Zhao and Yao *Boundary Value Problems* (2015) 2015:221
DOI 10.1186/s13661-015-0481-7

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Optimal boundary conditions for the Navier-Stokes fluid in a bounded domain with a thin layer

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Abstract

In this paper, we derive the optimal boundary conditions of the Navier-Stokes fluid in a bounded domain with a thin layer. We get the effects of the layer by investigating the limiting δ, the thickness of the layer shrinking to zero. In the two-dimensional case, we derive effective boundary conditions by the limit of u and p on the boundary of the uncoated body.

MSC: 45M05; 76D07; 76M50

Keywords: homogenization; Stokes fluid; boundary conditions

1 Introduction

We consider the mathematical problem arising from fluids by applying a thin layer of special area to a body to be protected. The body is assumed to be a two-dimensional smooth and bounded domain, denoted by Ω_1. The layer, denoted by Ω_2, has a thickness of $\delta \ll 1$. Then the whole domain $\Omega = \bar{\Omega}_1 \cup \Omega_2$.

We study the Navier-Stokes fluid of the coated body

$$\begin{cases} \frac{\partial u^\delta}{\partial t} + (u^\delta \cdot \nabla)u^\delta - \operatorname{div}(\mathcal{A}\nabla u^\delta) + \nabla p^\delta = f(x,t) & \text{in } \Omega, \\ \operatorname{div} u^\delta = 0 \quad \text{in } \Omega, \\ u^\delta|_{t=0} = u_0^\delta(x), \qquad u^\delta|_{\partial\Omega} = 0, \quad \text{for } t > 0, \end{cases} \tag{1}$$

where u^δ and p^δ are the velocity and the pressure of the fluid, respectively. $f(x,t)$ is the force term. $\mathcal{A}$ is a 2×2 matrix, which is positive and symmetry. For simplicity, we assume the interior body Ω_1 is homogeneous. More precisely, we assume that

$$A(x) = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \forall x \in \Omega_1. \tag{2}$$

Within the coating Ω_2, we assume the tensor $\mathcal{A}$ takes the form

$$\mathcal{A}(x)\boldsymbol{\nu} = \sigma\boldsymbol{\nu}, \qquad \mathcal{A}(x)\tau = \mu\tau, \quad \forall x \in \Omega_2, \tag{3}$$

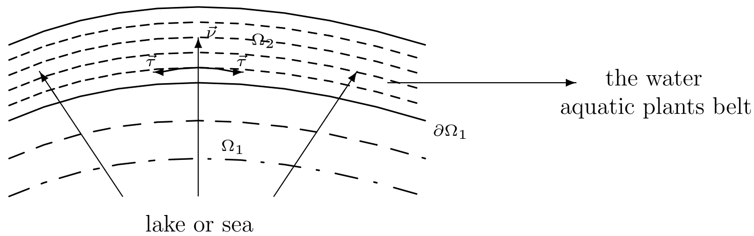

Figure 1 A model of a lake or a sea near the shore.

where σ and μ are positive constants, $\boldsymbol{\nu} = \nabla d(x)$ is the unit outward normal to $\partial\Omega$ and $\tau \perp \boldsymbol{\nu}$ is a unit tangent vector. Note that (3) implies

$$\mathcal{A}(x) = \sigma \boldsymbol{\nu} \otimes \boldsymbol{\nu} + \mu\tau \otimes \tau.$$

This model describes an incompressible Navier-Stokes fluid in a composed domain, which contains two different parts, a microscopic one (Ω_2) and a macroscopic one (Ω_1). In the macroscopic part, the fluid is considered to be isotropic. The microscopic part is anisotropic and surrounds the macroscopic part and has the thickness δ, which is small when compared to the size of Ω_1. The real world applications may include lake or sea shore protection and desert stabilization: to protect a shoreline from erosion, along the shoreline in the water aquatic plants and trees may be planted, and large rocks and concrete blocks may be placed (see Figure 1); similarly, along the edge of a desert trees may be planted to prevent the growth of the desert. If these barriers are placed periodically and parallel to the shoreline/desert edge, then after homogenization the viscous tensor in the thin layer Ω_2 takes the form of $\mathcal{A}(x)\nabla u^\delta$ with $\mathcal{A}(x)$ satisfying (4).

In the three-dimensional case of an anisotropic fluid, the viscous tensor, as a symmetric 3×3 matrix, has three orthogonal eigenvectors and three eigenvalues. Each of these eigenvalues measures the impact rate of the anisotropic fluid in the corresponding eigendirection. To protect $\partial\Omega_1$ from eroding, it is desirable that the eigenvector corresponding to the smallest eigenvalue is orthogonal to the boundary of the body in order to directly confront the ambient effect. The other eigenvalues (in the tangential directions) may not need to be small. Motivated by these considerations, in this paper, we introduce the notion of 'optimally aligned layer' (a similar conception as in [1]), by which we mean that at every point $x \in \Omega_2$, the vector normal to $\partial\Omega_1$ at the projection of x is always an eigenvector of the viscous tensor $\mathcal{A}(x)$.

The static case of such a problem was studied by Rosencrans *et al.* [1–4] and Li [5] for the heat equation on a coated body under different boundary conditions. In fluid mechanics, Miksis and Davis [6] studied a Navier-Stokes fluid flows over another one over rough and coated surfaces. Jäger and Mikelic [7] considered the boundary conditions on the contact interface between a porous medium and a free fluid. Kohn and Vogelius [8, 9] studied the effective models in a thin plate with rapidly varying thickness, he found that those models are different according to the different scaling. Braides *et al.* [10] got the effective results on inhomogeneous thin layers in 2D and 3D. Ansini and Braides [11] applied their effective results on oscillating boundary to thin films. Other important work on the domain coarsening in thin layers was done by Niethammer and Otto [12]. Moreover, the effect of the strong interaction of two-dimensional, long, internal gravity waves propagating on neighboring pycnoclines in a stratified fluid was recently studied in [13]. For other work, we refer the interested reader to [14–16] and the references therein.

In this paper, we considered an incompressible fluid in a bounded domain with a thin layer in 2D. The aim of this paper is to find the homogenized model of (1) and corresponding optimal boundary conditions as the thickness δ tends to zero. The main difficulty we met here is to investigate the optimal boundary conditions for the homogenized model. To this end, we define a boundary layer equations between the macroscopic domain and the coating. We obtained the optimal boundary conditions by using a special test function constructed by the boundary layer solution.

The paper is organized as follows. In Section 2, we review some classical results and state our main results. In Section 3, we give the local parameter transformation. In Section 4, we prove the main results in this paper.

2 Preliminaries and the main results

In this paper, we assume that the boundaries of $\partial\Omega_1$ and Ω are both smooth enough when $\delta > 0$ is sufficiently small. Then we assume that the distance function $d(x)$, the unit normal vector $\nu(x)$, and the unit tangent vector $\tau(x)$ are also smooth enough on $\bar{\Omega}_2$. We denote the time-space domain by $\Omega_T = \Omega \times (0, T)$.

