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Dynamical analysis of the permanent-magnet synchronous motor chaotic system

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Abstract

This paper is concerned with some dynamics of the permanent-magnet synchronous motor chaotic system based on Lyapunov stability theory and optimization theory. The innovation of the paper lies in that we derive a family of mathematical expressions of globally exponentially attractive sets for this chaotic system with respect to system parameters. Numerical simulations confirm that theoretical analysis results are correct.

Keywords: permanent-magnet synchronous motor; chaotic attractors; Lyapunov stability; numerical simulations

1 Introduction

Since Lorenz *et al.* were the first to investigate the Lorenz equations in 1963, chaotic systems have played an important role in a variety of industrial fields [1–8]. As is well known, the research on chaos is not limited to the fields of mathematics and physics. It is found that chaos widely exists in the fields of meteorology, medicine, computer science, economics, mechanical engineering, cryptography, and so on [9–19]. However, it was not until the 1990s that chaos has gradually attracted enough attention due to the findings in practical engineering. From the point of view of the potential application of chaos theory in practical engineering, many efforts have been made to study chaos in the past 20 years.

This paper mainly focuses on the chaotic system model from a permanent-magnet synchronous motor (PMSM) which is a nonlinear, multivariable, and strong coupling system. A permanent-magnet synchronous motor is a kind of highly efficient and high-powered motor, which has been widely used in the industry. Usually, the dynamics of a PMSM is modeled as a three-dimensional autonomous differential equation [20, 21]. Dynamical behaviors of the PMSM, such as periodic solutions, chaos phenomena, phase portraits, bifurcation diagrams, Lyapunov exponents, chaos anti-control and chaos synchronization, have been widely studied in [20, 21].

In recent years, dynamical behaviors of chaotic systems, such as stability, periodic solutions, circuit implementation, image encryption algorithm, chaos synchronization, chaos attractors, heteroclinic orbits and homoclinic orbits, have been extensively investigated [22–24]. However, little seems to be known about the global exponential attractive set of chaotic systems [22–24]. Despite the fact that many qualitative and quantitative results on the permanent-magnet synchronous motor system have been obtained [20, 21], there is a fundamental question that has not been completely answered so far: is there a global exponential attractive set for the permanent-magnet synchronous motor system? Global exponential attractive sets play an important role in dynamical systems. The global exponential attractive set is also very important for engineering applications, since it is very difficult to predict the existence of hidden attractors and they can lead to crashes [10]. Therefore, how to get the global attractive sets of a chaotic dynamical system is particularly significant both for theoretical research and practical applications. In [25, 26], one shows that Lyapunov functions can be used to study chaos synchronization. However, Lyapunov-like functions used in [16, 18, 25, 26] cannot be used to study the global attractive sets for the permanent-magnet synchronous motor system. In this paper, a new Lyapunov-like function is constructed to investigate the global attractive sets of the permanent-magnet synchronous motor system.

Motivated by the above discussion, we will investigate the global attractive sets of the permanent-magnet synchronous motor system. The meaning of the contribution of this article is that not only do we derive a family of mathematical expressions of global exponential attractive sets for permanent-magnet synchronous motor systems in [20, 21] with respect to the parameters of the system, but we also get the rate of the trajectories of the system going from the exterior of the trapping set to the interior of the trapping set.

The rest of the paper is organized as follows. The permanent-magnet synchronous motor (PMSM) model is given in Section 2. In Section 3, we prove that there exist global exponential attractive sets for the chaotic PMSM system. Some numerical simulations are also given in Section 3. Section 4 gives conclusions.

