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Jackson's (-1) -Bessel functions with the Askey-Wilson algebra setting

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University, P.O. Box 2455, Riyadh,
11451, Saudi Arabia**Abstract**

This work is devoted to the study of some functions arising from a limit transition of the Jackson q -Bessel functions when q tends to -1 . These functions coincide with the so-called *cas* function for particular values of parameters. We prove that there are eigenfunctions of differential-difference operators of Dunkl-type. Also we consider special cases of the Askey-Wilson algebra $AW(3)$, which have these operators (up to constants) as one of their three generators and whose defining relations are given in terms of anti-commutators.

MSC: 33D45; 33D80**Keywords:** q -special functions; difference-differential equations; Askey-Wilson algebra

1 Introduction

In [1], Vinet and Zhedanov introduced new families of orthogonal polynomials by considering appropriate limits when q tends to -1 of the little and big q -Jacobi polynomials. In this work we will study the limit when q tends to -1 of the three q -analogs of the Bessel function, which are introduced by Jackson [2–4]. The first and the second q -Bessel functions are reconsidered and rewritten in modern notations by Ismail [5]. The third q -Bessel function is rediscovered later by Hahn [6] and Exton [7]. This function has an interpretation as matrix elements of irreducible representations of the quantum group of plane motions $E_q(2)$ and satisfies an orthogonality relation that makes it more suitable for harmonic analysis [8–10]. Of course, when q tends to 1, the Jackson q -Bessel functions tend to the standard Bessel function [11].

In this paper we will show that the limit when q tends to -1 of the third q -Bessel functions leads to a new type of nonsymmetric Bessel functions satisfying first order differential-difference equation. Also these functions coincide for a particular value of its parameters with the *cas* function [12]. Furthermore, by using the limit transition from little q -Jacobi polynomials to the third q -Bessel function and from q -Laguerre to the second q -Bessel function we construct a q -Bessel version of the Askey-Wilson $AW(3)$ algebra.

Notations Throughout we assume $-1 < q < 1$. For q -Pochhammer symbols and q -hypergeometric series we use the notation of [13],

$$(a; q)_0 := 1, \quad (a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k), \quad n = 1, \dots, \infty. \quad (1.1)$$

The basic hypergeometric series are defined by

$${}_r\Phi_s \left(\begin{matrix} a_2, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| q; z \right) := \sum_{k=0}^{+\infty} \frac{(a_1, \dots, a_r; q)_k}{(q, b_1, \dots, b_s; q)_k} ((-1)^k q^{\binom{k}{2}})^{1+s-r} z^k, \tag{1.2}$$

where

$$(a_1, \dots, a_r; q)_n := (a_1; q)_n \cdots (a_r; q)_n.$$

2 Askey-Wilson relations for the little q -Jacobi polynomials

2.1 Little q -Jacobi polynomials

The little q -Jacobi polynomial is defined by [14], (18.27(iv))

$$p_n(x) = p_n(x; a, b|q) := {}_2\Phi_1 \left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix} \middle| q; qx \right). \tag{2.1}$$

For $0 < a < q^{-1}$ and $b < q^{-1}$, the polynomials $\{p_n(x)\}_n$ satisfy the orthogonality relations

$$\frac{(aq; q)_\infty}{(abq^2; q)_\infty} \sum_{k=0}^{\infty} p_m(q^k) p_n(q^k) (aq)^k \frac{(bq; q)_k}{(q; q)_k} = \frac{(1-abq)(aq)^n}{1-abq^{2n+1}} \frac{(q, bq; q)_n}{(aq, abq; q)_n} \delta_{m,n}.$$

