS/CFT story for Brownian motion, a mathematical description of this motion is given by the Langevin equation [13–15], which has the generalized form

$$\dot{p}(t) = - \int_{-\infty}^t \gamma(t - \acute{t}) p(\acute{t}) + R(t) + K(t), \tag{14}$$

where p is the momentum of the Brownian particle. The terms on the right-hand side of (14) correspond to the friction, random, and external force, respectively, and $\gamma(t)$ is the kernel function. In a first approximation, one can assume the following averages for the random force:

$$\langle R(t) \rangle = 0, \quad \langle R(t)R(\acute{t}) \rangle = \kappa_0 \delta(t - \acute{t}), \tag{15}$$

where κ_0 is a constant which, due to the fluctuation–dissipation, is related to the friction coefficient γ_0 through

$$\kappa_0 = \frac{\gamma_0}{2mT}. \tag{16}$$

This comes from the fact that the frictional and random forces have the same origin at the microscopic level (collision with the fluid constituents).

The time evolution of the displacement squared for long times (or low frequencies) can be derived by computing the two-point correlation function, so we find as a result the following relation:

$$\langle s(t)^2 \rangle = \langle [x(t) - x(0)]^2 \rangle \approx 2Dt, \quad \text{for } t \gg \frac{1}{\gamma_0} \tag{17}$$

where $D = \frac{T}{\gamma_0 m}$ is the diffusion constant and we have assumed that $\langle m\dot{x}^2 \rangle = T$.

After applying the Fourier transformation for relation (14) one can obtain

$$p(\omega) = \frac{R(\omega) + K(\omega)}{\gamma[\omega] - i\omega}. \tag{18}$$

If we take the statistical average of (18), then we have

$$\langle p(\omega) \rangle = \mu(\omega)K(\omega), \quad \mu(\omega) \equiv \frac{1}{\gamma[\omega] - i\omega}; \tag{19}$$

here $\mu(\omega)$ is known as the admittance. So we can determine this quantity by measuring the response $p(\omega)$ to an external force.

The power spectrum, $I_O(\omega)$, is defined for a quantity $O(t)$ by

$$I_O(\omega) = \int_{-\infty}^{\infty} dt \langle O(t_0)O(t_0 + t) \rangle e^{i\omega t}, \tag{20}$$

and it is related to the two-point function because of the Wiener–Khintchine theorem,

$$\langle O(\omega)O(\acute{\omega}) \rangle = 2\pi \delta(\omega + \acute{\omega})I_O(\omega). \tag{21}$$

For the case without external force, i.e., $K = 0$, by using (18) and after some algebra, one gets

$$\kappa(\omega) = I_R(\omega) = \frac{I_p(\omega)}{|\mu(\omega)|^2}. \tag{22}$$

This will be important for checking the validity of the fluctuation–dissipation theorem.

3.2 Brownian motion in the bulk

An external quark on the boundary theory can be realized as the endpoint of an open string which is hung from the boundary and dips into the black-hole horizon. The dynamics of this string is governed by the Nambu–Goto action [17–20],

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{-\det g_{ab}}, \tag{23}$$

where $g_{ab} = G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu$ denotes the induced metric on the worldsheet. We set $\tau = t$ and $\sigma = r$ to work in the static gauge. Our string embedding is then given by $X^\mu(t, r) = (t, r, \psi(t, r))$. This string stretches along the r direction and has small fluctuations in the transverse direction $\psi(t, r)$. For $\psi(t, r) = c$, one can check that this yields a trivial solution which corresponds to a quark in equilibrium with a thermal bath. In this case the mass of the particle can easily be computed from the tension of the string,

$$m = \frac{1}{2\pi\alpha'} \int_{r_+}^{r_b} dr \sqrt{-g_{tt}g_{rr}} = \frac{1}{2\pi\alpha'} (r_b - r_+) \approx \frac{1}{2\pi\alpha'} r_b, \tag{24}$$

for $r_b \gg r_+$,

where r_b is the position of a boundary and r_+ is defined by the radius of the horizon of the black hole.