2 Permanent-magnet synchronous motor model

A permanent-magnet synchronous motor (PMSM) is a kind of highly efficient and high-powered motor, which has been widely used in the industry. The model of the PMSM, as described in [20], is as follows:

$$\begin{cases} \frac{di_d}{dt_1} = (u_d - R_1 i_d + \omega L_q i_q)/L_d, \\ \frac{di_q}{dt_1} = (u_q - R_1 i_q - \omega L_d i_d + \omega \psi_r)/L_q, \\ \frac{d\omega}{dt_1} = [n_p \psi_r i_q + n_p (L_d - L_q) i_d i_q - T_L - \beta\omega]/J, \end{cases} \tag{1}$$

where i_d, i_q and ω are the state variables, which represent currents and motor angular frequency, respectively; u_d and u_q represent the direct- and quadrature-axis stator voltage components, respectively; J represents the polar moment of inertia; T_L represents the external load torque; β represents the viscous damping coefficient; R_1 represents the stator winding resistance; L_d and L_q represent the direct- and quadrature-axis stator inductors, respectively; ψ_r represents the permanent-magnet flux, and n_p represents the number of pole-pairs, the parameters L_d, L_q, J, T_L, R_1, ψ_r, β are positive.

In [20], by applying an affine transformation, $X = \lambda Y$, and time-scaling transformation, $t_1 = \tau t$, where

$$X = \begin{bmatrix} i_d & i_q & \omega \end{bmatrix}^T, \qquad Y = \begin{bmatrix} x & y & z \end{bmatrix}^T,$$

$$\lambda = \begin{bmatrix} \lambda_d & 0 & 0 \\ 0 & \lambda_q & 0 \\ 0 & 0 & \lambda_\omega \end{bmatrix} = \begin{bmatrix} bk & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & \frac{1}{\tau} \end{bmatrix}, \quad b = \frac{L_q}{L_d}, k = \frac{\beta}{n_p \tau \psi_r}, \tau = \frac{L_q}{R_1},$$

the model (1) is written as [20]

$$\begin{cases} \frac{dx}{dt} = -\frac{L_q}{L_d}x + yz + \tilde{u}_d, \\ \frac{dy}{dt} = -y - xz + \gamma z + \tilde{u}_q, \\ \frac{dz}{dt} = \sigma(y - z) + \varepsilon xy - \tilde{T}_L, \end{cases} \tag{2}$$

where

$$\gamma = \frac{n_p\psi_r^2}{R_1\beta}, \qquad \sigma = \frac{L_q\beta}{R_1 J}, \qquad \tilde{u}_q = \frac{n_p L_q \psi_r u_q}{R_1^2\beta}, \qquad \tilde{u}_d = \frac{n_p L_q \psi_r u_d}{R_1^2\beta},$$

$$\varepsilon = \frac{L_q\beta^2(L_d - L_q)}{L_d J n_p \psi_r^2}, \qquad \tilde{T}_L = \frac{L_q^2 T_L}{R_1^2 J}, \qquad n_p = 1.$$

The PMSM system (2) with smooth-air-gap case ($L_q = L_d$) is written as [21]

$$\begin{cases} \frac{dx}{dt} = -x + yz + \tilde{u}_d, \\ \frac{dy}{dt} = -y - xz + \gamma z + \tilde{u}_q, \\ \frac{dz}{dt} = \sigma(y - z) - \tilde{T}_L, \end{cases} \tag{3}$$

where x, y and z are the new variables of the system (3), and the parameters γ and σ are positive constants.

When $\tilde{u}_d = 0$, $\tilde{u}_q = 0$, $\tilde{T}_L = 0$, where this case can be considered as the case that, after a period of operation, the external inputs of the system (3) are removed, the PMSM system (3) is written as [21]

$$\begin{cases} \frac{dx}{dt} = -x + yz, \\ \frac{dy}{dt} = -y - xz + \gamma z, \\ \frac{dz}{dt} = \sigma(y - z), \end{cases} \tag{4}$$

where x, y and z are the new variables of the system (4), and the parameters γ and σ are positive constants. There exist complex nonlinear dynamical behaviors in the system (4) including chaos and periodic orbit. The butterfly chaotic attractor of the system (4) with $\gamma = 100$ and $\sigma = 10$ in the $xoyz$ space is shown in Figure 1. Chaotic attractors of

