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New mean value theorems and generalization of Hadamard inequality via coordinated m-convex functions

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Abstract

We derive new mean value theorems for functionals associated with Hadamard inequality for convex functions on the coordinates. We present some Hadamard type inequalities and related results for m-convex functions on the coordinates.

Keywords: convex functions; m-convex functions; convex functions on coordinates; Hadamard inequality; mean value theorems

1 Introduction

We start with the very basic concept of a convex function that has been seen as important ever since it was defined.

Definition 1.1 A real valued function $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, is called convex if

$$f\bigl(\lambda x + (1-\lambda)y\bigr) \leq \lambda f(x) + (1-\lambda)f(y), \tag{1}$$

where $\lambda \in [0,1]$, and $x, y \in I$.

One cannot deny the importance of convex functions. It is used by many mathematicians in many fields of mathematics such as functional analysis, mathematical statistics, and complex analysis (see for example [1–4] and references therein). In the field of inequalities, convex functions play a unique role. Most of the new inequalities till now defined are results and consequences of (1). The most cardinal and classical inequality for convex functions is stated in the following.

Theorem 1.2 *Let $f : I \to \mathbb{R}$ be convex function and $a, b \in I$ with $a < b$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{2}$$

This famous integral inequality can be traced back to the papers presented by Hermite [5] and Hadamard [6]. Researchers have used inequality (2) several times for giving generalization and modification of Hadamard type inequalities using different modification

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of convex functions. For refinements, counterparts, and generalizations see for example [3, 7–19].

Recently, many authors have considered a convex function on the coordinates to give the Hadamard inequality on the coordinates and its different variants. Also they have given many results associated with it (*e.g.* see [20–23]).

Definition 1.3 Let $\Delta^2 := [a,b] \times [c,d] \subset \mathbb{R}^2$ with $a < b$ and $c < d$. A function $f : \Delta^2 \to \mathbb{R}$ is called convex on the coordinates if the partial mapping $f_y : [a,b] \to \mathbb{R}$, $f_y(u) := f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) := f(x,v)$ are convex, where they are defined for all $y \in [c,d]$ and $x \in [a,b]$.

We will keep the notation $\Delta^2 = [a,b] \times [c,d]$ throughout this paper.

Recall that a mapping $f : \Delta^2 \to \mathbb{R}$ is convex in Δ^2 if, for $(x,y),(u,v) \in \Delta^2$ and $\alpha \in [0,1]$, the following inequality holds:

$$f\bigl(\alpha(x,y) + (1-\alpha)(u,v)\bigr) \le \alpha f(x,y) + (1-\alpha)f(u,v).$$

It can be seen that every convex mapping $f : \Delta^2 \to \mathbb{R}$ is convex on the coordinates but the converse is not true. Dragomir gave the Hadamard inequality for a rectangle in the plane for convex functions on the coordinates (see [11]).

Theorem 1.4 *Suppose that $f : \Delta^2 \to \mathbb{R}$ is convex on the coordinates on Δ^2. Then one has the following inequalities*:

$$\begin{aligned}
&f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\quad\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\
&\quad\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dx\,dy \\
&\quad\le \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx + \frac{1}{b-a}\int_a^b f(x,d)\,dx + \frac{1}{d-c}\int_c^d f(a,y)\,dy\right. \\
&\qquad\left. + \frac{1}{d-c}\int_c^d f(b,y)\,dy\right] \\
&\quad\le \frac{1}{4}\bigl[f(a,c) + f(a,d) + f(b,c) + f(b,d)\bigr].
\end{aligned}$$

In [24], Toader defined the concept of m-convexity, an intermediate between the usual convexity and star shape properties.

Definition 1.5 The function $f : [0,b] \to \mathbb{R}$ is said to be m-convex, where $m \in [0,1]$, if for every $x, y \in [0,b]$ and $t \in [0,1]$ we have

$$f\bigl(tx + m(1-t)y\bigr) \le tf(x) + m(1-t)f(y). \tag{3}$$

In [25] using inequality (2) and (3) Dragomir and Toader gave the following Hadamard type inequality for m-convex functions.