Let $\mathcal{D}(\Omega)$ be the space of $\mathcal{C}^\infty$ functions with compact support contained in Ω. We define the space

$$\begin{aligned} &\mathcal{V} = \{u \in \mathcal{D}(\Omega), \operatorname{div} u = 0\}, \\ &\mathcal{H} = \{\text{the closure of } \mathcal{V} \text{ in } L^2(\Omega)\}, \\ &\mathcal{W} = \{\text{the closure of } \mathcal{V} \text{ in } H_0^1(\Omega)\}. \end{aligned}$$

Throughout the paper, we make assumptions on Ω, $\mathcal{A}(x)$ and $f(x)$ as follows:

1. Ω is bounded, open and convex with smooth boundary (at least C^2);
2. $\mathcal{A}(x) \in C^2(\Omega)$;
3. $f(x) \in L^2(\Omega_T)$.

Definition 2.1 A function pair $\{u^\delta, p^\delta\} \in L^2(0, T; \mathcal{W}) \times L^2(0, T; H^1(\Omega))$ is said to be a weak solution to (1) if

$$\int_\Omega u^\delta \cdot \eta + \int_{\Omega_T} \left(-u^\delta \cdot \eta_t - u^\delta \otimes u^\delta : \nabla\eta + \nabla\eta : \mathcal{A}\nabla u^\delta\right) = \int_{\Omega_T} f \cdot \eta,$$

for all $\eta \in L^2(0, T; \mathcal{V})$ with $\eta(x, t) = 0$ at $t = 0, T$.

Lemma 2.1 *Let $\Omega \subset R^2$ be an open bounded domain. $u_0^\delta \in L^2(\Omega)$ and $f \in L^2(\Omega_T)$. Then up to a constant,* (1) *determines a unique weak solution pair $\{u^\delta, p^\delta\}$ in the sense of Definition* 2.1 *and the following estimates hold*:

$$\text{(i)} \quad \max_{t\in[0,T]} \int_\Omega |u^\delta|^2\,dx + 2\int_{\Omega_T} \nabla u^\delta : \mathcal{A}\nabla u^\delta\,dx\,dt \le C;$$

$$\text{(ii)} \quad \int_{\Omega_T} |u_t^\delta|^2\,dx\,d\tau + \max_{t\in[0,T]} \int_\Omega \nabla u^\delta : \mathcal{A}\nabla u^\delta\,dx \le C;$$

(iii) $\int_{\Omega_T} t\left|u_t^\delta\right|^2 \,dx\,d\tau + \max_{t\in[0,T]} t \int_\Omega \nabla u^\delta : \mathcal{A}\nabla u^\delta \,dx \le C$;

(iv) $\|p^\delta\|_{L^2(\Omega\times(0,T))} \le C$,

where C is independent of δ.

Proof It is obvious that the tensor $\mathcal{A}(x)$ is positive, symmetric, and coercive. Note that $\partial\Omega$ is sufficiently smooth, the energy estimates can be proved formally multiplying (1) by u^δ for (i), u_t^δ for (ii), and tu_t^δ for (iii). To prove (iv), we use the div and curl decomposition. Let P and Q be two orthogonal operators. For all f, we have

$$f = P(f) + Q(f),$$

where P maps the divergence-free functions to themselves. Then we can write (1) in the following way:

$$u_t^\delta + P\big(\big(u^\delta\cdot\nabla\big)u^\delta\big) - P\big(\operatorname{div}\big(\mathcal{A}\nabla u^\delta\big)\big) = P(f).$$

The classical result (see [17]) shows that u_t^δ is bounded in $L^2(0,T;\mathcal{W}')$. By (1), we see that ∇p is bounded in $L^2(0,T;H^{-1}(\Omega))$. Finally, by using the Necćas inequality (see[18], Chapter 1, Proposition 1.2), we see that p is bounded in $L^2(\Omega\times(0,T))$ and (iv) holds.

From those *a priori* estimates, by using the Faedo-Galerkin method, we can establish the existence result. □

Now, we state our main results in this paper.

Theorem 2.1 *Under the hypotheses of Lemma* 2.1, *suppose $\mathcal{A}(x)$ is given by* (2) *or* (3) *and u_0^δ converges strongly to $u_0(x)$ in the corresponding space. We also assume that $\sigma(\delta)\ge\mu(\delta)\ge\delta$. Let $\{u^\delta,p^\delta\}$ be the solution pair of* (1). *Then up to a constant, there exists a unique function pair $\{u,p\}$ such that*

$$u^\delta\to u \quad \textit{weakly in } L^2\big(0,T;H^1(\Omega_1)\big), \qquad p^\delta\to p \quad \textit{weakly in } L^2\big(\Omega_1\times(0,T)\big)$$

as $\delta\to0$, where $\{u,p\}$ is the solution pair of the following homogenized problem:

$$\begin{cases} u_t + (u\cdot\nabla)u - \triangle u + \nabla p = f(x,t) \quad \textit{in } \Omega_1\times(0,T),\\ \operatorname{div} u = 0 \quad \textit{in } \Omega_1\times(0,T),\\ u|_{t=0} = u_0(x), \quad \forall x\in\Omega_1, \end{cases} \tag{4}$$

together with the following homogenized boundary conditions:

1. *If* $\lim_{\delta\to0}\frac{\sigma^{\frac{3}{2}}}{\delta} = 0$, *the boundary condition would be*

$$\frac{\partial u}{\partial\nu} - (u\otimes u + pI)\nu = 0 \quad \textit{on } \partial\Omega_1. \tag{5}$$

2. *If* $\lim_{\delta\to0}\frac{\sigma^{\frac{3}{2}}}{\delta} = \alpha\in(0,+\infty)$,

2(i). *if* $\lim_{\delta\to 0}\frac{\mu}{\sigma}=0$, *the boundary condition would be*

$$\frac{\partial u}{\partial \nu}-(u\otimes u+pI)\nu=\alpha\nu\cdot ue_2 \quad \text{on } \partial\Omega_1; \tag{6}$$

2(ii). *if* $\lim_{\delta\to 0}\frac{\mu}{\sigma}=\beta\in(0,1]$, *the boundary condition would be*

$$\frac{\partial u}{\partial \nu}-(u\otimes u+pI)\nu=\alpha\sqrt{\beta}\tau\cdot ue_1+\alpha\nu\cdot ue_2 \quad \text{on } \partial\Omega_1. \tag{7}$$

3. *If* $\lim_{\delta\to 0}\frac{\sigma^{\frac{3}{2}}}{\delta}=+\infty$,

3(i). *if* $\lim_{\delta\to 0}\frac{\mu}{\sigma}=0$, *the boundary condition would be*

$$u(x,t)\cdot\nu=0 \quad \text{on } \partial\Omega_1; \tag{8}$$

3(ii). *if* $\lim_{\delta\to 0}\frac{\mu}{\sigma}=\beta\in(0,1]$, *the boundary condition would be*

$$u(x,t)=0 \quad \text{on } \partial\Omega_1. \tag{9}$$

Theorem 2.2 *Under the hypotheses of Lemma* 2.1, *suppose* $\mathcal{A}(x)$ *is given by* (2) *or* (3) *and* u_0^δ *converges strongly to* $u_0(x)$ *in the corresponding space. We assume that* $\delta\le\sigma(\delta)\le\mu(\delta)\le M\sigma(\delta)$ *for some constants* $M>0$. *Let* $\{u^\delta,p^\delta\}$ *be the solution pair of* (1). *The function pair* $\{u,p\}$ *is defined in Theorem* 2.1. *Then* (4) *still holds as* $\delta\to 0$ *together with the following homogenized boundary conditions*:

1. *If* $\lim_{\delta\to 0}\frac{\mu^{\frac{3}{2}}}{\delta}=0$, *the optimal boundary condition would be*

$$\frac{\partial u}{\partial \nu}-(u\otimes u+pI)\nu=0 \quad \text{on } \partial\Omega_1. \tag{10}$$

2. *If* $\lim_{\delta\to 0}\frac{\mu^{\frac{3}{2}}}{\delta}=\alpha\in(0,+\infty)$,

2(i). *if* $\lim_{\delta\to 0}\frac{\sigma}{\mu}=0$, *the optimal boundary condition would be*

$$\frac{\partial u}{\partial \nu}-(u\otimes u+pI)\nu=0 \quad \text{on } \partial\Omega_1; \tag{11}$$

2(ii). *if* $\lim_{\delta\to 0}\frac{\sigma}{\mu}=\beta\in(0,1]$, *the optimal boundary condition would be*

$$\frac{\partial u}{\partial \nu}-(u\otimes u+pI)\nu=\beta\tau\cdot ue_1+\beta^{\frac{3}{2}}\nu\cdot ue_2 \quad \text{on } \partial\Omega_1. \tag{12}$$

3. *If* $\lim_{\delta\to 0}\frac{\mu^{\frac{3}{2}}}{\delta}=+\infty$,

3(i). *if* $\lim_{\delta\to 0}\frac{\sigma}{\mu}=0$, *the optimal boundary condition would be*

$$u(x,t)\cdot\tau=0 \quad \text{on } \partial\Omega_1; \tag{13}$$

3(ii). *if* $\lim_{\delta\to 0}\frac{\sigma}{\mu}=\beta\in(0,1]$, *the optimal boundary condition would be*

$$u(x,t)=0 \quad \text{on } \partial\Omega_1. \tag{14}$$

Remark 2.1 The results in Theorem 2.1 show that boundary conditions on $\partial\Omega_1$ mainly depend on the relationship between δ and σ. More precisely, there exists a critical index between δ and σ ($\delta \backsim \sigma^{\frac{3}{2}}$). The same result was found by Mikelic [19, 20], where the authors considered effective models in a domain with a thin layer between two porous media.

Definition 2.2 A function pair $\{u,p\}$ is said to be a weak solution to (4) if $(u,p) \in L^2(0,T;\mathcal{W}) \times L^2(\Omega_1 \times (0,T))$ and satisfies

$$\mathcal{L}(u,\eta) = -\int_{\Omega_1} u_0(x)\cdot\eta(x,0) + \int_0^T \int_{\Omega_1} \bigl(-u\cdot\eta_t - u^\delta \otimes u^\delta : \nabla\eta + \nabla u : \nabla\eta - f\cdot\eta\bigr) = 0,$$

for any $\eta \in C^\infty(\bar{\Omega}_1 \times (0,T])$ with $\operatorname{div}\eta = 0$. Also, u satisfies the following weak boundary conditions:

1. $\frac{\partial u}{\partial \nu} - (u\otimes u + pI)\nu = 0$ on $\partial\Omega_1$ if

$$\mathcal{L}(u,\eta) = 0. \tag{15}$$

2. $\frac{\partial u}{\partial \nu} - (u\otimes u + pI)\nu = \alpha u\cdot \nu e_2$ or $\frac{\partial u}{\partial \nu} - (u\otimes u + pI)\nu = \alpha\sqrt{\beta}\tau\cdot ue_1 + \alpha\nu\cdot ue_2$ on $\partial\Omega_1$ (resp. $\frac{\partial u}{\partial \nu} - (u\otimes u + pI)\nu = \beta\tau\cdot ue_1 + \beta^{\frac{3}{2}}\nu\cdot ue_2$) if

$$\begin{aligned} &\mathcal{L}(u,\eta) = -\int_0^l \alpha u\cdot\nu e_2\eta\,dr \quad\text{or}\quad \mathcal{L}(u,\eta) = -\int_0^l (\alpha\sqrt{\beta}\tau\cdot ue_1 + \alpha u\cdot\nu e_2)\eta\,dr \\ &\left(\text{resp. } \mathcal{L}(u,\eta) = -\int_0^l \bigl(\beta\tau\cdot ue_1 + \beta^{\frac{3}{2}}\nu\cdot ue_2\bigr)\eta\,dr\right). \end{aligned} \tag{16}$$

3. $u = 0$ or $u\cdot\nu = 0$ (resp. $u\cdot\tau = 0$) on $\partial\Omega_1$ if

$$\begin{aligned} &\int_0^l (\sqrt{\beta}\tau\cdot ue_1 + \nu\cdot ue_2)\cdot\eta\,dr = 0 \quad\text{or} \\ &\int_0^l \nu\cdot ue_2\cdot\eta\,dr = 0 \qquad \left(\text{resp. } \int_0^l \tau\cdot ue_2\cdot\eta\,dr = 0\right). \end{aligned} \tag{17}$$

Theorem 2.3 *Suppose $\partial\Omega_1$ is smooth enough* (*at least $\partial\Omega_1 \in C^2$ for example*), $u_0 \in L^2(\Omega_1)$ *and $f \in L^2(\Omega_T)$. Then* (4) *with any of the boundary condition* (5)$\backsim$(9) (*resp.* (10)$\backsim$(14)) *has one weak solution in the sense of Definition* 2.2.

Proof To establish the existence result, it is sufficient give the *a priori* estimates on different boundary conditions.

1. $\frac{\partial u}{\partial \nu} - (u\otimes u + pI)\nu = 0$ on $\partial\Omega_1$.

Multiplying (4) by u and integrating over $\Omega_1 \times (0,t)$, we have

$$\int_{\Omega_1} |u|^2\,dx + 2\int_0^t\int_{\Omega_1} |\nabla u|^2\,dx\,d\tau = \int_{\Omega_1} |u_0|^2\,dx + 2\int_0^t\int_{\Omega_1} f\cdot u\,dx\,d\tau.$$

We obtain the *a priori* estimate $u \in L^\infty(0,T;L^2(\Omega_1)) \cap L^2(0,T;H^1(\Omega_1))$ by the Young inequality and the Fredrich inequality.

2. $\frac{\partial u}{\partial \nu} - (u\otimes u + pI)\nu = \alpha u\cdot\nu e_2$ on $\partial\Omega_1$.

Multiplying (4) by u and integrating over $\Omega_1 \times (0,t)$, we have

$$\begin{aligned}&\int_{\Omega_1} |u|^2\,dx + 2\int_0^t \int_{\Omega_1} |\nabla u|^2\,dx\,d\tau \\ &\quad\leq \int_{\Omega_1} |u_0|^2\,dx + 2\alpha\int_0^t \int_{\partial\Omega_1} |u|^2\,dS\,d\tau + 2\int_0^t \int_{\Omega_1} f\cdot u\,dx\,d\tau.\end{aligned}$$

To obtain the *a priori* estimate, we only consider the second term on the right-hang side. For each $\varepsilon_0 > 0$, by the boundary trace imbedding theorem (see [21], Chapter 5, (5.36)) and the interpolation inequality (see [21], Chapter 5, (5.2)), we have

$$\|u\|_{L^2(\partial\Omega_1)} \leq K_1\|u\|_{W^{\frac{1}{2},2}(\Omega_1)} \leq K\bigl(\varepsilon\|\nabla u\|_{L^2(\Omega_1)} + \varepsilon^{-\frac{1}{4}}\|u\|_{L^2(\Omega_1)}\bigr),$$

for any $0 < \varepsilon < \varepsilon_0$.