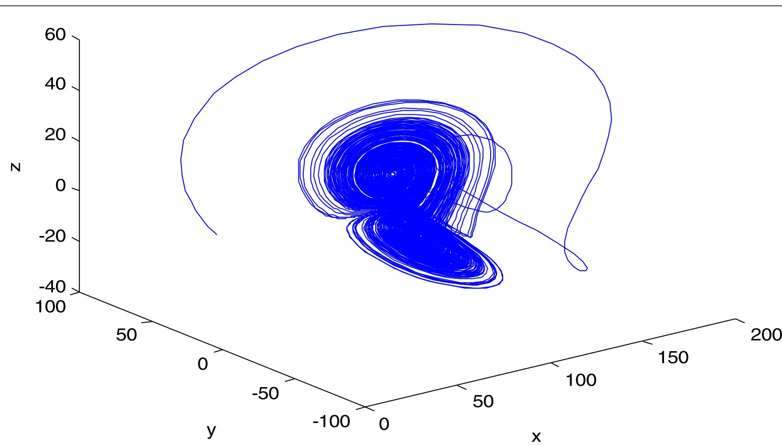

Figure 1 Chaotic attractor of system (4) in the *xoyz* space.

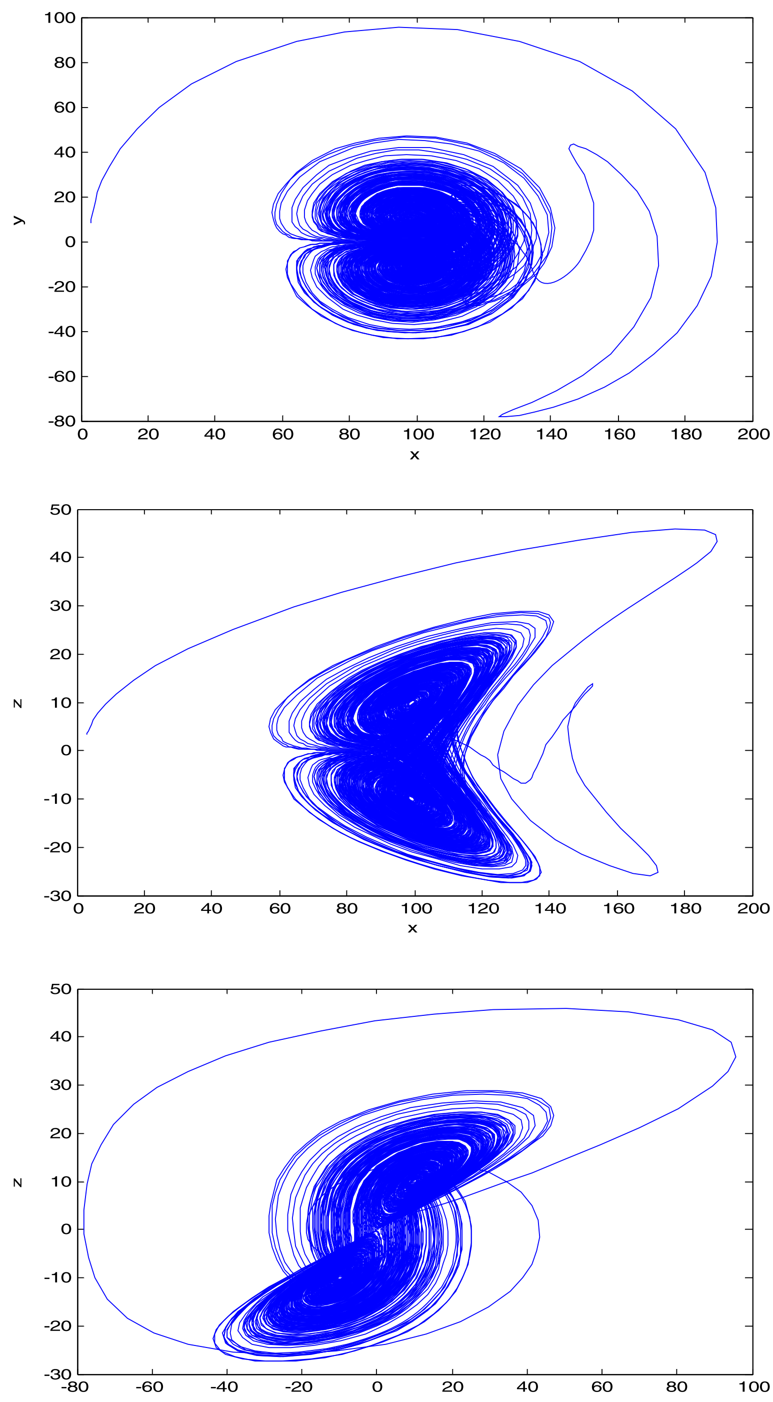

Figure 2 Projections of chaotic attractors of system (4) onto planes.

the system (4) with $\gamma = 100$ and $\sigma = 10$ on the x-y, x-z, and y-z planes are shown in Figure 2.