Theorem 1.6 *Let $f:[0,\infty)\to\mathbb{R}$ be m-convex function with $m\in(0,1]$ and $0\le a<b$. If $f\in L_1[a,b]$, then one has the following inequalities*:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{1}{b-a}\int_a^b \frac{f(x)+mf(\frac{x}{m})}{2}\,dx \\ &\le \frac{m+1}{4}\left[\frac{f(a)+f(b)}{2}+m\left(\frac{f(\frac{a}{m})+f(\frac{b}{m})}{2}\right)\right]. \end{aligned} \tag{4}$$

They also gave the following related results to the Hadamard type inequality for m-convex functions.

Theorem 1.7 *Let $f:[0,\infty)\to\mathbb{R}$ be an m-convex function with $m\in(0,1]$. If $0\le a<b<\infty$ and $f\in L_1[a,b]$, then one has the inequality*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \le \min\left\{\frac{f(a)+mf(\frac{b}{m})}{2},\frac{f(b)+mf(\frac{a}{m})}{2}\right\}.$$

Theorem 1.8 *Let $f:[0,\infty)\to\mathbb{R}$ be an m-convex function with $m\in(0,1]$. If $0\le a<b<\infty$ and f is differentiable on $(0,\infty)$, then one has the inequality*

$$\begin{aligned} \frac{f(mb)}{m}-\frac{b-a}{2}f'(mb) &\le \frac{1}{b-a}\int_a^b f(x)\,dx \\ &\le \frac{(b-ma)f(d)-(a-mb)f(a)}{2(b-a)}. \end{aligned}$$

In this paper, new mean value theorems of Cauchy type for functionals associated with nonnegative differences of the Hadamard inequality on the coordinates are proved. Generalized results related to the Hadamard inequality for m-convex functions on the coordinates are also given.

2 Mean value theorems

We know that if a function f is twice differentiable on an interval I then it is convex on I if and only if its second order derivative is nonnegative. If a function $f(X):=f(x,y)$ has continuous second order partial derivatives on interior of Δ^2 then it is convex on Δ^2 if the Hessian matrix

$$H_f(X)=\begin{pmatrix} \frac{\partial^2 f(X)}{\partial x^2} & \frac{\partial^2 f(X)}{\partial y\partial x} \\ \frac{\partial^2 f(X)}{\partial x\partial y} & \frac{\partial^2 f(X)}{\partial y^2} \end{pmatrix}$$

is nonnegative definite, that is, $\mathbf{v}H_f(X)\mathbf{v}^\tau$ is nonnegative for all real nonnegative vector $\mathbf{v}$ (see [3], p.11).

It is easy to see that $f:\Delta^2\to\mathbb{R}$ is coordinated convex on Δ^2 iff

$$f_x''(y)=\frac{\partial^2 f(x,y)}{\partial y^2} \quad\text{and}\quad f_y''(x)=\frac{\partial^2 f(x,y)}{\partial x^2}$$

are nonnegative for all interior points (x,y) in Δ^2.

For a real valued function $f : \Delta^2 \to \mathbb{R}$ we define

$$\mathcal{H}(f) = \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) dy\right] - f\left(\frac{a+b}{2}, \frac{c+d}{2}\right). \tag{5}$$

One can note that $\mathcal{H}(f) \geq 0$ if f is convex on the coordinates in Δ^2.

To give the mean value theorems of Cauchy type, we need the following lemma.

Lemma 2.1 *Let $f : \Delta^2 \to \mathbb{R}$ be a function such that*

$$m_1 \leq \frac{\partial^2 f(x,y)}{\partial x^2} \leq M_1 \quad \text{and} \quad m_2 \leq \frac{\partial^2 f(x,y)}{\partial y^2} \leq M_2$$

for all interior points (x, y) in Δ^2.

Consider the functions $g, h : \Delta^2 \to \mathbb{R}$ defined as

$$g(x,y) = \frac{1}{2}\max\{M_1, M_2\}\left(x^2 + y^2\right) - f(x,y)$$

and

$$h(x,y) = f(x,y) - \frac{1}{2}\min\{m_1, m_2\}\left(x^2 + y^2\right).$$

Then g and h are coordinated convex in Δ^2.