There is a q -difference equation for this polynomial of the form

$$Y_{a,b,q}f(x) = \lambda_n f(x), \tag{2.2}$$

where

$$(Y_{a,b,q}f) := a(bq - x^{-1})(f(qx) - f(x)) + (1 - x^{-1})(f(q^{-1}x) - f(x)) \tag{2.3}$$

and

$$\lambda_n = q^{-n}(1 - q^n)(1 - abq^{n+1}). \tag{2.4}$$

The Askey-Wilson algebra $AW(3)$ involves a nonzero scalar q and three parameters ω_1 , ω_2 , and ω_3 , it was introduced by Zhedanov [15] as an associative algebra generated by X , Y , and Z subject to the following commutation relations:

$$\begin{aligned} YX - qXY &= \mu_3 Z + \omega_3, & ZY - qYZ &= \mu_2 X + \omega_2, \\ XZ - qZX &= \mu_1 Y + \omega_1. \end{aligned} \tag{2.5}$$

There is a central element Q , which is explicitly given as a polynomial of degree 3 in terms of X , Y , and Z [15]

$$Q = (q^2 - 1)YXZ + \mu_1 Y^2 + \mu_2 qX^2 + \mu_3 Z^2 + (q + 1)(\omega_1 Y + \omega_2 qX + \omega_3 Z). \tag{2.6}$$

The limit of orthogonal polynomials in the Askey scheme as $q \rightarrow 1$ corresponds to the limit $q \rightarrow 1$ of $AW(3)$ to some classical algebras. In particular, the related Askey-Wilson

algebra for the little q -Jacobi polynomial is generated by X, Y, Z with relations [1]

$$YX - qXY = Z + \omega_3, \quad ZY - qYZ = X + \omega_2, \quad XZ - qZX = 0, \tag{2.7}$$

where

$$\omega_2 = -\frac{1+b}{(1+q)b}, \quad \omega_3 = -\frac{1+a}{(1+q)\sqrt{ab}}. \tag{2.8}$$

There is a representation on the space of polynomials of the little q -Jacobi $AW(3)$ defined by relations (2.7) with structure constants given in (2.8) as follows:

$$\begin{aligned} (Yf)(x) &:= \mu(Y_{a,b,q} + 1 + qab)f(x), & (Xf)(x) &:= xf(x), \\ (Zf)(x) &:= \frac{1-x}{q\sqrt{ab}}f(q^{-1}x), \end{aligned} \tag{2.9}$$

where

$$\mu = -\frac{1}{(1-q^2)\sqrt{ab}}.$$

The Casimir operator

$$Q = (q^2 - 1)YXZ + q^2X^2 + Z^2 + (q + 1)(\omega_2qX + \omega_3Z) \tag{2.10}$$

takes the value $Q = -b^{-1}$.

2.2 Little (-1) -Jacobi polynomials

The little (-1) -Jacobi polynomials $P_n^{(\alpha,\beta,-1)}(x)$ have been introduced and investigated in [1] as limits of the little q -Jacobi polynomials (2.1)

$$\lim_{\varepsilon \rightarrow 0} p_n(x; -e^{\varepsilon\alpha}, -e^{\varepsilon\beta} | -e^{\varepsilon}) = P_n^{(\alpha,\beta,-1)}(x). \tag{2.11}$$

We recall here their basic properties [1]. The polynomial $P_n^{(\alpha,\beta,-1)}(x)$ satisfies the following difference-differential equation

$$(Y_{\alpha,\beta,-1}f)(x) = \lambda_n f(x), \tag{2.12}$$

where

$$(Y_{\alpha,\beta,-1}f)(x) = (x-1)f'(-x) + (\alpha + \beta - \alpha x^{-1})\frac{f(x) - f(-x)}{2}$$

and

$$\lambda_n = \begin{cases} -n, & \text{if } n \text{ is even,} \\ \alpha + \beta + n + 1, & \text{if } n \text{ is odd.} \end{cases}$$

These polynomials have the following expressions in terms of the hypergeometric series:

For n , even

$$P_n^{(\alpha, \beta, -1)}(x) = {}_2F_1\left(-\frac{n}{2}, \frac{\alpha + \beta + n + 2}{2}; x^2\right) + \frac{nx}{\alpha + 1} {}_2F_1\left(1 - \frac{n}{2}, \frac{\alpha + \beta + n + 2}{2}; x^2\right),$$

and for n odd

$$P_n^{(\alpha, \beta, -1)}(x) = {}_2F_1\left(-\frac{n-1}{2}, \frac{\alpha + \beta + n + 1}{2}; x^2\right) - \frac{(\alpha + \beta + n + 1)x}{\alpha + 1} {}_2F_1\left(-\frac{n-1}{2}, \frac{\alpha + \beta + n + 3}{2}; x^2\right).$$

We introduce the operators

$$(Yf)(x) := (Y_{\alpha, \beta, -1}f)(x) - \frac{1}{2}(\alpha + \beta + 1)f(x), \quad (Xf)(x) := xf(x),$$

$$(Zf)(x) := (x - 1)f(-x).$$

In [1], it was shown that these operators are closed in the framework of the Askey-Wilson algebra and they satisfy the commutation relations

$$\{X, Y\} = Z + \alpha, \quad \{X, Z\} = 0, \quad \{Y, Z\} = Y + \beta, \tag{2.13}$$

with $\{A, B\} = AB + BA$ denoting as usual the anti-commutator of A and B .

The Casimir operator is

$$Q = Y^2 + Z^2 \tag{2.14}$$

and takes the value

$$Q = 1.$$

3 The nonsymmetric Hankel transform

For Bessel functions $J_\alpha(x)$ see [14], Chapter 10, and the references given therein. Let us consider the normalized Bessel function $\mathcal{J}_\alpha(x)$, which is given by

$$\mathcal{J}_\alpha(x) := \Gamma(\alpha + 1)(2/x)^\alpha J_\alpha(x). \tag{3.1}$$

Then

$$\mathcal{J}_\alpha(x) = \sum_{k=0}^\infty \frac{(-\frac{1}{4}x^2)^k}{(\alpha + 1)_k k!} = {}_0F_1\left(\alpha + 1; -\frac{1}{4}x^2\right) \quad (\alpha > -1).$$

$\mathcal{J}_\alpha(x)$ is an entire function and has the simple properties and special cases [14], Section 10.16.9

$$\mathcal{J}_\alpha(x) = \mathcal{J}_\alpha(-x), \quad \mathcal{J}_\alpha(0) = 1, \quad \mathcal{J}_{-1/2}(x) = \cos x, \quad \mathcal{J}_{1/2}(x) = \frac{\sin x}{x}.$$

The function $x \mapsto \mathcal{J}_\alpha(\lambda x)$ satisfies also the eigenvalue equation [14], Section 10.13.5:

$$\left(\frac{d^2}{dx^2} + \frac{2\alpha + 1}{x} \frac{d}{dx}\right) \mathcal{J}_\alpha(\lambda x) = -\lambda^2 \mathcal{J}_\alpha(\lambda x).$$

The *Hankel transform* pair [14], Section 10.22(v), for f in a suitable function class, is given by

$$\begin{cases} \widehat{f}(\lambda) = \int_0^\infty f(x) \mathcal{J}_\alpha(\lambda x) \frac{x^{2\alpha+1} dx}{2^{\alpha+1/2} \Gamma(\alpha+1)}, \\ f(x) = \int_0^\infty \widehat{f}(\lambda) \mathcal{J}_\alpha(\lambda x) \frac{\lambda^{2\alpha+1} d\lambda}{2^{\alpha+1/2} \Gamma(\alpha+1)}. \end{cases} \tag{3.2}$$

Now consider the so-called *nonsymmetric Bessel function*, also called *Dunkl-type Bessel function*, in the rank one case (see [16], Section 4):

$$\mathcal{E}_\alpha(x) := \mathcal{J}_\alpha(x) + \frac{ix}{2(\alpha+1)} \mathcal{J}_{\alpha+1}(x). \tag{3.3}$$