If the scalar $\psi(t, r)$ does not fluctuate too far from its equilibrium values ($\psi(t, r) = 0$), we can expand the Nambu–Goto action up to quadratic order in the perturbations [3] as

$$S_{NG} \approx -\frac{1}{4\pi\alpha'} \int dt dr r^2 \left[f(r)\psi'^2 - \frac{1}{f(r)}\dot{\psi}^2 \right], \tag{25}$$

where $\psi' \equiv \partial_r \psi$ and $\dot{\psi} \equiv \partial_t \psi$.

If we consider an external force on the Brownian particle as in (14), we can obtain the admittance from the response of particle to this force. This situation can easily be realized by turning on world-volume electric field on the flavor D-brane at $r = r_b$. Since the endpoint of sting is charged, this is the same as adding a boundary term to the Nambu–Goto action, $S = S_{NG} + S_{EM}$, where

$$S_{EM} = \oint (A_t + A_\psi \dot{\psi}) dt. \tag{26}$$

This will affect the motion of the external quark on the boundary and will not play any role for the string dynamics on the bulk. We leave this part of the action for now but we will need it later.

The Nambu–Goto action (25) near the horizon limit $r \sim r_+$ becomes

$$S_{NG} \approx -\frac{1}{4\pi\alpha'} \int dt dr_* r_+^2 [\dot{\psi}'^2 - \dot{\psi}^2]. \tag{27}$$

Here, the primes stand for derivatives with respect to the tortoise coordinates r_* , which is defined by

$$dr_* = \frac{dr}{f(r)}. \tag{28}$$

Thus, the equation of motion is then

$$(\partial_{r_*}^2 - \partial_t^2)\psi = 0, \tag{29}$$

which shows that in the near horizon limit ψ behaves like massless Klein–Gordon scalars. If we define $\psi(t, r) = e^{-i\omega t} g_\omega(r)$, then the two independent solutions to Eq. (29) are

$$\psi^{\text{out}}(t, r) = e^{-i\omega t} g^{\text{out}}(r) \sim e^{-i\omega(t-r_*)} \tag{30}$$

$$\psi^{\text{in}}(t, r) = e^{-i\omega t} g^{\text{in}}(r) \sim e^{-i\omega(t+r_*)}. \tag{31}$$

Following the standard quantization of quantum fields in curved spacetime, we obtain a mode expansion of the form

$$\psi(t, r) = \int_0^\infty \frac{d\omega}{2\pi} [a_\omega u_\omega(t, r) + a_\omega^\dagger u_\omega(t, r)^*], \tag{32}$$

with

$$u_\omega(t, r) = \eta [g^{\text{out}}(r) + \delta g^{\text{in}}(r)] e^{-i\omega t},$$

$$[a_\omega, a_{\dot{\omega}}^\dagger] = 2\pi\delta(\omega - \dot{\omega}). \tag{33}$$

η and δ are constants that are found by requiring the normalization of modes through the standard Klein–Gordon inner product and the boundary condition at $r = r_b$, respectively. The string modes satisfy the Bose–Einstein distribution:

$$\langle a_\omega^\dagger a_{\dot{\omega}} \rangle = \frac{2\pi\delta(\omega - \dot{\omega})}{e^{\beta\omega} - 1}, \tag{34}$$

with $\beta = \frac{1}{T}$. Using this and the mode expansion given in (33), we compute

$$\begin{aligned} \langle \Psi(t)\Psi(0) \rangle &= \langle \psi(t, r_b)\psi(0, r_b) \rangle \\ &= \int_0^\infty \frac{d\omega}{2\pi} \frac{1}{e^{\beta\omega} - 1} [u_\omega(t, r_b)u_\omega(0, r_b)^* \\ &\quad + u_\omega(t, r_b)^*u_\omega(0, r_b)], \end{aligned} \tag{35}$$