Proof Since

$$\frac{\partial^2 g(x,y)}{\partial x^2} = \max\{M_1, M_2\} - \frac{\partial^2 f(x,y)}{\partial x^2} \geq 0$$

and

$$\frac{\partial^2 g(x,y)}{\partial y^2} = \frac{\partial^2 f(x,y)}{\partial y^2} - \min\{m_1, m_2\} \geq 0$$

for all interior points (x, y) in Δ^2, g is convex on the coordinates in Δ^2.

Similarly one can prove that h is convex on the coordinates in Δ^2. □

Theorem 2.2 *Let $f : \Delta^2 \to \mathbb{R}$ be a function, which has continuous partial derivatives of second order in Δ^2 and $q(x, y) := x^2 + y^2$. Then there exist (η_1, ξ_1) and (η_2, ξ_2) in the interior of Δ^2 such that*

$$\mathcal{H}(f) = \frac{1}{2}\frac{\partial^2 f(\eta_1, \xi_1)}{\partial x^2}\mathcal{H}(q)$$

and

$$\mathcal{H}(f) = \frac{1}{2}\frac{\partial^2 f(\eta_2, \xi_2)}{\partial x^2}\mathcal{H}(q),$$

provided that $\mathcal{H}(q) \neq 0$.

Proof Since f has continuous partial derivatives of second order in compact set Δ^2, there exist real numbers m_1, m_2, M_1, and M_2 such that

$$m_1 \leq \frac{\partial^2 f(x,y)}{\partial x^2} \leq M_1 \quad \text{and} \quad m_2 \leq \frac{\partial^2 f(x,y)}{\partial y^2} \leq M_2.$$

Now consider functions g defined in Lemma 2.1. As g is convex on the coordinates in Δ^2,

$$\mathcal{H}(g) \geq 0,$$

that is,

$$\mathcal{H}\left(\frac{1}{2}\max\{M_1, M_2\}q - f(x,y)\right) \geq 0.$$

From this we get

$$2\mathcal{H}(f) \leq \max\{M_1, M_2\}\mathcal{H}(q). \tag{6}$$

On the other hand, for the function h, one has

$$\min\{m_1, m_2\}\mathcal{H}(q) \leq 2\mathcal{H}(f). \tag{7}$$

As $\mathcal{H}(q) \neq 0$, combining inequalities (6) and (7), we get

$$\min\{m_1, m_2\} \leq \frac{2\mathcal{H}(f)}{H(q)} \leq \max\{M_1, M_2\}.$$

Then there exist (η_1, ξ_1) and (η_2, ξ_2) in the interior of Δ^2 such that

$$\frac{2\mathcal{H}(f)}{H(q)} = \frac{\partial^2 f(\eta_1, \xi_1)}{\partial x^2} \quad \text{and} \quad \frac{2\mathcal{H}(f)}{H(q)} = \frac{\partial^2 f(\eta_2, \xi_2)}{\partial y^2}.$$

Hence the required result follows. □

3 Hadamard type inequalities for *m*-convex function on two coordinates

In this section we give Hadamard type inequalities for m-convex functions on two coordinates. First of all we give the definitions of m-convex functions in two coordinates.

Definition 3.1 Let $\Delta = [0,b] \times [0,d] \subset [0,\infty)^2$, then a function $f : \Delta \to \mathbb{R}$, will be called m-convex on the coordinates if the partial mappings $f_y : [0,b] \to \mathbb{R}$, $f_y(u) := f(u,y)$, and $f_x : [0,d] \to \mathbb{R}, f_x(v) := f(x,v)$, are m-convex on $[0,b]$ and $[0,d]$, respectively.

In the following we give the Hadamard type inequality for m-convex functions on the coordinates.