In particular,

$$\mathcal{E}_{-1/2}(x) = e^{ix}.$$

The nonsymmetric Hankel transform pair takes the form

$$\begin{cases} \widehat{f}(\lambda) = \int_{\mathbb{R}} f(x) \mathcal{E}_\alpha(\lambda x) \frac{|x|^{2\alpha+1} dx}{2^{\alpha+1} \Gamma(\alpha+1)}, \\ f(x) = \int_{\mathbb{R}} \widehat{f}(\lambda) \mathcal{E}_\alpha(\lambda x) \frac{|\lambda|^{2\alpha+1} d\lambda}{2^{\alpha+1} \Gamma(\alpha+1)}. \end{cases} \tag{3.4}$$

The transform pair (3.4) follows immediately from (3.2). For some given α let us define the differential-reflection operator

$$(T_\alpha f)(x) := f'(x) + \left(\alpha + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x}, \tag{3.5}$$

called the *Dunkl operator* for root system A_1 (see [17], Definition 4.4.2). We have the eigenvalue equation

$$T_\alpha(\mathcal{E}_\alpha(\lambda x)) = i\lambda \mathcal{E}_\alpha(\lambda x). \tag{3.6}$$

If in (3.6) we substitute (3.3), compare even and odd parts, and then substitute (3.1), then we see that (3.6) is equivalent to a pair of lowering and raising differentiation formulas for Bessel functions (see [14], (10.6.2)):

$$J'_\alpha(x) - \frac{\alpha}{x} J_\alpha(x) = -J_{\alpha+1}(x), \quad J'_{\alpha+1}(x) + \frac{\alpha+1}{x} J_{\alpha+1}(x) = J_\alpha(x).$$

The double degeneration of the double affine Hecke algebra \mathcal{H}'' is generated by D , Z , and s with relations [18]

$$sZs^{-1} = -Z, \quad sDs^{-1} = -D, \quad [D, Z] = 1 + 2ks. \tag{3.7}$$

The operators D , s , and Z act on the polynomial $f(x)$ as

$$(Df)(x) = (T_\alpha f)(x), \quad (sf)(x) = f(-x), \quad (Zf)(x) = xf(x). \tag{3.8}$$

The operators defined in (3.8) are also known as para-Bose operators and the algebra (3.7) is equivalent to the para-Bose algebra [19, 20].

Another important generalization of the exponential function is the so-called cas function, which is given by [12, 21]

$$\text{cas}(x) = \cos(x) + \sin(x). \tag{3.9}$$

It is evident that the function $y(x) = \text{cas}(\lambda x)$ satisfies

$$(\Lambda y)(x) = \lambda y(x), \quad y(0) = 1, \tag{3.10}$$

where $\Lambda = s\partial$ and ∂ is the derivative operator.

4 Limit $q \rightarrow -1$ of the third q -Bessel function

The third q -Bessel function was introduced in [6, 7], see also [7, 9, 10], and is defined by

$$J_v^{(3)}(x; q) := \frac{(q^{v+1}; q)_\infty}{(q; q)_\infty} \left(\frac{1}{2}x\right)^v {}_1\Phi_1\left(\begin{matrix} 0 \\ q^{v+1} \end{matrix} \middle| q; \frac{1}{4}qx^2\right) \quad (x > 0).$$

We will consider a slightly different function $\mathcal{J}_3(x, a; q)$, called a normalized third q -Bessel function, which is defined by

$$\mathcal{J}_3(x, a; q) = {}_1\Phi_1\left(\begin{matrix} 0 \\ aq \end{matrix} \middle| q; qx\right). \tag{4.1}$$

It is easy to see that

$$\lim_{q \rightarrow 1} \mathcal{J}_3((1 - q)^2 x, q^\alpha; q) = \mathcal{J}_\alpha(2\sqrt{x}).$$

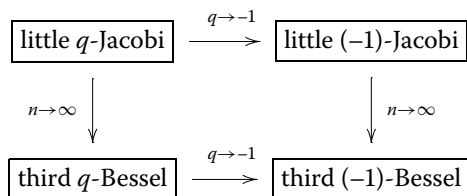
The function $\mathcal{J}_3(\lambda x, a; q)$ is a solution of the q -difference

$$(Y_{a,q}f)(x) = -\lambda f(x), \tag{4.2}$$

where

$$(Y_{a,q}f)(x) := \frac{a}{x}(f(qx) - f(x)) + \frac{1}{x}(f(q^{-1}x) - f(x)). \tag{4.3}$$

Next, we describe the construction of new function by the limiting process from normalized third q -Bessel as $q \rightarrow -1$ and from the (-1) -Jacobi polynomials as $n \rightarrow \infty$. Moreover, we have the following diagram for limit relations between these special functions and orthogonal polynomials.