The periodic and chaos phenomena, phase portraits, bifurcation diagrams, Lyapunov exponents, chaos anti-control of the permanent-magnet synchronous motors (2), (3) and (4) are widely studied in [20, 21] in detail. But the global exponential attractive sets of systems (2)-(4) are still unknown. Our principal aim here is to investigate the global exponential attractive sets of (2), (3) and (4).

3 Dynamics of the PMSM

In this section, we will discuss the global exponential attractive sets of PMSM system (2), (3) and (4). We have the following results.

Theorem 1 *For* $\forall\lambda > \frac{1}{|\varepsilon|} > 0$, $L_q > 0$, $L_d > 0$, $\sigma > 0$, *with*

$$V_\lambda(X) = (1+\lambda\varepsilon)y^2 + (x-\gamma-\lambda\gamma\varepsilon-\lambda\sigma)^2 + \lambda z^2,$$
$$\theta = \min(a,\sigma,1) > 0, \qquad a = \frac{L_q}{L_d} > 0,$$
$$L_\lambda = \frac{1}{\theta}\left\{\frac{[u_d - a(\gamma+\lambda\gamma\varepsilon+\lambda\sigma)]^2}{a} + (1+\lambda\varepsilon)u_q^2 + \frac{\lambda\tilde{T}_L^2}{\sigma}\right\}, \qquad X(t) = \bigl(x(t),y(t),z(t)\bigr).$$

When $V_\lambda(X(t)) > L_\lambda$, $V_\lambda(X(t_0)) > L_\lambda$, *we can get the exponential estimate of the system* (2), *given by*

$$V_\lambda\bigl(X(t)\bigr) - L_\lambda \le \bigl(V_\lambda\bigl(X(t_0)\bigr) - L_\lambda\bigr)e^{-\theta(t-t_0)}.$$

That is to say, the set

$$\Omega_\lambda = \left\{(x,y,z)\middle|(1+\lambda\varepsilon)y^2 + (x-\gamma-\lambda\gamma\varepsilon-\lambda\sigma)^2 + \lambda z^2 \le L_\lambda, \forall\lambda > \frac{1}{|\varepsilon|} > 0\right\} \tag{5}$$

is the global exponential attractive set of the permanent-magnet synchronous motor system (2).

Proof Let us define

$$f(x) = -ax^2 + 2u_d x,$$
$$h(y) = -(1+\lambda\varepsilon)y^2 + 2(1+\lambda\varepsilon)u_q y,$$
$$g(z) = -\sigma\lambda z^2 - 2\lambda\tilde{T}_L z - 2u_d(\gamma+\lambda\gamma\varepsilon+\lambda\sigma),$$

then we get

$$\max_{x\in R} f(x) = \frac{u_d^2}{a},$$
$$\max_{y\in R} h(y) = (1+\lambda\varepsilon)u_q^2,$$
$$\max_{z\in R} g(z) = \frac{\lambda\tilde{T}_L^2}{\sigma} - 2u_d(\gamma+\lambda\gamma\varepsilon+\lambda\sigma).$$

Define the following Lyapunov-like function:

$$V_\lambda(X) = (1+\lambda\varepsilon)y^2 + (x-\gamma-\lambda\gamma\varepsilon-\lambda\sigma)^2 + \lambda z^2 \quad \left(\forall\lambda > \frac{1}{|\varepsilon|} > 0\right).$$