Choosing ε small enough, we obtain the same estimate as above by Gronwall's inequality. By the boundary condition $\frac{\partial u}{\partial \nu} - (u\otimes u + pI)\nu = \alpha\sqrt{\beta}\tau\cdot ue_1 + \alpha\nu\cdot ue_2$, we can get the same estimate following the steps as above.

3. $u = 0$ or $u\cdot\nu = 0$ on $\partial\Omega_1$.

The same estimate can be obtained by multiplying u in (4) and integrating over $\Omega_1 \times (0,t)$.

With this estimate, we can construct an approximate solution by the Faedo-Galerkin method and follow the steps in [17] (Chapter 1, (6.4)) to obtain the existence result. □

3 The parameter transformation model

To investigate the boundary conditions on $\partial\Omega_1$, it is convenient to write the system in another coordinate system. We introduce the parameter transformation as [1].

3.1 The parameter transformation

Before proving the main theorems, we introduce the parameter transformation near $\partial\Omega_1$. Let $l = |\partial\Omega_1|$ be the arc-length of $\partial\Omega_1$. By using the arc-length $r \in [0,l)$, we define the mapping

$$\mathbf{p}(r) : [0,l) \to \partial\Omega_1,$$

in the contour-clockwise fashion.

For all small $\delta > 0$, we also define the mapping

$$x = X(r,s) = \mathbf{p}(r) + s\boldsymbol{\nu}(r), \quad (r,s) \in [0,l) \times (-\delta,\delta),$$

where $\boldsymbol{\nu}(r)$ is the unit outer normal vector of $\partial\Omega_1$ at $\mathbf{p}(r)$. It is well known that X is a diffeomorphism. We also have

$$\Omega_2 = X\bigl([0,l) \times (0,\delta)\bigr).$$

We denote by (r,s) the inverse of the mapping $x = X(r,s)$. We denote by $\tau(r)$ the unit tangent vector $\frac{d\mathbf{p}}{dr}$ of $\partial\Omega_1$ at $\mathbf{p}(r)$, and write the curvature of $\partial\Omega_1$ at $\mathbf{p}(r)$ as $\kappa(r)$. Then we have the Frenet equations,

$$\boldsymbol{\tau}'(r) = -\kappa(r)\boldsymbol{\nu}(r), \qquad \boldsymbol{\nu}'(r) = \kappa(r)\boldsymbol{\tau}(r),$$
$$X_r(r,s) = \bigl(1 + s\kappa(r)\bigr)\boldsymbol{\tau}(r), \qquad X_s(r,s) = \boldsymbol{\nu}(r).$$

We assume that $X(r,s)$ is l-periodic in r. It is obvious that

$$dx = (1 + s\kappa)\,dr\,ds, \qquad X(r,s) = X(x + l, s).$$

Near $\partial\Omega_1$, in the (r,s) coordinate system, we have

$$\begin{cases} u_t^\delta = \frac{\tilde{\mu}}{1+s\kappa(r)}(\frac{u_r^\delta}{1+s\kappa(r)})_r + \frac{\tilde{\sigma}}{(1+s\kappa(r))}(1 + s\kappa(r)u_s^\delta)_s \\ \qquad + \frac{u_r^\delta}{1+s\kappa}(\tau \cdot u^\delta) + u_s^\delta(\nu \cdot u^\delta) \\ \qquad - \frac{\tau}{1+s\kappa(r)}p_r^\delta - \boldsymbol{\nu}p_s^\delta + f(r,s,t), \\ \boldsymbol{\tau} \cdot u_r^\delta + (1 + s\kappa(r))\boldsymbol{\nu} \cdot u_s^\delta = 0, \end{cases} \tag{18}$$

complemented with the boundary condition

$$u^\delta|_{s=\delta} = 0, \tag{19}$$

where

$$\tilde{\mu} = \begin{cases} 1 & \text{in } \Omega_1, \\ \mu & \text{in } \Omega_2, \end{cases} \qquad \tilde{\sigma} = \begin{cases} 1 & \text{in } \Omega_1, \\ \sigma & \text{in } \Omega_2. \end{cases}$$

Moreover, $u^\delta(r,s,t)$ is l-periodic in r, namely,

$$u^\delta(r,s,t) = u^\delta(r + l, s, t).$$

Lemma 3.1 *Let $u_0^\delta(r,s) \in L^2(\Omega)$ and δ be small enough. Then*

$$\int_0^T \int_0^l \int_0^\delta \bigl(\tilde{\mu}\bigl|u_r^\delta\bigr|^2 + \tilde{\sigma}\bigl|u_s^\delta\bigr|^2\bigr)\,ds\,dr\,dt \le C,$$

where C does not depend on δ.

Proof By Lemma 2.1(i) and the parameter transformation, the lemma can be proved. □

3.2 Boundary layer problem

A classical way of finding boundary conditions is by using the matched asymptotic method (see [22]). The main idea is to supplement the problem by an inner system in which the independent variables are stretched out in order to capture the behavior in the neighborhood of the boundary. As for our problem, this approach uses the limit rule, by which asymptotic behavior of the outward in the neighborhood of the boundary has to be equal to asymptotic behavior of the inner.

Let $g(r,t) \in C_0^\infty(\bar{\Omega}_1 \times [0,+\infty))$ be a solenoidal field with compact support. We extend it to the domain $\Omega \times [0,+\infty)$ via the functions φ and π defined by the following system:

$$\begin{cases} -\mu\varphi_{rr} - \sigma\varphi_{ss} + \mu\mathbb{B}\nabla\pi = 0 \quad \text{in } R \times (0,\delta), \\ \sqrt{\mu}\varphi_{1r} + \sqrt{\sigma}\varphi_{2s} = 0, \quad \forall r \in R \\ \varphi(r,0,t) = g(x(r),t), \qquad \varphi(r,\delta,t) = 0, \qquad \int_{R\times(0,\delta)} \pi \, dr \, ds = 0, \end{cases} \tag{20}$$

where φ, π are l-periodic in r and $\mathbb{B} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \sqrt{\frac{\sigma}{\mu}} \end{smallmatrix}\right)$.

Suppressing the t dependence, we write

$$\Phi(r,s) = \varphi\left(r, s\sqrt{\frac{\sigma}{\mu}}, t\right), \qquad \Pi(r,s) = \pi\left(r, s\sqrt{\frac{\sigma}{\mu}}, t\right), \qquad h := \delta\sqrt{\frac{\mu}{\sigma}}.$$

Then Φ and Π satisfy the following system:

$$\begin{cases} -\Phi_{rr} - \Phi_{ss} + \nabla\Pi = 0 \quad \text{in } R \times (0,h), \\ \Phi_{1r} + \Phi_{2s} = 0, \qquad \Phi(r,h) = 0, \qquad \Phi(r,0) = g(r,t), \qquad \int_{R\times(0,\delta)} \Pi \, dr \, ds = 0, \end{cases} \tag{21}$$

where Φ, Π are also l-periodic in r.

The existence result and the regularity result are given by the following lemma (see [23]).