The upper limit in the previous diagram is studied by [1]. Next we are concerned only by the three other limits.

There is a well-known limit from Jacobi polynomials to Bessel functions, see [14], (18.11.5),

$$\lim_{n \rightarrow \infty} \frac{\Gamma(\alpha + 1)}{n^\alpha} P_n^{(\alpha, \beta)} \left(1 - \frac{x^2}{2n^2} \right) = \mathcal{J}_\alpha(x). \tag{4.4}$$

The q -analog of this limit transition starts with the little q -Jacobi polynomials (2.1). From Proposition A.1 in [9] we have

$$\lim_{n \rightarrow \infty} p_n(q^n x; a, b|q) = \mathcal{J}_3(x, a; q). \tag{4.5}$$

The operators $X, Y,$ and Z defined in (2.9) have also a limit for $n \rightarrow \infty$ after the rescaling $x \rightarrow q^n x$. More precisely, let us denote

$$\begin{aligned} (Xf)(x) &:= \lim_{n \rightarrow \infty} q^{-n} (Xf)(q^n x), \\ (Yf)(x) &:= \lim_{n \rightarrow \infty} \sqrt{ab} q^n (Yf)(q^n x), \\ (Zf)(x) &:= \lim_{n \rightarrow \infty} \sqrt{ab} (Zf)(q^n x). \end{aligned}$$

Then in the limit the operators $X, Y,$ and Z are given by

$$(Xf)(x) = xf(x), \quad (Yf)(x) = \frac{1}{q^2 - 1} (Y_{a,q} f)(x), \quad (Zf)(x) = q^{-1} f(q^{-1} x), \tag{4.6}$$

and (2.13) become

$$YX - qXY = Z + \omega_3, \quad ZY - qYZ = 0, \quad XZ - qZX = 0, \tag{4.7}$$

where

$$\omega_3 = -\frac{1 + a}{1 + q}. \tag{4.8}$$

The resulting algebra generated by $X, Y,$ and Z with (4.7) and (4.8) is called the q -Bessel $AW(3)$ algebra. In this case the Casimir operator becomes

$$Q = (q^2 - 1)YXZ + Z^2 - (1 + a)Z \tag{4.9}$$

and takes the value $Q = -a$.

Equations (4.7) hold in the limit $q \rightarrow -1$. Indeed, let us take the parametrization

$$a = -e^{\varepsilon(2\alpha+1)} \quad \text{and} \quad q = -e^\varepsilon.$$

Since

$$f(-e^{\pm\varepsilon} z) = f(-z) \mp zf(-z)\varepsilon + o(\varepsilon),$$

$$q = -1 - \varepsilon + o(\varepsilon),$$

$$a = -1 - (2\alpha + 1)\varepsilon + o(\varepsilon),$$

the q -difference equation (4.2) tends formally as $\varepsilon \rightarrow 0$ to the differential-difference equation

$$(Y_\alpha f)(x) = \lambda f(x), \tag{4.10}$$

where

$$(Y_\alpha f)(x) = f'(-x) + \left(\alpha + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x}. \tag{4.11}$$

The operator Y_α is a difference-differential operator of the first order containing reflection terms. Notice the important property of the operator Y_α : it sends the linear space of polynomials of dimension $n + 1$ to the space of dimension n . In particular, this means that there are no polynomial eigenfunction of the operator Y_α .