Differentiating the above Lyapunov-like function $V_\lambda(X)$ with respect time t along the trajectory of system (2) yields

$$\begin{aligned}
\left.\frac{dV_\lambda(X)}{dt}\right|_{(2)} &= 2(x-\gamma-\lambda\gamma\varepsilon-\lambda\sigma)\frac{dx}{dt}+2(1+\lambda\varepsilon)y\frac{dy}{dt}+2\lambda z\frac{dz}{dt} \\
&= 2(x-\gamma-\lambda\gamma\varepsilon-\lambda\sigma)(-ax+yz+\tilde{u}_d)+2(1+\lambda\varepsilon)y(-y-xz+\gamma z+\tilde{u}_q) \\
&\quad +2\lambda z\big[\sigma(y-z)+\varepsilon xy-\tilde{T}_L\big] \\
&= -2ax^2-2(1+\lambda\varepsilon)y^2+2(1+\lambda\varepsilon)u_q y-2\lambda\sigma z^2-2\lambda\tilde{T}_L z \\
&\quad +2\big[u_d+a(\gamma+\lambda\gamma\varepsilon+\lambda\sigma)\big]x+-2u_d(\gamma+\lambda\gamma\varepsilon+\lambda\sigma) \\
&= -2ax^2+2u_d x+2a(\gamma+\lambda\gamma\varepsilon+\lambda\sigma)x-2(1+\lambda\varepsilon)y^2+2(1+\lambda\varepsilon)u_q y \\
&\quad -2\lambda\sigma z^2-2\lambda\tilde{T}_L z-2u_d(\gamma+\lambda\gamma\varepsilon+\lambda\sigma) \\
&= -ax^2+2a(\gamma+\lambda\gamma\varepsilon+\lambda\sigma)x-ax^2+2u_d x-(1+\lambda\varepsilon)y^2-(1+\lambda\varepsilon)y^2 \\
&\quad +2(1+\lambda\varepsilon)u_q y-\lambda\sigma z^2-\lambda\sigma z^2-2\lambda\tilde{T}_L z-2u_d(\gamma+\lambda\gamma\varepsilon+\lambda\sigma) \\
&= -ax^2+2a(\gamma+\lambda\gamma\varepsilon+\lambda\sigma)x+f(x)-(1+\lambda\varepsilon)y^2+h(y)-\lambda\sigma z^2+g(z) \\
&= -a\big[x-(\gamma+\lambda\gamma\varepsilon+\lambda\sigma)\big]^2+a(\gamma+\lambda\gamma\varepsilon+\lambda\sigma)^2-(1+\lambda\varepsilon)y^2-\lambda\sigma z^2 \\
&\quad +f(x)+h(y)+g(z) \\
&\le -\theta V_\lambda(X)+a(\gamma+\lambda\gamma\varepsilon+\lambda\sigma)^2+f(x)+h(y)+g(z) \\
&\le -\theta V_\lambda(X)+a(\gamma+\lambda\gamma\varepsilon+\lambda\sigma)^2+\frac{u_d^2}{a}+(1+\lambda\varepsilon)u_q^2 \\
&\quad +\frac{\lambda\tilde{T}_L^2}{\sigma}-2u_d(\gamma+\lambda\gamma\varepsilon+\lambda\sigma) \\
&\le -\theta V_\lambda(X)+\frac{[u_d-a(\gamma+\lambda\gamma\varepsilon+\lambda\sigma)]^2}{a}+(1+\lambda\varepsilon)u_q^2+\frac{\lambda\tilde{T}_L^2}{\sigma} \\
&\le -\theta\big[V_\lambda(X)-L_\lambda\big]<0,
\end{aligned}$$

which is equivalent to

$$\left.\frac{dV_\lambda(X(t))}{dt}\right|_{(2)} \le -\theta\big(V_\lambda(X)-L_\lambda\big)<0. \tag{6}$$

Integrating both sides of equation (6) yields

$$\begin{aligned}
V_\lambda\big(X(t)\big) &\le V_\lambda\big(X(t_0)\big)e^{-\theta(t-t_0)}+\int_{t_0}^{t}\theta L_\lambda e^{-\theta(t-\tau)}\,d\tau \\
&= V_\lambda\big(X(t_0)\big)e^{-\theta(t-t_0)}+L_\lambda\big(1-e^{-\theta(t-t_0)}\big),
\end{aligned}$$

and if $V_\lambda(X(t))>L_\lambda$, $V_\lambda(X(t_0))>L_\lambda$, we have the inequality for system (2) given by

$$V_\lambda\big(X(t)\big)-L_\lambda \le \big(V_\lambda\big(X(t_0)\big)-L_\lambda\big)e^{-\theta(t-t_0)}. \tag{7}$$