Theorem 3.2 *Let $\Delta = [0,b] \times [0,d] \subset [0,\infty)^2$ with $b,d > 0$ and $f : \Delta \to \mathbb{R}$ be m-convex on the coordinates in Δ with $m \in (0,1]$. If $f_x \in L_1[0,d]$ and $f_y \in L_1[0,b]$, $0 \leq a < b$, $0 \leq c < d$.*

Then we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\\
&\quad\le\frac{1}{2(b-a)(d-c)}\int_a^b\int_c^d\left(2f(x,y)+m\left(f\left(x,\frac{y}{m}\right)+f\left(\frac{x}{m},y\right)\right)\right)dy\,dx\\
&\quad\le\frac{(m+1)^2}{16}\left[f(a,c)+f(b,c)+f(a,d)+f(b,d)+m\left(f\left(\frac{a}{m},c\right)+f\left(\frac{b}{m},c\right)\right.\right.\\
&\qquad\left.+f\left(\frac{a}{m},d\right)+f\left(\frac{b}{m},d\right)+f\left(a,\frac{c}{m}\right)+f\left(a,\frac{d}{m}\right)+f\left(b,\frac{c}{m}\right)+f\left(b,\frac{d}{m}\right)\right)\\
&\qquad\left.+m^2\left(f\left(\frac{a}{m},\frac{c}{m}\right)+f\left(\frac{b}{m},\frac{c}{m}\right)+f\left(\frac{a}{m},\frac{d}{m}\right)+f\left(\frac{b}{m},\frac{d}{m}\right)\right)\right].
\end{aligned}$$

Proof Since mapping $f:\Delta\to\mathbb{R}$ is m-convex on the coordinates, the functions f_x and f_y are m-convex on $[0,d]$ and $[0,b]$, respectively. Using (4) for the function f_y we have

$$\begin{aligned}
f_y\left(\frac{a+b}{2}\right)&\le\frac{1}{b-a}\int_a^b\left(\frac{f_y(x)+mf_y(\frac{x}{m})}{2}\right)dx\\
&\le\frac{m+1}{4}\left[\frac{f_y(a)+f_y(b)}{2}+m\frac{f_y(\frac{a}{m})+f_y(\frac{b}{m})}{2}\right],
\end{aligned}$$

that is,

$$\begin{aligned}
f\left(\frac{a+b}{2},y\right)&\le\frac{1}{b-a}\int_a^b\left(\frac{f(x,y)+mf(\frac{x}{m},y)}{2}\right)dx\\
&\le\frac{m+1}{4}\left[\frac{f(a,y)+f(b,y)}{2}+m\frac{f(\frac{a}{m},y)+f(\frac{b}{m},y)}{2}\right].
\end{aligned}\tag{8}$$

From this one has

$$\begin{aligned}
&\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\\
&\quad\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\left(\frac{f(x,y)+mf(\frac{x}{m},y)}{2}\right)dy\,dx\\
&\quad\le\frac{m+1}{4(d-c)}\int_c^d\left[\frac{f(a,y)+f(b,y)}{2}+m\frac{f(\frac{a}{m},y)+f(\frac{b}{m},y)}{2}\right]dy.
\end{aligned}\tag{9}$$

Similarly using (4) for the function f_x we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx\\
&\quad\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\left(\frac{f(x,y)+mf(x,\frac{y}{m})}{2}\right)dy\,dx\\
&\quad\le\frac{m+1}{4(b-a)}\int_a^b\left[\frac{f(x,c)+f(x,d)}{2}+m\frac{f(x,\frac{c}{m})+f(x,\frac{d}{m})}{2}\right]dx.
\end{aligned}\tag{10}$$

By adding (9) and (10), we get

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy \\
&\quad\le \frac{1}{2(b-a)(d-c)}\int_a^b\int_c^d \left(2f(x,y)+m\left(f\left(x,\frac{y}{m}\right)+f\left(\frac{x}{m},y\right)\right)\right)dy\,dx \\
&\quad\le \frac{m+1}{4}\left[\frac{1}{b-a}\int_a^b\left(\frac{f(x,c)+f(x,d)}{2}+m\frac{f(x,\frac{c}{m})+f(x,\frac{d}{m})}{2}\right)dx\right. \\
&\qquad\left.+\frac{1}{d-c}\int_c^d\left(\frac{f(a,y)+f(b,y)}{2}+m\frac{f(\frac{a}{m},y)+f(\frac{b}{m},y)}{2}\right)dy\right]. \qquad (11)
\end{aligned}$$