Lemma 3.2 *There exists a unique solution pair* $\{\Phi,\Pi\} \in C^\infty(R\times(0,h)) \times C^\infty(R\times(0,h))$ *to* (10) *such that*

$$\Phi \in C^m\big(R\times(0,h)\big), \qquad \Pi \in C^m\big(R\times(0,h)\big),$$

for any integer $m > 0$. *Moreover, the following estimates hold*:

$$\|\Phi\|_{C^m(R\times(0,h))} \le C\|g\|_{C^m(R\times(0,h))},$$
$$\|\Pi\|_{C^m(R\times(0,h))} \le C\|g\|_{C^m(R\times(0,h))}.$$

For any $0 < h \le 1$, we have

$$\begin{aligned} \left|\Phi_s(r,0) + \frac{g(r)}{h}\right| &= \left|\frac{1}{h}\int_0^h \big[\Phi_s(r,0) - \Phi_s(r,s)\big]\,ds\right| \\ &\le h\big\|\Phi_{ss}(r,s)\big\|_{L^\infty(R\times(0,h))} \\ &\le h\big\|g(r)\big\|_{L^\infty(R)}, \end{aligned}$$

which implies that

$$\Phi_s(r,0) = -\frac{1}{h}g(r) + O(h) = \frac{1}{h}\big(-g(r) + O\big(h^2\big)\big).$$

It follows from $\sigma\varphi_s(r,0,t) = \sqrt{\mu\sigma}\,\Phi_s(r,0)$ and the definition of h that

$$\sigma\varphi_s(r,0,t) = \frac{\sigma}{\delta}\big(-g(r) + O\big(h^2\big)\big). \tag{22}$$

4 Proof of main theorems

In this section, we put our emphasis on the proofs of Theorem 2.1 and Theorem 2.2. They contain two parts:

1. Recover the system in Ω_1.
2. Investigate the optimal boundary conditions on $\partial\Omega_1$.

Proof of the first part in Theorem 2.1 *and Theorem* 2.2 By Lemma 2.1 and the standard compactness theorem, we have $u^\delta \rightharpoonup u$ weakly in $L^2(0,T;H^1(\Omega_1))$ and $p^\delta \rightharpoonup p$ weakly in $L^2(\Omega_1 \times (0,T))$ as $\delta \to 0$. Next, we choose the test function

$$\eta(r,s,t) = \begin{cases} g & \text{in } \Omega_1, \\ \sqrt{\mu}\tau\varphi_1 + \frac{\sqrt{\sigma}\nu}{1+s\kappa}\varphi_2 & \text{in } \Omega_2, \end{cases} \tag{23}$$

where $\varphi = (\varphi_1, \varphi_2)^\tau$ is the solution of the boundary layer problem. Note that $\operatorname{div}_{x_1,x_2} \eta(r,s,t) = 0$ in $\Omega_1 \cup \Omega_2$ and $\eta(r,s,t)|_{\partial\Omega} = 0$. Multiplying (1) by $\eta(r,s,t)$ and integrating over $\Omega \times (0,T)$, we have

$$\int_0^T \int_\Omega f \cdot \eta = \int_0^T \int_\Omega \eta \cdot \left[u_t^\delta + \left(u^\delta \cdot \nabla\right)u^\delta - \operatorname{div}\left(\mathcal{A}\nabla u^\delta\right) + \nabla p^\delta\right]. \tag{24}$$

To avoid the influence on $\partial\Omega_1$, we set $g = 0$ on $\partial\Omega_1$, then (15) admits a unique solution $\varphi = 0$. Then we have

$$\int_0^T \int_{\Omega_1} f \cdot g = \int_0^T \int_{\Omega_1} g \cdot \left[u_t^\delta + \left(u^\delta \cdot \nabla\right)u^\delta - \operatorname{div}\left(\mathcal{A}\nabla u^\delta\right) + \nabla p^\delta\right].$$

Passing to the limit $\delta \to 0$ and using the uniform bounds of those quantities, we recover the system and the initial condition

$$u_t + (u \cdot \nabla)u - \triangle u + \nabla p = f \quad \text{in } \Omega_1, \qquad u(x,0) = u_0. \tag{25}$$

At last, by using the same test function, we have

$$0 = \int_{\Omega_1} \operatorname{div} u^\delta g = -\int_{\Omega_1} u^\delta \cdot \nabla g \to -\int_{\Omega_1} u \cdot \nabla g,$$

which implies

$$\operatorname{div} u = 0. \tag{26}$$

The first part is then proved. □

Note that (25) and (26) are not determined because of lacking of boundary conditions. In the following, we investigate the boundary conditions on $\partial\Omega_1$ satisfied by u.

Proof of Theorem 2.1 As in [1], suppressing the t dependence, we write (24) in the following way:

$$I_1^\delta - \int_\Omega f \cdot \eta = I_2^\delta,$$

where

$$I_1^\delta = -\int_{\Omega_1} \eta_t \cdot u^\delta + \int_{\Omega_1} \nabla\eta : \nabla u^\delta - \int_{\Omega_1} \eta_0 \cdot u_0^\delta + \int_{\Omega_1} \nabla p^\delta \cdot \eta + \int_{\Omega_1} \left(u^\delta \cdot \nabla\right) u^\delta \cdot \eta,$$

$$I_2^\delta = \int_{\Omega_2} \eta_t \cdot u^\delta - \int_{\Omega_2} \nabla\eta : \mathcal{A}\nabla u^\delta + \int_{\Omega_2} \eta_0 \cdot u_0^\delta - \int_{\Omega_2} \nabla p^\delta \cdot \eta - \int_{\Omega_2} \left(u^\delta \cdot \nabla\right) u^\delta \cdot \eta.$$

Denote

$$\begin{aligned} B_1^\delta &= -\int_{\Omega_2} \nabla\eta : \mathcal{A}\nabla u^\delta \\ &= -\int_0^l \int_0^\delta (1+s\kappa)\left[\sigma \eta_s u_s^\delta + \frac{\mu}{(1+s\kappa)^2}\eta_r u_r^\delta\right] \\ &= B_{11}^\delta + B_{12}^\delta, \end{aligned} \tag{27}$$

where

$$B_{11}^\delta = -\int_0^l \int_0^\delta \left(\sigma \eta_s u_s^\delta + \mu \eta_r u_r^\delta\right), \qquad B_{12}^\delta = \int_0^l \int_0^\delta s\kappa \left(\frac{\mu}{1+s\kappa}\eta_r u_r^\delta - \sigma \eta_s u_s^\delta\right). \tag{28}$$

Also

$$\begin{aligned} B_2^\delta &= -\int_0^l \int_0^\delta \left[\tau p_r + \nu(1+sk)p_s\right] \cdot \left(\sqrt{\mu}\tau\varphi_1 + \frac{\sqrt{\sigma}\nu}{1+s\kappa}\varphi_2\right) dr\, ds \\ &= -\int_0^l \int_0^\delta (\sqrt{\mu} p_r \varphi_1 + \sqrt{\sigma} p_s \varphi_2)\, dr\, ds \\ &= \sqrt{\sigma} \int_0^l p(r,0,t)\varphi_2(r,0,t)\, dr. \end{aligned} \tag{29}$$