By definition, taking the limit on both sides of the above inequality as $t \to +\infty$ results in

$$\overline{\lim_{t\to+\infty}} V_\lambda\big(X(t)\big) \le L_\lambda. \tag{8}$$

Namely,

$$\Omega_\lambda = \left\{(x,y,z)\middle|(1+\lambda\varepsilon)y^2 + (x-\gamma-\lambda\gamma\varepsilon-\lambda\sigma)^2 + \lambda z^2 \le L_\lambda, \forall\lambda > \frac{1}{|\varepsilon|} > 0\right\}$$

is the global exponential attractive set of the permanent-magnet synchronous motor system (2).

This completes the proof. □

Theorem 2 *For* $\forall\lambda > 0$, $\sigma > 0$, *with*

$$V_\lambda(X) = y^2 + (x-\gamma-\lambda\sigma)^2 + \lambda z^2, \qquad \theta_0 = \min(\sigma,1) > 0,$$
$$M_\lambda = \frac{1}{\theta_0}\left\{\big[u_d - (\gamma+\lambda\sigma)\big]^2 + u_q^2 + \frac{\lambda\tilde{T}_L^2}{\sigma}\right\}, \qquad X(t) = \big(x(t),y(t),z(t)\big).$$

When $V_\lambda(X(t)) > M_\lambda$, $V_\lambda(X(t_0)) > M_\lambda$, *we can get the exponential estimate of the system* (3), *given by*

$$V_\lambda\big(X(t)\big) - M_\lambda \le \big(V_\lambda\big(X(t_0)\big) - M_\lambda\big)e^{-\theta_0(t-t_0)}.$$

That is to say, the set

$$\Psi_\lambda = \big\{(x,y,z)|y^2 + (x-\gamma-\lambda\sigma)^2 + \lambda z^2 \le M_\lambda, \forall\lambda > 0\big\} \tag{9}$$

is the global exponential attractive set of the permanent-magnet synchronous motor system (3).

Proof Let us define

$$f_1(x) = -x^2 + 2u_d x,$$
$$h_1(y) = -y^2 + 2u_q y,$$
$$g_1(z) = -\sigma\lambda z^2 - 2\lambda\tilde{T}_L z - 2u_d(\gamma+\lambda\sigma),$$

then we can get

$$\max_{x\in R} f_1(x) = u_d^2,$$
$$\max_{y\in R} h_1(y) = u_q^2,$$
$$\max_{z\in R} g_1(z) = \frac{\lambda\tilde{T}_L^2}{\sigma} - 2u_d(\gamma+\lambda\sigma).$$

Define the following Lyapunov-like function:

$$V_\lambda(X) = y^2 + (x - \gamma - \lambda\sigma)^2 + \lambda z^2, \quad \forall\lambda > 0.$$

Differentiating the above Lyapunov-like function $V_\lambda(X)$ with respect time t along the trajectory of system (3) yields