For fixed y using the m-convexity of f_y we have

$$f(x,y)\le\frac{f(x,y)+mf(\frac{x}{m},y)}{2}. \qquad (12)$$

Performing the average integral over the interval $[a,b]$ and using (4) we get

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x,y)\,dx &\le \frac{1}{b-a}\int_a^b\left(\frac{f(x,y)+mf(\frac{x}{m},y)}{2}\right)dx \\
&\le \frac{m+1}{4}\left[\frac{f(a,y)+f(b,y)}{2}+m\frac{f(\frac{a}{m},y)+f(\frac{b}{m},y)}{2}\right]. \qquad (13)
\end{aligned}$$

Similarly for fixed x using the m-convexity of f_x one has

$$\begin{aligned}
\frac{1}{d-c}\int_c^d f(x,y)\,dy &\le \frac{1}{d-c}\int_c^d\left(\frac{f(x,y)+mf(x,\frac{y}{m})}{2}\right)dy \\
&\le \frac{m+1}{4}\left[\frac{f(x,c)+f(x,d)}{2}+m\frac{f(x,\frac{c}{m})+f(x,\frac{d}{m})}{2}\right]. \qquad (14)
\end{aligned}$$

Considering (13) for $y=c,d$, (14) for $x=a,b$, then (13) for $y=\frac{c}{m},\frac{d}{m}$, (14) for $x=\frac{a}{m},\frac{b}{m}$ to multiply later with m. Adding all these inequalities, we obtain

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b\left(\frac{f(x,c)+f(x,d)}{2}+m\frac{f(x,\frac{c}{m})+f(x,\frac{d}{m})}{2}\right)dx \\
&\qquad+\frac{1}{d-c}\int_c^d\left(\frac{f(a,y)+f(b,y)}{2}+m\frac{f(\frac{a}{m},y)+f(\frac{b}{m},y)}{2}\right)dy \\
&\quad\le\frac{1}{2}\left[\frac{1}{b-a}\int_a^b\left(\frac{f(x,c)+mf(\frac{x}{m},c)+f(x,d)+mf(\frac{x}{m},d)}{2}\right.\right. \\
&\qquad\left.+m\frac{f(x,\frac{c}{m})+mf(\frac{x}{m},\frac{c}{m})+f(x,\frac{d}{m})+mf(\frac{x}{m},\frac{d}{m})}{2}\right)dx \\
&\qquad+\frac{1}{d-c}\int_c^d\left(\frac{f(a,y)+mf(a,\frac{y}{m})+f(b,y)+mf(b,\frac{y}{m})}{2}\right. \\
&\qquad\left.\left.+m\frac{f(\frac{a}{m},y)+mf(\frac{a}{m},\frac{y}{m})+f(\frac{b}{m},y)+mf(\frac{b}{m},\frac{y}{m})}{2}\right)dy\right]
\end{aligned}$$

$$\begin{aligned}&\le \frac{1}{4}\left[f(a,c)+f(b,c)+f(a,d)+f(b,d)+m\left(f\left(\frac{a}{m},c\right)+f\left(\frac{b}{m},c\right)\right.\right.\\&\quad+f\left(\frac{a}{m},d\right)+f\left(\frac{b}{m},d\right)+f\left(a,\frac{c}{m}\right)+f\left(a,\frac{d}{m}\right)+f\left(b,\frac{c}{m}\right)+f\left(b,\frac{d}{m}\right)\Big)\\&\quad\left.+m^2\left(f\left(\frac{a}{m},\frac{c}{m}\right)+f\left(\frac{b}{m},\frac{c}{m}\right)+f\left(\frac{a}{m},\frac{d}{m}\right)+f\left(\frac{b}{m},\frac{d}{m}\right)\right)\right].\end{aligned}\tag{15}$$

Now combining inequalities in (11) and (15), we get the last two inequalities of the theorem. □