Now, let us introduce the operators

$$(Xf)(x) := xf(x), \quad (Yf)(x) := (Y_\alpha f)(x), \quad (Zf)(x) := f(-x). \tag{4.12}$$

Then it is elementary to verify that the operators X, Y, Z satisfy the relations

$$\{X, Y\} = -Z - 2\alpha - 1, \quad \{X, Z\} = 0, \quad \{Y, Z\} = 0, \tag{4.13}$$

which corresponds to the $AW(3)$ algebra with parameters

$$q = -1, \quad \omega_3 = -2\alpha - 1, \quad \mu_3 = -1, \quad \omega_2 = \omega_3 = \mu_1 = \mu_2 = 0.$$

It is easily verified that the Casimir operator is

$$Q = Z^2. \tag{4.14}$$

In the case of the realization (4.12) of the operators X, Y, Z , the Casimir operator becomes the identity operator.

Theorem 4.1 *For each $\lambda \in \mathbb{C}$. The differential-difference equation (4.10) under the initial condition $y(0) = 1$, admits unique C^∞ -solution denoted $\mathcal{J}_{\alpha,-1}(\lambda x)$, which is expressed in terms of the normalized q -Bessel function (3.1) by*

$$\mathcal{J}_{\alpha,-1}(\lambda x) = \mathcal{J}_\alpha(\lambda x) + \frac{\lambda x}{2(\alpha + 1)} \mathcal{J}_{\alpha+1}(\lambda x). \tag{4.15}$$

Proof From the decomposition in the form $f = f_1 + f_2$ where f_1 is even and f_2 is odd. Equation (3.1) is equivalent to the following system:

$$\begin{cases} -f_1'(x) = \lambda f_2(x), \\ f_2'(x) + \frac{2\alpha+1}{x} f_2(x) = \lambda f_1(x), \\ f_1(0) = 1, \quad f_1'(0) = 0. \end{cases}$$

Thus

$$\begin{cases} -f_1'(x) = \lambda f_2(x), \\ f_1''(x) + \frac{2\alpha+1}{x} f_1'(x) = -\lambda^2 f_1(x), \\ f_1(0) = 1, \quad f_1'(0) = 0. \end{cases}$$

Hence,

$$f_1(x) = \mathcal{J}_\alpha(\lambda x) \quad \text{and} \quad f_2(x) = -\frac{1}{\lambda} \frac{d}{dx} \mathcal{J}_\alpha(\lambda x). \quad \square$$

In particular, for $\alpha = -1/2$, the (-1) -Bessel function $\mathcal{J}_{\alpha,-1}(\lambda x)$ coincides with the cas function (3.9)

$$\mathcal{J}_{-1/2,-1}(\lambda x) = \text{cas}(\lambda x).$$

The function $\mathcal{J}_{\alpha,-1}(\lambda x)$ is related to the Dunkl function (3.3) by

$$\mathcal{J}_{\alpha,-1}(\lambda x) = \frac{1}{2} \left((1+i)\mathcal{E}_\alpha(\lambda x) + (1-i)\mathcal{E}_\alpha(-\lambda x) \right). \tag{4.16}$$

The (-1) -Bessel transform pair takes the form

$$\begin{cases} \widehat{f}(\lambda) = \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)} \int_{-\infty}^{\infty} f(x) \mathcal{J}_{\alpha,-1}(-\lambda x) |x|^{2\alpha+1} dx, \\ f(x) = \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)} \int_{-\infty}^{\infty} \widehat{f}(\lambda) \mathcal{J}_{\alpha,-1}(\lambda x) |\lambda|^{2\alpha+1} d\lambda. \end{cases} \tag{4.17}$$

The transform pair (4.17) follows immediately from (3.2) by putting $f(x) = f_1(x) + x f_2(x)$ in (4.17) with f_1 and f_2 even.