From the above relation we can easily derive the general form of the momentum correlator,

$$\begin{aligned} \langle : p(t)p(0) : \rangle &= -m^2\partial_t^2 \langle : \psi(t, r_b)\psi(0, r_b) : \rangle, \\ &= \int_0^\infty \frac{d\omega}{2\pi} \frac{2m^2\omega^2|\eta|^2 \cos(\omega t)}{e^{\beta\omega} - 1} |g^{\text{out}}(r_b) + \delta g^{\text{in}}(r_b)|^2. \end{aligned} \tag{36}$$

Eventually, the displacement square can be obtained from the relation (35) as follows:

$$\begin{aligned} S^2(t) &= \langle : [\Psi(t) - \Psi(0)]^2 : \rangle \\ &= \int_0^\infty \frac{d\omega}{2\pi} \frac{8|\eta|^2 \sin^2(\frac{\omega t}{2})}{e^{\beta\omega} - 1} |g^{\text{out}}(r_b) + \delta g^{\text{in}}(r_b)|^2. \end{aligned} \tag{37}$$

4 Brownian motion in Hairy black holes

4.1 String dynamics in Hairy black holes and the response function

By having the knowledge of last sections at hand, we are now able to realize the Brownian motion of a particle moving on the boundary of hairy black holes. The dual state of such a particle is an open string which extends from the boundary to the horizon of the hairy black holes. We can obtain the dynamics of this string through the Nambu–Goto action (25). The equation of motion derived from this action is

$$\frac{\partial}{\partial \rho} \left[\rho^2 f(\rho) \frac{\partial g_\omega(\rho)}{\partial \rho} \right] + \frac{4B^2 \rho^2 \omega^2}{f(\rho)} g_\omega(\rho) = 0, \tag{38}$$

where we have defined $\rho = \frac{r}{r_h}$ and used $\psi(t, r) = e^{-i\omega t} g_\omega(r)$. In general, one can check that it is not possible to solve the above equation analytically for any kind of $f(r)$ defined in Sect. 2. However, we can employ the low frequency approximation by means of the matching technique [17,20–22]. To find the solutions in this way, we consider three regimes:

- (I) the near horizon solution ($\rho \sim 1$) for arbitrary ν (where $\nu = \frac{l^2\omega}{r_h}$),
- (II) the solution for arbitrary ρ in the limit $\nu \ll 1$, and
- (III) the asymptotic $\rho \rightarrow \infty$ solution for arbitrary ν ,

and find the approximate solutions for each of regimes. Then we match them to leading order in ν . We implement the above method for each kind of uncharged and charged hairy black holes separately. In this section, we begin our discussion with the special case of a conformal black hole, then we continue on an uncharged black hole with a general mass. We study the charged hairy black hole at the end of this section.

4.1.1 Uncharged black hole with the special mass (conformal black hole)

In the conformal black hole, with $f(r)$ defined through Eq. (12), we expect two solutions in regime (I) of the form of Eqs. (31) and (32). In the tortoise coordinate system utilized in those solutions, the (t, r_*) part of the metric is conformally flat and the equation of motion near the horizon has a behavior similar to the wave equation in flat space. Thus, we choose solutions (31) and (32), where the parameter r_* has the following definition:

$$r_* \sim \frac{2l^2}{9B} \ln(2B - r) \tag{39}$$

near the horizon, and $r = 2B$, for the conformal black hole. In fact, in the near horizon regime, our equation of motion for this kind of black hole reduces to

$$g_\nu(\rho)'' + \frac{1}{\rho - 1} g_\nu(\rho)' + \frac{16\nu^2}{81(\rho - 1)^2} g_\nu(\rho) = 0, \tag{40}$$

and consequently independent solutions are obtained:

$$g_\nu^{\text{out/in}}(\rho) = e^{\pm \frac{4i\nu}{9} \ln(\rho-1)} = 1 \pm \frac{4i\nu}{9} \ln(\rho - 1) + O(\nu^2). \tag{41}$$