Also

$$\begin{aligned} B_3^\delta &= -\int_0^l \int_0^\delta \left[(1+s\kappa)\sqrt{\mu}\varphi_1\left(u_2^\delta u_{1s}^\delta - u_1^\delta u_{2s}^\delta\right) + \frac{\sqrt{\sigma}\varphi_2}{1+s\kappa}\left(u_1^\delta u_{2r}^\delta - u_2^\delta u_{1r}^\delta\right)\right] dr\, ds \\ &= B_{31}^\delta + B_{32}^\delta, \\ B_{31}^\delta &= -\int_0^l \int_0^\delta \left[\sqrt{\mu}\varphi_1\left(u_2^\delta u_{1s}^\delta - u_1^\delta u_{2s}^\delta\right) + \sqrt{\sigma}\varphi_2\left(u_1^\delta u_{2r}^\delta - u_2^\delta u_{1r}^\delta\right)\right] dr\, ds \\ &= B_{311}^\delta + B_{312}^\delta. \end{aligned} \tag{30}$$

It is obvious that

$$\begin{aligned} &\left|B_{311}^\delta\right| \le C\sqrt{\mu}, \qquad \left|B_{312}^\delta\right| \le C\sqrt{\sigma}, \\ &B_{32}^\delta = -\int_0^l \int_0^\delta s\kappa\left[\sqrt{\mu}\varphi_1\left(u_2^\delta u_{1s}^\delta - u_1^\delta u_{2s}^\delta\right) - \frac{\sqrt{\sigma}\varphi_2}{1+s\kappa}\left(u_1^\delta u_{2r}^\delta - u_2^\delta u_{1r}^\delta\right)\right] dr\, ds \\ &\quad = B_{321}^\delta + B_{322}^\delta. \end{aligned} \tag{31}$$

We also have

$$\left|B_{321}^\delta\right| \le C\delta\sqrt{\mu}, \qquad \left|B_{322}^\delta\right| \le C\delta\sqrt{\sigma}. \tag{32}$$

In Ω_2, η satisfies

$$\begin{aligned}
-\mu\eta_{rr} - \sigma\eta_{ss} = {} & -\mu\sqrt{\mu}\tau\pi_r - \frac{\sigma\sqrt{\mu}\nu}{1+s\kappa}\pi_s - \mu\sqrt{\mu}(\tau_r\varphi_1)_r - \mu\sqrt{\mu}\tau_r\varphi_{1r} \\
& - \left[\left(\frac{\mu\sqrt{\sigma}\nu}{1+s\kappa}\right)_r\varphi_2\right]_r - \left(\frac{\mu\sqrt{\sigma}\nu}{1+s\kappa}\right)_r\varphi_{2r} \\
& - \left[\left(\frac{\sigma\sqrt{\sigma}\nu}{1+s\kappa}\right)_s\varphi_2\right]_s - \left(\frac{\sigma\sqrt{\sigma}\nu}{1+s\kappa}\right)_s\varphi_{2s}.
\end{aligned} \tag{33}$$

Multiplying (33) by η and integrating over $[0,l)\times(0,\delta)$ and elaborated computation based on (33), and by integration by parts, we have

$$\begin{aligned}
& \int_0^l\int_0^\delta \mu|\eta_r|^2 + \sigma|\eta_s|^2 \\
& = -\sigma\mu\int_0^l \varphi_1(r,0,t)\varphi_{1s}(r,0,t) - \sigma^2\int_0^l \varphi_2(r,0,t)\varphi_{2s}(r,0,t) \\
& \quad + \sigma\sqrt{\mu\sigma}\int_0^l \pi(r,0,t)\varphi_2(r,0,t) + 2\mu\sigma\int_0^l\int_0^\delta \kappa\left|\varphi_2(r,0,t)\right|^2 \\
& \quad + \int_0^l\int_0^\delta\left[\mu^2 - \frac{\mu\sigma}{(1+s\kappa)^2}\right]\pi\varphi_{1r} - \sigma\sqrt{\mu\sigma}\int_0^l\int_0^\delta \frac{2\kappa}{(1+s\kappa)^3}\pi\varphi_2 \\
& \quad + \mu^2\int_0^l\int_0^\delta \kappa^2|\varphi_1|^2 - \mu\sigma\int_0^l\int_0^\delta\left(\frac{\kappa}{1+s\kappa}\right)^2|\varphi_2|^2 \\
& \quad + \mu\sigma\int_0^l\int_0^\delta\left|\left(\frac{1}{1+s\kappa}\right)_r\varphi_2\right|^2 + \sigma^2\int_0^l\int_0^\delta\left|\left(\frac{1}{1+s\kappa}\right)_s\varphi_2\right|^2.
\end{aligned} \tag{34}$$

Multiplying (33) by u^δ and integrating over $[0,l)\times(0,\delta)$, we have

$$\begin{aligned}
& \int_0^l\int_0^\delta (\mu\eta_{rr} + \sigma\eta_{ss})\cdot u^\delta \\
& = 2\sigma\sqrt{\sigma}\int_0^l \kappa\nu\varphi_2(r,0,t)\cdot u^\delta(r,0,t) \\
& \quad + \sigma\sqrt{\mu}\int_0^l \nu\pi(r,0,t)u^\delta(r,0,t) + \mu\sqrt{\mu}\int_0^l\int_0^\delta \tau_r\pi\cdot u^\delta \\
& \quad + \mu\sqrt{\mu}\int_0^l\int_0^\delta \tau\pi\cdot u_r^\delta + \sigma\sqrt{\mu}\int_0^l\int_0^\delta \frac{\nu}{1+s\kappa}\pi\cdot u_s^\delta \\
& \quad - \sigma\sqrt{\mu}\int_0^l\int_0^\delta \frac{\nu\kappa}{(1+s\kappa)^2}\pi\cdot u^\delta - 2\mu\sqrt{\mu}\int_0^l\int_0^\delta \kappa\nu\varphi_1\cdot u_r^\delta \\
& \quad + 2\mu\sqrt{\sigma}\int_0^l\int_0^\delta\left(\frac{\nu}{1+s\kappa}\right)_r\varphi_2\cdot u_r^\delta + 2\sigma\sqrt{\sigma}\int_0^l\int_0^\delta\left(\frac{\nu}{1+s\kappa}\right)_s\varphi_2\cdot u_s^\delta \\
& \quad - \mu\sqrt{\mu}\int_0^l\int_0^\delta \kappa'\nu\varphi_1\cdot u^\delta - \mu\sqrt{\mu}\int_0^l\int_0^\delta \kappa^2\tau\varphi_1\cdot u^\delta \\
& \quad + \mu\sqrt{\sigma}\int_0^l\int_0^\delta\left(\frac{\nu}{1+s\kappa}\right)_{rr}\varphi_2\cdot u^\delta + \sigma\sqrt{\sigma}\int_0^l\int_0^\delta\left(\frac{\nu}{1+s\kappa}\right)_{ss}\varphi_2\cdot u^\delta.
\end{aligned} \tag{35}$$