The (-1) -Bessel function $\mathcal{J}_{\alpha,-1}(\lambda x)$ can be obtained also as limit case of the normalized third q -Bessel function (4.1)

$$\lim_{\varepsilon \rightarrow 0} \mathcal{J}_3(x, -e^{\varepsilon(2\alpha+1)}; -e^\varepsilon) = \mathcal{J}_{\alpha,-1}(x). \tag{4.18}$$

Indeed, from (4.1) we can expand $\mathcal{J}(x, -e^{\varepsilon(2\alpha+1)}; -e^\varepsilon)$ in a power series of x as follows:

$$\mathcal{J}(x, -e^{\varepsilon(2\alpha+1)}; -e^\varepsilon) = \sum_{n=0}^{\infty} c_{n,\alpha}(\varepsilon) x^n,$$

where

$$c_{n,\alpha}(\varepsilon) = (-1)^n \frac{(-1)^{\frac{n(n-1)}{2}} (1 - e^{2\varepsilon})^n}{4^n (e^{\varepsilon(2\alpha+2)}; -e^\varepsilon; -e^\varepsilon)_n}.$$

Then the result follows from the following elementary limits involving q -shifted factorials:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \varepsilon^{-[n/2]} (-e^{\varepsilon\alpha}; -e^\varepsilon)_n &= (-1)^{[n/2]} 2^n ((\alpha + 1)/2)_{[n/2]}, \\ \lim_{\varepsilon \rightarrow 0} \varepsilon^{-[(n+1)/2]} (e^{\varepsilon\alpha}; -e^\varepsilon)_n &= (-1)^{[(n+1)/2]} 2^n (\alpha/2)_{[(n+1)/2]}. \end{aligned}$$

5 The second q -Bessel function case

5.1 q -Laguerre polynomials

The q -Laguerre polynomials $\{L_n(x, a; q)\}_n$ are defined by

$$L_n(x, a; q) := \frac{(aq; q)_n}{(q; q)_n} \phi_1 \left(\begin{matrix} q^{-n} \\ aq \end{matrix} \middle| q, -aq^{n+1}x \right), \tag{5.1}$$

we have used slightly different notations (see [14], (18.27.15)). They satisfy the recurrence relations

$$\begin{aligned} -aq^{2n+1}xL_n(x, a; q) &= (1 - q^{n+1})L_{n+1}(x, a; q) - [(1 - q^{n+1}) \\ &\quad + q(1 - aq^n)]L_n(x, a; q) + q(1 - aq^n)L_{n-1}(x, a; q). \end{aligned} \tag{5.2}$$

There is a q -difference equation of the form

$$(L_{a,q}y)(x) = -a(1 - q^n)y(x), \tag{5.3}$$

where

$$(L_{a,q}y)(x) := a(1 + x^{-1})y(qx) - [x^{-1} + a(1 + x^{-1})]y(x) + x^{-1}y(q^{-1}x). \tag{5.4}$$

When $q \rightarrow 1$ ($a = q^\alpha$) the q -Laguerre polynomial $L_n(x, a; q)$ becomes the ordinary Laguerre polynomial

$$L_n^\alpha(x) := \frac{(\alpha + 1)_n}{n!} {}_1F_1 \left(\begin{matrix} -n \\ \alpha + 1 \end{matrix} \middle| x \right). \tag{5.5}$$

There is a limit transition from little q -Jacobi to q -Laguerre (see [22], Section 4.12.2),

$$L_n(x, a; q) = \lim_{b \rightarrow \infty} \frac{(qa; q)_\infty}{(q; q)_\infty} p_n \left(-\frac{x}{qb}, a, b \middle| q \right). \tag{5.6}$$

Starting with the operators $X, Y,$ and Z given by (2.9) we can also obtain the following operators:

$$\begin{aligned} (Xf)(x) &:= -q \lim_{b \rightarrow \infty} b(Xf)(-x/qb), \\ (Yf)(x) &:= \sqrt{a} \lim_{b \rightarrow \infty} \frac{1}{\sqrt{b}} (Yf)(-x/qb), \\ (Zf)(x) &:= -q \lim_{b \rightarrow \infty} \sqrt{ab} (Zf)(-x/qb). \end{aligned}$$