We can expand $g_\nu(\rho)$ as a power series in ν for regime (II); then we have

$$g_\nu(\rho) = g_\nu^{(0)}(\rho) + \nu g_\nu^{(1)}(\rho) + \nu^2 g_\nu^{(2)}(\rho) + \dots \tag{42}$$

The first term can be derived from solving the following equation:

$$\frac{\partial}{\partial \rho} \left[\rho^2 f(\rho) \frac{\partial g_\nu^{(0)}(\rho)}{\partial \rho} \right] = 0. \tag{43}$$

The general solution in this regime for any kind of $f(\rho)$ is given by

$$g_\nu^{(0)} = B_1 + B_2 \int \frac{d\rho}{\rho^2 f(\rho)}. \tag{44}$$

In the special case of a conformal black hole we attain

$$g_\nu^{(0)}(\rho) = B_1 + B_2 \left[\frac{8}{9} \ln(2\rho + 1) - \ln \rho + \frac{1}{9} \ln(\rho - 1) - \frac{2}{3(2\rho + 1)} \right]. \tag{45}$$

In the regime (III), one has $f(\rho) \rightarrow \frac{4B^2\rho^2}{l^2}$ for $\rho \rightarrow \infty$, and then Eq. (38) becomes

$$\frac{\partial}{\partial \rho} \left[\rho^4 \frac{\partial g_\nu(\rho)}{\partial \rho} \right] + \nu^2 g_\nu(\rho) = 0. \tag{46}$$

The general solution to the above equation can be found as a series expansion in $\frac{1}{\rho}$. For leading order terms we have

$$g_\nu(\rho) = C_1 \left(1 + \frac{\nu^2}{2\rho^2} \right) + C_2 \frac{i\nu^3}{3\rho^3} + O\left(\frac{1}{\rho^4}\right). \tag{47}$$

In order to have asymptotic solutions at low frequencies, we require the coefficients. So we expand (45) near the horizon and match the solution with (41) to find B_1 and B_2 . From (41) and (45) it follows that

$$B_1^{(\text{out/in})} = 1 \mp \frac{4i\nu}{9} (8 \ln 3 - 2),$$

$$B_2^{(\text{out/in})} = \pm 4i\nu. \tag{48}$$

Finally, expanding the solution in (45) for $\rho \rightarrow \infty$ yields

$$g_\nu^{(0)}(\rho) = B_1 + B_2 \left[\frac{8}{9} \ln 2 + \frac{-1}{12\rho^3} \right]. \tag{49}$$

By comparing Eq. (49) with (47) and using (48) we obtain

$$C_1^{(\text{out/in})} = 1 \mp \frac{4i\nu}{9} \left(8 \ln \frac{3}{2} - 2 \right),$$

$$C_2^{(\text{out/in})} = \mp \frac{1}{\nu^2}. \tag{50}$$

Thus, the asymptotic solutions, in the low frequency limit, and for corresponding modes of outgoing and incoming wavefunctions at the horizon, are obtained:

$$g_\nu^{\text{out/in}}(\rho) = 1 \mp \frac{4i\nu}{9} \left(8 \ln \frac{3}{2} - 2 \right) \left(1 + \frac{\nu^2}{2\rho^2} \right) \mp \frac{i\nu}{3\rho^3} + O\left(\frac{1}{\rho^4}\right). \tag{51}$$

We now turn our attention to the case that an external force is imposed on the Brownian particle. We can compute the admittance from the response of the particle to this force. As mentioned in Sect. 2, in this case, because of the new boundary condition, we must modify our equation of motion as

$$\frac{\partial \mathcal{L}}{\partial \psi_i'} = F_i, \quad \text{where } F_i = -(F_{it} + F_{ij} \psi^j), \tag{52}$$

with $F_{it} = \partial_i A_t - \partial_t A_i$ and $F_{ij} = \partial_i A_j - \partial_j A_i$. For the general metric background defined in Eq. (2) we get

$$F_{t\psi} = \frac{r_h f(\rho) \rho^2 \partial_\rho \psi}{2\pi \dot{\alpha}} \quad \text{at} \quad \rho = \rho_b. \tag{53}$$