Theorem 3.3 *Let* $\Delta = [0,b]\times[0,d]\subset[0,\infty)^2$ *with* $b,d>0$ *and* $f:\Delta\to\mathbb{R}$ *be m-convex on the coordinate in* Δ *with* $m\in(0,1]$. *If* $f_x\in L_1[0,d]$ *and* $f_y\in L_1[0,b]$, *then*

$$\begin{aligned}2f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&\le\frac{1}{b-a}\int_a^b\left(\frac{f(x,\frac{c+d}{2})+mf(\frac{x}{m},\frac{c+d}{2})}{2}\right)dx\\&\quad+\frac{1}{c-d}\int_c^d\left(\frac{f(\frac{a+b}{2},y)+mf(\frac{a+b}{2},\frac{y}{m})}{2}\right)dy.\end{aligned}\tag{16}$$

Proof By using (8) for $y=\frac{c+d}{2}$

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\le\frac{1}{b-a}\int_a^b\left(\frac{f(x,\frac{c+d}{2})+mf(\frac{x}{m},\frac{c+d}{2})}{2}\right)dx.\tag{17}$$

Applying the first inequality in (4) for f_x on $[c,d]$ we have

$$f\left(x,\frac{c+d}{2}\right)\le\frac{1}{c-d}\int_c^d\left(\frac{f(x,y)+mf(x,\frac{y}{m})}{2}\right)dy.$$

Put $x=\frac{a+b}{2}$ we get

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\le\frac{1}{c-d}\int_c^d\left(\frac{f(\frac{a+b}{2},y)+mf(\frac{a+b}{2},\frac{y}{m})}{2}\right)dy.\tag{18}$$

By adding (17) and (18) we get (16). □

Remark 3.4 By putting $m=1$ in Theorem 3.2 and Theorem 3.3 and combining we get inequalities in Theorem 1.4.

Theorem 3.5 *Let* f, f_x, *and* f_y *be defined as in Theorem* 3.2. *Then one has the following inequality*:

$$\begin{aligned}&\frac{2}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx\\&\quad\le\min\left\{\frac{1}{b-a}\int_a^b\left(\frac{f(x,c)+mf(x,\frac{d}{m})}{2}\right)dx,\right.\\&\quad\left.\frac{1}{b-a}\int_a^b\left(\frac{f(x,d)+mf(x,\frac{c}{m})}{2}\right)dx\right\}\end{aligned}$$

$$+ \min\left\{ \frac{1}{d-c} \int_c^d \left(\frac{f(a,y) + mf(\frac{b}{m},y)}{2} \right) dy, \right.$$
$$\left. \frac{1}{d-c} \int_c^d \left(\frac{f(b,y) + mf(\frac{a}{m},y)}{2} \right) dy \right\}. \tag{19}$$

Proof Since mapping $f : \Delta \to \mathbb{R}$ is m-convex on the coordinates, the functions f_x and f_y are m-convex on $[0,d]$ and $[0,b]$, respectively. Thus we have by Theorem 1.7

$$\frac{1}{b-a} \int_a^b f_y(x)\,dx \le \min\left\{ \frac{f_y(a) + f_y(\frac{b}{m})}{2}, \frac{f_y(b) + f_y(\frac{a}{m})}{2} \right\},$$

that is,

$$\frac{1}{b-a} \int_a^b f(x,y)\,dx \le \min\left\{ \frac{f(a,y) + f(\frac{b}{m},y)}{2}, \frac{f(b,y) + f(\frac{a}{m},y)}{2} \right\}.$$

Performing the average integral over the interval $[c,d]$

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dx$$
$$\le \min\left\{ \frac{1}{d-c} \int_c^d \left(\frac{f(a,y) + f(\frac{b}{m},y)}{2} \right) dy, \frac{1}{d-c} \int_c^d \left(\frac{f(b,y) + f(\frac{a}{m},y)}{2} \right) dy \right\}. \tag{20}$$

Similarly for f_x one has

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dx$$
$$\le \min\left\{ \frac{1}{b-a} \int_a^b \left(\frac{f(x,c) + f(x,\frac{d}{m})}{2} \right) dx, \frac{1}{b-a} \int_a^b \left(\frac{f(x,d) + f(x,\frac{c}{m})}{2} \right) dx \right\}. \tag{21}$$