Then

$$\begin{aligned} (Xf)(x) &:= xf(x), \\ (Yf)(x) &:= \frac{q}{q^2 - 1} (L_{a,q} + a)f(x), \\ (Zf)(x) &:= -f(q^{-1}x), \end{aligned} \tag{5.7}$$

where the operator $L_{a,q}$ is defined in (5.4). A simple computation shows that the operators X, Y, Z satisfy the relations

$$YX - qXY = Z + \omega_3, \quad ZY - qYZ = \omega_2, \quad XZ - qZX = 0, \tag{5.8}$$

where

$$\omega_2 = \frac{aq}{1+q}, \quad \omega_3 = \frac{q(1+a)}{1+q}. \tag{5.9}$$

The Casimir operator

$$Q = (q^2 - 1)YXZ + Z^2 + (q + 1)(\omega_2qX + \omega_3Z) \tag{5.10}$$

takes the value $Q = -aq^2$.

5.2 Second Jackson's q -Bessel function

The second Jackson's q -Bessel function is defined as follows:

$$J_\nu^{(2)}(x; q) := \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \left(\frac{1}{2}x\right)^\nu {}_0\Phi_1\left(q^{\nu+1} \middle| q; -\frac{1}{4}q^{\nu+1}x^2\right) \quad (x > 0).$$

This notation is from [5] and is deferent from Jackson's notation [2–4]. The classical Bessel function J_ν is recovered by letting $q \rightarrow 1$ in $J_\nu^{(2)}(x; q)$. Similarly to (4.1), we defined the second normalized q -Bessel function $\mathcal{J}_2(x; a; q)$ by

$$\mathcal{J}_2(x; a; q) = {}_0\Phi_1\left(qa \middle| q; -qax\right). \tag{5.11}$$

There is a well-known limit from q -Laguerre [22] to the second normalized q -Bessel function as $n \rightarrow \infty$

$$\mathcal{J}_2(x; a; q) = \lim_{n \rightarrow \infty} L_n(x, a; q). \tag{5.12}$$

From (5.12) and (5.3) is not difficult to establish the q -difference equation for $\mathcal{J}_2(\lambda x, a; q)$

$$(Y_{a,q,2}y)(x) = -a\lambda y(x), \tag{5.13}$$

where

$$(Y_{a,q,2}y)(x) = \frac{aq}{x}y(x) - q\frac{a+1}{x}y(q^{-1}x) + \frac{q}{x}y(q^{-2}x). \tag{5.14}$$

Furthermore, the q -Bessel operator $Y_{a,q,2}$ is related to the q -Laguerre operator $L_{a,q}$ defined in (5.4) by

$$(L_{a,q} + a)f(q^{-1}x) = (Y_{a,q,2} + a)f(x). \tag{5.15}$$

This allows us to construct a Askey-Wilson algebra type that has the q -Bessel operator $Y_{a,q,2}$ as one of its three generators. A straightforward computation shows that the operators X, Y, Z given by

$$\begin{aligned} (Xf)(x) &:= xf(x), \\ (Yf)(x) &:= \frac{q}{q^2-1}(Y_{a,q,2} + a)f(x), \\ (Zf)(x) &:= -f(q^{-1}x), \end{aligned} \tag{5.16}$$

satisfy the relations

$$YX - q^2XY = \mu_1Z + \mu_2X + \mu_3, \quad ZY - qYZ = \mu_4Z, \quad XZ - qZX = 0, \tag{5.17}$$

where

$$\mu_1 = -q \frac{1+a}{(1+q)}, \quad \mu_2 = -aq, \quad \mu_3 = -aq^2, \quad \mu_4 = -\frac{aq}{(1+q)}. \tag{5.18}$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements

The third author would like to extend his sincere appreciation to the Deanship of Scientific Research at King Saud University for funding this Research group No. (RG-1435-026). The work of the second author is supported by King Saud University, Riyadh, DSFP, through the grant MATH 01.

Received: 18 January 2015 Accepted: 19 August 2015 Published online: 29 August 2015

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