The general solution for ψ is a combination of ingoing and outgoing modes. In the semiclassical approximation, because of Hawking radiation [32,33] outgoing modes are always excited. Then we can write $\langle \psi \rangle = \langle A^{(in)} \rangle e^{-i\omega t} g^{(in)}(\rho)$. By using this relation in (53) and $F_{t\psi} = E_0 e^{-i\omega t}$ we obtain

$$A^{(in)} = \frac{2\pi \dot{\alpha} E_0}{r_h f(\rho) \rho^2 g^{(in)}(\rho)} \Big|_{\rho=\rho_b}. \tag{54}$$

The position of particle is given by

$$\begin{aligned} \langle \Psi(t) \rangle &= \langle \psi(t, \rho_b) \rangle = \langle A^{(in)} e^{-i\omega t} g^{(in)}(\rho_b) \rangle \\ &= e^{-i\omega t} \frac{2\pi \dot{\alpha} E_0 g^{(in)}(\rho_b)}{r_h f(\rho) \rho^2 g^{(in)}(\rho)} \Big|_{\rho = \rho_b}, \end{aligned} \tag{55}$$

and then

$$\langle p(t) \rangle = e^{-i\omega t} \frac{-2im\omega\pi\dot{\alpha}E_0g^{(in)}(\rho_b)}{r_h f(\rho)\rho^2 g^{(in)}(\rho)} \Big|_{\rho=\rho_b}. \tag{56}$$

The above relation leads to the following result for the admittance:

$$\mu(\omega) = \frac{-2im\omega\pi\dot{\alpha}g^{(in)}(\rho_b)}{r_h f(\rho)\rho^2 g^{(in)}(\rho)} \Big|_{\rho=\rho_b}. \tag{57}$$

For the conformal black hole in the zero frequency and $\frac{r_b}{r_h} \gg 1$ limits, we get

$$\mu(0) = \frac{9\dot{\alpha}m}{32\pi l^4 T^2} = \frac{1}{\gamma_0} = t_{\text{relax}} \tag{58}$$

4.1.2 Uncharged black hole with the general mass

The uncharged black hole, with $f(r)$ defined through relation (7), can be investigated in a similar way to the conformal black hole in the last section. Let us begin with the tortoise coordinate,

$$r_* = \frac{l^2}{r_h} \left[\frac{\ln(\rho-1)}{(a-1)(b-1)} + \frac{a \ln(\rho-a)}{(a-b)(a-1)} - \frac{b \ln(\rho-b)}{(a-b)(b-1)} \right], \tag{59}$$

with $a = \frac{r_1}{r_+}$ and $b = \frac{r_2}{r_+}$. Thus, we find our solutions in this regime near the horizon to have the following form:

$$\psi^{(out/in)}(t, r) \sim e^{-i\omega(t \pm r_*)} \sim e^{-i\omega(t \pm \frac{\omega l^2 \ln(\rho-1)}{r_+(a-1)(b-1)}}. \tag{60}$$

In the regime (II), from Eq. (54) for this kind of black hole, one can obtain

$$\begin{aligned} g_v^0(\rho) &= B_1 + B_2 \left[\frac{-\ln(\rho)}{ab} + \frac{\ln(\rho-1)}{(a-1)(b-1)} \right. \\ &\quad \left. + \frac{\ln(\rho-a)}{a(a-b)(a-1)} - \frac{\ln(\rho-b)}{b(a-b)(b-1)} \right]. \end{aligned} \tag{61}$$

By expanding (61) and comparing it with the above relation in the limit $\rho \rightarrow 1$, B_1 and B_2 are easily derived as follows:

$$B_1 = 1 \mp \frac{iv}{(a-b)} \left[\frac{\ln(1-a)}{a(a-1)} - \frac{\ln(1-b)}{b(b-1)} \right], \quad B_2 = \pm iv. \tag{62}$$