For the desired result we add inequalities (20) and (21). □

Theorem 3.6 *Let $f_x : [0,d] \subset [0,\infty) \to \mathbb{R}$, $f_x(u) = f(x,u)$ and $f_y : [0,b] \subset [0,\infty) \to \mathbb{R}$, $f_y(v) = f(v,y)$ be partial m-convex mappings with $m \in (0,1]$. If $0 \le a < b < \infty$, $0 \le c < d < \infty$, also if $f_x \in L_1[0,d]$ and $f_y \in L_1[0,b]$ with f_y and f_x are differentiable on $(0,\infty)$, then we have the following inequality*:

$$\frac{1}{2m}\left(\frac{1}{b-a} \int_a^b f(x,md)\,dx + \frac{1}{d-c} \int_c^d f(mb,y)\,dy \right)$$
$$- \frac{1}{2}\left(\frac{d-c}{b-a} \int_a^b \frac{\partial f(x,md)}{\partial y}\,dx + \frac{b-a}{d-c} \int_c^d \frac{\partial f(mb,y)}{\partial x}\,dy \right)$$
$$\le \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dy\,dx \le \frac{1}{4(b-a)(d-c)}$$
$$\times \left[(d-mc) \int_a^b f(x,d)\,dx + (b-ma) \int_c^d f(b,y)\,dy \right.$$
$$\left. - \left((c-md) \int_a^b f(x,c)\,dx + (a-mb) \int_c^d f(a,y)\,dy \right) \right]. \tag{22}$$

Proof Since the functions f_x and f_y are m-convex on $[0,d]$ and $[0,b]$, respectively, and the functions f_x and f_y are differentiable on $(0,\infty)$, applying Theorem 1.8 we have

$$\begin{aligned}\frac{f_x(md)}{m} - \frac{d-c}{2}f_x'(md) &\le \frac{1}{d-c}\int_c^d f_x(y)\,dy \\ &\le \frac{(d-mc)f_x(d)-(c-md)f_x(c)}{2(d-c)}\end{aligned}$$

and

$$\begin{aligned}\frac{f_y(mb)}{m} - \frac{b-a}{2}f_y'(mb) &\le \frac{1}{b-a}\int_a^b f_y(x)\,dy \\ &\le \frac{(b-ma)f_y(b)-(a-mb)f_y(a)}{2(b-a)}.\end{aligned}$$

This gives us

$$\begin{aligned}\frac{f(x,md)}{m} - \frac{d-c}{2}\frac{\partial f(x,md)}{\partial y} &\le \frac{1}{d-c}\int_c^d f(x,y)\,dy \\ &\le \frac{(d-mc)f(x,d)-(c-md)f(x,c)}{2(d-c)}\end{aligned} \tag{23}$$

and

$$\begin{aligned}\frac{f(mb,y)}{m} - \frac{b-a}{2}\frac{\partial f(mb,y)}{\partial x} &\le \frac{1}{b-a}\int_a^b f(x,y)\,dx \\ &\le \frac{(b-ma)f(b,y)-(a-mb)f(a,y)}{2(b-a)}.\end{aligned} \tag{24}$$

Integrating (23) over $[a,b]$ and (24) over $[c,d]$ we get

$$\begin{aligned}&\frac{1}{b-a}\int_a^b \frac{f(x,md)}{m}\,dx - \frac{d-c}{2(b-a)}\int_a^b \frac{\partial f(x,md)}{\partial y}\,dx \\ &\quad\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx \\ &\quad\le \frac{1}{b-a}\int_a^b \left(\frac{(d-mc)f(x,d)-(c-md)f(x,c)}{2(d-c)}\right)dx\end{aligned} \tag{25}$$

and

$$\begin{aligned}&\frac{1}{d-c}\int_c^d \frac{f(mb,y)}{m}\,dy - \frac{b-a}{2(d-c)}\int_c^d \frac{\partial f(mb,y)}{\partial x}\,dy \\ &\quad\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx \\ &\quad\le \frac{1}{d-c}\int_c^d \left(\frac{(b-ma)f(b,y)-(a-mb)f(a,y)}{2(b-a)}\right)dy,\end{aligned} \tag{26}$$

respectively.

Adding (25) and (26) we get (22). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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