Integrating by parts, combining with (22), (23), and (35), we have

$$\begin{aligned}
B_{11}^{\delta} &= \int_0^l \int_0^\delta (\mu\eta_{rr} + \sigma\eta_{ss}) \cdot u^\delta + \sigma \int_0^l \eta_s(r,0,t) \cdot u^\delta(r,0,t)\,dr \\
&= -\frac{\sigma}{\delta}\sqrt{\mu}\int_0^l \tau g_1(r,t)u^\delta(r,0,t)\,dr - \frac{\sigma}{\delta}\sqrt{\sigma}\int_0^l \nu g_2(r,t)u^\delta(r,0,t)\,dr \\
&\quad + \sigma\sqrt{\sigma}\int_0^l \nu\kappa\varphi_2(r,0,t)u^\delta(r,0,t)\,dr - \sigma\sqrt{\mu}\int_0^l \nu\pi(r,0,t)u^\delta(r,0,t)\,dr \\
&\quad - \mu\sqrt{\mu}\int_0^l\int_0^\delta \tau_{rr}\varphi_1 \cdot u^\delta - 2\mu\sqrt{\mu}\int_0^l\int_0^\delta \tau_r\varphi_1 \cdot u_r^\delta \\
&\quad - \mu\sqrt{\mu}\int_0^l\int_0^\delta \tau_r\pi \cdot u^\delta - \mu\sqrt{\mu}\int_0^l\int_0^\delta \tau\pi \cdot u_r^\delta \\
&\quad - \mu\sqrt{\sigma}\int_0^l\int_0^\delta \left(\frac{\nu}{1+s\kappa}\right)_{rr}\varphi_2 \cdot u^\delta - 2\mu\sqrt{\sigma}\int_0^l\int_0^\delta \left(\frac{\nu}{1+s\kappa}\right)_{r}\varphi_2 \cdot u_r^\delta \\
&\quad - \sigma\sqrt{\sigma}\int_0^l\int_0^\delta \left(\frac{\nu}{1+s\kappa}\right)_{ss}\varphi_2 \cdot u^\delta - 2\sigma\sqrt{\sigma}\int_0^l\int_0^\delta \left(\frac{\nu}{1+s\kappa}\right)_{s}\varphi_2 \cdot u_s^\delta \\
&\quad - \sigma\sqrt{\mu}\int_0^l\int_0^\delta \left(\frac{\nu}{1+s\kappa}\right)_{s}\pi \cdot u^\delta - \sigma\sqrt{\mu}\int_0^l\int_0^\delta \frac{\nu}{1+s\kappa}\pi \cdot u_s^\delta + \frac{\sigma}{\delta}\sqrt{\mu}o(h^2).
\end{aligned} \tag{36}$$

By using the Hölder inequality and (34), we have

$$\left|B_{12}^\delta\right| \le C\delta\left(\int_0^l\int_0^\delta \mu|\eta_r|^2 + \sigma|\eta_s|^2\right)^{\frac{1}{2}}\left(\int_0^l\int_0^\delta \mu\left|u_r^\delta\right|^2 + \sigma\left|u_s^\delta\right|^2\right)^{\frac{1}{2}}. \tag{37}$$

Now we consider the limit to (29), (30), (36), and (37) as $\delta \to 0$. We study three cases:

1. If $\lim_{\delta\to 0}\frac{\sigma^{\frac{3}{2}}}{\delta} = 0$, in this case, $\lim_{\delta\to 0}\sigma = \lim_{\delta\to 0}\mu = 0$, we have

$$B_1^\delta = B_{11}^\delta + B_{12}^\delta \to 0, \qquad B_2^\delta \to 0, \qquad B_3^\delta \to 0.$$

We obtain the boundary condition

$$\frac{\partial u}{\partial \nu} = (u \otimes u + pI) \cdot \nu \quad \text{on } \partial\Omega_1.$$

2. If $\lim_{\delta\to 0}\frac{\sigma^{\frac{3}{2}}}{\delta} = \alpha \in (0,+\infty)$, in this case, $\lim_{\delta\to 0}\sigma = \lim_{\delta\to 0}\mu = 0$, we consider two different subcases.

2(i). If $\lim_{\delta\to 0}\frac{\mu}{\sigma} = 0$, we have

$$B_1^\delta = B_{11}^\delta + B_{12}^\delta \to -\alpha\int_0^l \nu g_2(r,t)u(r,0,t)\,dr, \qquad B_2^\delta \to 0, \qquad B_3^\delta \to 0.$$

Then the boundary condition is

$$\frac{\partial u}{\partial \nu} - (u \otimes u + pI) \cdot \nu = \alpha\nu \cdot u(r,0,t)e_2 \quad \text{on } \partial\Omega_1.$$

2(ii). If $\lim_{\delta\to 0}\frac{\mu}{\sigma}=\beta\in(0,1]$, we also have $B_2^\delta\to 0$ and $B_3^\delta\to 0$,

$$B_1^\delta\to -\alpha\sqrt{\beta}\int_0^l \tau g_1(r,t)u(r,0,t)\,dr-\alpha\int_0^l \nu g_2(r,t)u(r,0,t)\,dr,$$

which implies

$$\frac{\partial u}{\partial\nu}-(u\otimes u+pI)\cdot\nu=\alpha\sqrt{\beta}\tau\cdot ue_1+\alpha\nu\cdot ue_2\quad\text{on }\partial\Omega_1.$$

3. If $\lim_{\delta\to 0}\frac{\sigma^{\frac{3}{2}}}{\delta}=+\infty$, in this case, we also consider two different subcases.

3(i). If $\lim_{\delta\to 0}\frac{\mu}{\sigma}=0$, we have

$$\frac{\delta}{\sigma\sqrt{\sigma}}B_1^\delta=\frac{\delta}{\sigma\sqrt{\sigma}}B_{11}^\delta+\frac{\delta}{\sigma\sqrt{\sigma}}B_{12}^\delta.$$

Moreover, if $\sigma\geq M>0$,

$$\frac{\delta}{\sigma\sqrt{\sigma}}\left|B_2^\delta\right|\leq\frac{\delta}{M\sqrt{M}}\int_0^l\left|p^\delta(r,0,t)\varphi_2(r,0,t)\right|dr\to 0.$$

If $0<\sigma<M$, we have

$$\frac{\delta}{\sigma\sqrt{\sigma}}\left|B_2^\delta\right|\leq\frac{\delta}{\sigma\sqrt{\sigma}}\sqrt{M}\int_0^l\left|p^\delta(r,0,t)\varphi_2(r,0,t)\right|dr\to 0.$$

The limit of B_3^δ can be derived as B_2^δ and we have $\frac{\delta}{\sigma\sqrt{\sigma}}|B_3^\delta|\to 0$.

Then we obtain

$$\int_0^l \nu g_2(r,t)u^\delta(r,0,t)\,dr\to\int_0^l \nu g_2(r,t)u(r,0,t)\,dr=0,$$

which implies

$$u(r,0,t)\cdot\nu=0\quad\text{on }\partial\Omega_1.$$

3(ii). If $\lim_{\delta\to 0}\frac{\mu}{\sigma}=\beta\in(0,1]$, we have

$$\begin{aligned}\frac{\delta}{\sigma\sqrt{\sigma}}B^\delta &= \frac{\delta}{\sigma\sqrt{\sigma}}B_{11}^\delta+\frac{\delta}{\sigma\sqrt{\sigma}}B_{12}^\delta\\ &\to -\sqrt{\beta}\int_0^l \tau g_1(r,t)u(r,0,t)\,dr-\int_0^l \nu g_2(r,t)u(r,0,t)\,dr=0.\end{aligned}$$

Following the steps in 3(i), we also have

$$\frac{\delta}{\sigma\sqrt{\sigma}}B_2^\delta\to 0,\qquad\frac{\delta}{\sigma\sqrt{\sigma}}B_3^\delta\to 0.$$