For solutions in the regime (III), we use Eq. (46) and obtain similar solutions for the conformal black hole. As before, we expand the solutions in the regime (II) for $\rho \rightarrow \infty$ and compare them with solutions in the regime (III) to get coefficients C_1 and C_2 ; then we have

$$C_1 = B_1, \quad C_2 = \mp \frac{1}{v^2}. \tag{63}$$

Finally, our asymptotic solutions, in the low frequency limit can be obtained as

$$\begin{aligned} g_v^{(out/in)}(\rho) &\sim 1 \mp \frac{iv}{(a-b)} \left[\frac{\ln(1-a)}{a(a-1)} - \frac{\ln(1-b)}{b(b-1)} \right] \\ &\quad \times \left(1 + \frac{v^2}{2\rho^2} \right) \mp \frac{iv}{3\rho^3}. \end{aligned} \tag{64}$$

Now, if we consider the case that an external force is imposed on the Brownian particle, we can derive the admittance by using the relation (57); therefore we get

$$\mu(0) = \frac{2\pi\dot{\alpha}'m}{r_+^2}, \tag{65}$$

where we assumed the zero frequency and the $\frac{r_b}{r_+} \gg 1$ limit. In the above relation, the parameter r_+ is related to the temperature and mass of the black hole through the following equation:

$$4\pi T r_+^3 - 3M r_+^2 = -M^2 l^2. \tag{66}$$

4.1.3 Strings in the charged hairy black hole

In the case of a charged hairy black hole, the method of solving the equation of motion is the same as in the last section. It means that we must work in the same three regimes as introduced before. The first step is to derive the tortoise coordinate from the relation (28). The behavior of this parameter near the horizon helps to derive the solutions in the regime (I). In the next step we obtain solutions in the regime (II), through

the relation (44). By expanding these solutions near the horizon and matching them with the solutions in regime (I), one can obtain the exact solutions in regime (II). The final step is to derive the asymptotic solutions from expanding the solutions in the regime (II) for $\rho \rightarrow \infty$ and comparing them with Eq. (46) in regime (III). In working through the above processes for a charged hairy black hole, we encountered some problems. In fact, solving the integrals (28) and (44) for $f(r)$ defined through (3) is rather difficult. It seems that for this kind of black hole, we will achieve a similar relation for the admittance, i.e., $\mu(0) = \frac{2\pi\alpha'm}{r_+^2}$. So, the admittance parameter will be related to the charge, mass, and temperature of the black hole (e.g. Eq. (6) for the charged BTZ black hole). This statement is an opinion and we would like to confirm it in future work.

4.2 Displacement square and the fluctuation–dissipation theorem

This section is devoted to the computation of the displacement squared of the external quark and to the investigation of the fluctuation–dissipation theorem. The knowledge of the computation of the displacement squared is to be found in Sect. 3 for an arbitrary background. We use this knowledge for our hairy metric backgrounds. To start with, we consider the case that the fluctuating electric fields are turned off, i.e., the case of free Brownian motion. According to (53) this is

$$\frac{r_h f(\rho)\rho^2 \partial_\rho \psi}{2\pi\alpha'} = 0 \quad \text{at} \quad \rho = \rho_b. \tag{67}$$

We know that $\psi(t, r)$ can be written as the sum of outgoing and ingoing modes (recall Eq. (33)),

$$\psi(t, \rho) = \eta \left[g^{(\text{out})}(\rho) + \delta g^{(\text{in})}(\rho) \right] e^{-i\omega t}. \tag{68}$$