$$\begin{aligned}
\left.\frac{dV_\lambda(X)}{dt}\right|_{(3)} &= 2(x - \gamma - \lambda\sigma)\frac{dx}{dt} + 2y\frac{dy}{dt} + 2\lambda z\frac{dz}{dt} \\
&= 2(x - \gamma - \lambda\sigma)(-x + yz + \tilde{u}_d) + 2y(-y - xz + \gamma z + \tilde{u}_q) \\
&\quad + 2\lambda z\big[\sigma(y - z) - \tilde{T}_L\big] \\
&= -2x^2 - 2y^2 + 2u_q y - 2\lambda\sigma z^2 - 2\lambda\tilde{T}_L z + 2\big[u_d + (\gamma + \lambda\sigma)\big]x \\
&\quad + -2u_d(\gamma + \lambda\sigma) \\
&= -2x^2 + 2u_d x + 2(\gamma + \lambda\sigma)x - 2y^2 + 2u_q y - 2\lambda\sigma z^2 - 2\lambda\tilde{T}_L z \\
&\quad - 2u_d(\gamma + \lambda\sigma) \\
&= -x^2 + 2(\gamma + \lambda\sigma)x - x^2 + 2u_d x - y^2 - y^2 \\
&\quad + 2u_q y - \lambda\sigma z^2 - \lambda\sigma z^2 - 2\lambda\tilde{T}_L z - 2u_d(\gamma + \lambda\sigma) \\
&= -x^2 + 2(\gamma + \lambda\sigma)x + f_1(x) - y^2 + h_1(y) - \lambda\sigma z^2 + g_1(z) \\
&= -\big[x - (\gamma + \lambda\sigma)\big]^2 + (\gamma + \lambda\sigma)^2 - y^2 - \lambda\sigma z^2 \\
&\quad + f_1(x) + h_1(y) + g_1(z) \\
&\leq -\theta_0 V_\lambda(X) + (\gamma + \lambda\sigma)^2 + f_1(x) + h_1(y) + g_1(z) \\
&\leq -\theta_0 V_\lambda(X) + (\gamma + \lambda\sigma)^2 + u_d^2 + u_q^2 + \frac{\lambda\tilde{T}_L^2}{\sigma} - 2u_d(\gamma + \lambda\sigma) \\
&\leq -\theta_0 V_\lambda(X) + \big[u_d - (\gamma + \lambda\sigma)\big]^2 + u_q^2 + \frac{\lambda\tilde{T}_L^2}{\sigma} \\
&\leq -\theta_0\big[V_\lambda(X) - M_\lambda\big] < 0,
\end{aligned}$$

and integrating both sides of the above formula yields

$$\begin{aligned}
V_\lambda\big(X(t)\big) &\leq V_\lambda\big(X(t_0)\big)e^{-\theta_0(t-t_0)} + \int_{t_0}^{t} \theta_0 M_\lambda e^{-\theta_0(t-\tau)}\,d\tau \\
&= V_\lambda\big(X(t_0)\big)e^{-\theta_0(t-t_0)} + M_\lambda\big(1 - e^{-\theta_0(t-t_0)}\big),
\end{aligned}$$

while if $V_\lambda(X(t)) > M_\lambda$, $V_\lambda(X(t_0)) > M_\lambda$, we have the inequality for system (3) given by

$$V_\lambda\big(X(t)\big) - M_\lambda \leq \big(V_\lambda\big(X(t_0)\big) - M_\lambda\big)e^{-\theta_0(t-t_0)}. \tag{10}$$

Similarly, we get

$$\Psi_\lambda = \big\{(x, y, z) | y^2 + (x - \gamma - \lambda\sigma)^2 + \lambda z^2 \leq M_\lambda, \forall\lambda > 0\big\}$$

as the global exponential attractive set of the permanent-magnet synchronous motor system (3).

This completes the proof. □

Remark 1 (i) Currently the question is being actively discussed of the equivalence of various Lorenz-like systems and the possibility of universal consideration of their behavior [27–29] in view of the possibility of the reduction of such systems to the same normal form with the help of various reversible transformations. It is straightforward to obtain, simply by interchanging the state variables x and z in system (4), the PMSM model (4) can be written in the form of the Lorenz system (11) as follows:

$$\begin{cases} \frac{dx}{dt} = \sigma(y-x), \\ \frac{dy}{dt} = -y - xz + \gamma x, \\ \frac{dz}{dt} = -z + xy, \end{cases} \tag{11}$$

then, as a summary, the permanent-magnet synchronous motor system considered in (4), from a dynamical point of view, has an identical behavior to the Lorenz system. Thus, many known results on the localization and global exponential attractive sets of the Lorenz system can be used for the considered system (4) (see [5, 7, 30, 31] for a detailed discussion of the localization and global exponential attractive sets of the Lorenz system). Systems (2) and (3) are not equivalent to the Lorenz system [20, 21], the already known techniques do not work for the considered systems (2) and (3).

4 Conclusions

In this paper, the global attractive sets of the permanent-magnet synchronous motor have been obtained based on dynamical systems theory. This method can be applied to consider other chaotic systems. In the future we will conduct research on how to control the PMSM to avoid the chaotic behavior and protect the motors in practical applications.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have read and approved the final manuscript.

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