Then we obtain the boundary condition on u,

$$u(r,0,t)\cdot\tau|_{\partial\Omega_1}=0,\qquad u(r,0,t)\cdot\nu|_{\partial\Omega_1}=0,$$

which gives

$$u(r,0,t)=0 \quad \text{on } \partial\Omega_1.$$

Theorem 2.1 is proved. □

Proof of Theorem 2.2 Under the conditions in Theorem 2.2, (29), (33), and (34) are still satisfied. To obtain the boundary condition as $\delta \to 0$, we consider three cases:

1. If $\lim_{\delta\to 0} \frac{\mu^{\frac{3}{2}}}{\delta} = 0$, then $\lim_{\delta\to 0} \frac{\sigma\sqrt{\mu}}{\delta} = \lim_{\delta\to 0} \frac{\sigma\sqrt{\sigma}}{\delta} = 0$, $\lim_{\delta\to 0} \sigma = \lim_{\delta\to 0} \mu = 0$, we have

$$B_1^\delta = B_{11}^\delta + B_{12}^\delta \to 0, \qquad B_2^\delta \to 0, \qquad B_3^\delta \to 0.$$

Then we obtain the optimal boundary condition on u,

$$\frac{\partial u}{\partial \nu} - (u\otimes u + pI)\cdot \nu = 0 \quad \text{on } \partial\Omega_1.$$

2. If $\lim_{\delta\to 0} \frac{\mu^{\frac{3}{2}}}{\delta} = \alpha \in (0,+\infty)$, then $\lim_{\delta\to 0} \sigma = \lim_{\delta\to 0} \mu = 0$, and we consider two subcases.

2(i). If $\lim_{\delta\to 0} \frac{\sigma}{\mu} = 0$, then we have $\lim_{\delta\to 0} \frac{\sigma\sqrt{\sigma}}{\delta} \le \lim_{\delta\to 0} \frac{\sigma\sqrt{\mu}}{\delta} = \lim_{\delta\to 0}(\frac{\mu\sqrt{\mu}}{\delta} \times \frac{\sigma}{\mu}) = 0$, passing to the limit, we have

$$B_1^\delta = B_{11}^\delta + B_{12}^\delta \to 0, \qquad B_2^\delta \to 0, \qquad B_2^\delta \to 0.$$

Then we obtain the optimal boundary condition on u,

$$\frac{\partial u}{\partial \nu} - (u\otimes u + pI)\cdot \nu = 0 \quad \text{on } \partial\Omega_1.$$

2(ii). If $\lim_{\delta\to 0} \frac{\sigma}{\mu} = \beta \in (0,1]$, then we have $\lim_{\delta\to 0} \frac{\sigma\sqrt{\sigma}}{\delta} = \lim_{\delta\to 0}(\frac{\mu\sqrt{\mu}}{\delta} \times \frac{\sigma\sqrt{\sigma}}{\mu\sqrt{\mu}}) = \alpha\beta^{\frac{3}{2}}$, $\lim_{\delta\to 0} \frac{\sigma\sqrt{\mu}}{\delta} = \lim_{\delta\to 0}(\frac{\mu\sqrt{\mu}}{\delta} \times \frac{\sigma}{\mu}) = \alpha\beta$, passing to the limit, we have

$$B_1^\delta \to \alpha\beta \int_0^l (\tau\cdot ue_1)\cdot g\,dr + \alpha\beta^{\frac{3}{2}} \int_0^l (\nu\cdot ue_2)\cdot g\,dr, \qquad B_2^\delta \to 0, \qquad B_3^\delta \to 0.$$

Then we obtain the optimal boundary condition on u,

$$\frac{\partial u}{\partial \nu} - (u\otimes u + pI)\cdot \nu = \alpha\beta\tau\cdot ue_1 + \alpha\beta^{\frac{3}{2}}\nu\cdot ue_2 \quad \text{on } \partial\Omega_1.$$

3. If $\lim_{\delta\to 0} \frac{\mu^{\frac{3}{2}}}{\delta} = +\infty$, then $\lim_{\delta\to 0} \frac{\sigma\sqrt{\mu}}{\delta} = +\infty$. We consider two subcases.

3(i). If $\lim_{\delta\to 0} \frac{\sigma}{\mu} = 0$, then we have

$$\frac{\delta}{\sigma\sqrt{\mu}} B_1^\delta = \frac{\delta}{\sigma\sqrt{\mu}} B_{11}^\delta + \frac{\delta}{\sigma\sqrt{\mu}} B_{12}^\delta \to -\int_0^l (\tau\cdot ue_1)\cdot g\,dr.$$

Moreover, if $\mu \ge a > 0$,

$$\frac{\delta}{\sigma\sqrt{\mu}} |B_2^\delta| \le \frac{\delta}{a}\sqrt{M} \int_0^l \bigl|p^\delta(r,0,t)\varphi_2(r,0,t)\bigr|\,dr \to 0.$$

If $0 < \mu < M$, we have

$$\frac{\delta}{\sigma\sqrt{\mu}}\left|B_2^\delta\right| \le \frac{\delta}{\sigma\sqrt{\mu}}\sqrt{M}\int_0^l \left|p^\delta(r,0,t)\varphi_2(r,0,t)\right| dr \to 0.$$

The limit of B_3^δ can be derived as B_2^δ and we have $\frac{\delta}{\sigma\sqrt{\sigma}}|B_3^\delta| \to 0$. Then we obtain the optimal boundary condition on u,

$$u \cdot \tau = 0 \quad \text{on } \partial\Omega_1.$$

3(ii). If $\lim_{\delta\to 0}\frac{\sigma}{\mu} = \beta \in (0,1]$, then we have

$$\begin{aligned}\frac{\delta}{\sigma\sqrt{\mu}}B_1^\delta &= \frac{\delta}{\sigma\sqrt{\mu}}B_{11}^\delta + \frac{\delta}{\sigma\sqrt{\mu}}B_{12}^\delta \\ &\to -\int_0^l (\tau \cdot ue_1)\cdot g\,dr - \int_0^l (\sqrt{\beta}\nu \cdot ue_2)\cdot g\,dr.\end{aligned}$$

Following the steps in 3(i), we have

$$\frac{\delta}{\sigma\sqrt{\mu}}\left|B_2^\delta\right| \to 0.$$

Then we obtain the optimal boundary condition on u,

$$\tau \cdot ue_1 + \sqrt{\beta}\nu \cdot ue_2 = 0 \quad \text{on } \partial\Omega_1,$$

which implies

$$u = 0 \quad \text{on } \partial\Omega_1,$$

Theorem 2.2 is then proved. □

Remark 4.1 As stated in the introduction, to protect $\partial\Omega_1$ from eroding, it is desirable that the eigenvector corresponding to the smallest eigenvalue is orthogonal to the boundary of the body in order to directly confront the ambient effect. In our problem, it requires $\delta \le \sigma \le \mu$. The optimal boundary conditions to our problem are given in Theorem 2.2. We must acknowledge that we need the condition $\delta \le \sigma \le \mu \le M\sigma$ for some constants $M > 0$ so that we can ensure $0 < h \le 1$ as δ is small enough. If this condition does not hold, the estimate (22) will not be satisfied. The case that $h > 1$ we will consider in the future.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Acknowledgements
This work is supported by the research of high schools of Guangzhou (No. 1201420925) and GDNSF (No. 2014A030313526).

Received: 14 August 2015 Accepted: 7 November 2015 Published online: 01 December 2015

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