From (67), it is determined that

$$\delta = -\frac{g^{(\text{out})'}}{g^{(\text{in})'}}; \tag{69}$$

δ is a pure phase and $g^{(\text{out})} = g^{(\text{in})*}$. For hairy black holes with the metric background (2), we have $\eta = \sqrt{\frac{4\pi\alpha'}{\omega r_+^2}}$ and for uncharged black holes, to leading order in the frequency one finds $\delta = 1$ and then $|g^{(\text{out})} + \delta g^{(\text{in})}|^2 = 1 + O(v^2)$. Thus, we deduce the late-time behavior of the displacement squared from the low frequency limit of (37), i.e.,

$$S^2(t) = \frac{16\alpha'}{\beta r_+^2} \int_0^\infty d\omega \frac{\sin^2(\frac{\omega t}{2})}{\omega^2} \sim \frac{4\pi\alpha'}{\beta r_+^2} t. \tag{70}$$

Thus, we find that the diffusion constant defined as in (17) is given by

$$D = \frac{2\pi\alpha'T}{r_+^2}. \tag{71}$$

In the special case of conformal black hole, Eq. (70) reduces to

$$S^2(t) \sim \frac{9\alpha'}{16l^4\pi T} t, \tag{72}$$

and then $D = \frac{9\alpha'}{32l^4\pi T}$. For a general mass in the uncharged black hole, the radius of the horizon depends on the mass and temperature of the black hole, so, in this case, the diffusion constant has a dependence on mass and temperature of the black hole.

Now we have come to the end of this section, we turn our attention to an investigation of the fluctuation–dissipation theorem, see Eq. (16). In order to check this, we compute the random force autocorrelation in (23) to obtain the coefficient κ_0 . From (36) we can derive the correlator of the momentum as

$$\begin{aligned} \langle : p(t)p(0) : \rangle &= \int_0^\infty \frac{d\omega}{2\pi} \frac{2m^2\omega^2|\eta^2| \cos(\omega t)}{e^{\beta\omega} - 1} \\ &= \int_{-\infty}^\infty \frac{d\omega}{2\pi} \frac{4\pi T m^2 \alpha' \beta |\omega| e^{-i\omega t}}{r_+^2 (e^{\beta\omega} - 1)}. \end{aligned} \tag{73}$$

Therefore,

$$I_p(\omega) = \frac{4\pi T m^2 \alpha' \beta |\omega|}{r_+^2 (e^{\beta\omega} - 1)}. \tag{74}$$

By combining this with (65), at leading order one finds that

$$I_R(\omega) = \frac{r_+^2 T}{\pi\alpha'} + O(\omega); \tag{75}$$

this gives the coefficient $\kappa_0 = \frac{r_+^2 T}{\pi\alpha'}$, which in the special case of a conformal black hole is equal to $\kappa_0 = \frac{64\pi l^4 T^3}{9\alpha'}$. By checking Eq. (16), we see that the fluctuation–dissipation theorem holds in the plasma where the corresponding gravity is for a three dimensional uncharged hairy black hole. In the case of a charged hairy black hole, as for the uncharged black hole, we think that this theorem also holds. Because it seems that in the low frequency limit the $g(\rho)$ function for a charged black hole has a similar behavior to the $g(\rho)$ function for an uncharged black hole.

5 Conclusion

In this paper, we obtain the normalized asymptotic solutions (including outgoing and ingoing modes) to the equation of motion of uncharged hairy black hole at low frequencies. By

using those solutions, we derive the response function and correlation function for an uncharged black hole with general mass and special mass $M = \frac{3B^2}{l^2}$ separately. We found that the admittance and diffusion constant are dependent on the scalar parameter B and the mass of the black hole through the radius of the horizon. We prove that the fluctuation–dissipation theorem holds in the plasma where the corresponding gravity is for a three dimensional uncharged hairy black hole. In the case of a charged hairy black hole, we cannot get an explicit solution to the equation of motion, but we think that its behavior of as regards the asymptotic solution is similar to the uncharged case in the low frequency limit. It means that the dependence of the admittance and the diffusion constant on the radius of the horizon is as before and in the case of a charged black hole is related to the scalar parameter, charge, and mass of the black hole. This statement should be confirmed in future work.

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