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# Solutions of the equilibrium equations with finite mass subject

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**Abstract**
In this paper, we investigate the slow equilibrium equations with finite mass subject to a homogeneous Neumann type boundary condition. Based on an auxiliary function method and a differential inequality technique, the existence of equilibrium equations is obtained if the angular is bounded and the blow-up occurs in finite time.

**Keywords:** axisymmetric; Neumann type boundary condition; finite mass

## 1 Introduction

In 3-D space, the equilibrium equations for a self-gravitating fluid rotating about the $x_3$ axis with prescribed velocity $\Omega(r)$ can be written

$$\begin{cases} \nabla P = \rho\nabla(-\Phi + \int_0^r s\Omega^2(s)\,ds), \\ \Delta\Phi = 4\pi g\rho. \end{cases} \tag{1.1}$$

Here $\rho$, $g$, and $\Phi$ denote the density, gravitational constant, and gravitational potential, respectively, $P$ is the pressure of the fluid at a point $x \in \mathbb{R}^3$, $r = \sqrt{x_1^2 + x_2^2}$. We want to find axisymmetric equilibria and therefore always assume that $\rho(x) = \rho(r, x_3)$.

For a density $\rho$, from $(1.1)_2$ we can obtain the induced potential

$$\Phi_\rho(x) = -g\int \frac{\rho(y)}{|x-y|}\,dy. \tag{1.2}$$

Obviously, $\Phi_\rho$ is decreasing when $\rho$ is increasing.

In the study of this model, Auchmuty and Beals [1] proved the existence of the equilibrium solution if the angular velocity satisfies certain decay conditions. For constant angular velocity, Miyamoto [2] have found that there exists an equilibrium solution if the angular velocity is less than a certain constant and there is no equilibrium for large velocity. Huang and Liu [3] addressed the exact numbers of stationary solutions. Many other interesting results exist; see [4, 5].

In more general conditions than in [2], we prove that there exists an equilibrium solution under the following constraint set:

$$\mathcal{A}_M := \left\{\rho \,\middle|\, \rho \geq 0, \rho \text{ is axisymmetric}, \int \rho\,dx = M\right\}. \tag{1.3}$$

A standard method to obtain steady states is to prescribe the minimizer of the stellar energy functional. The main problem is to show the steady state has finite mass and compact support. To approach this problem, we define the energy functional,

$$F(\rho) := \int Q(\rho)\,dx - \int \rho J(r)\,dx - \frac{g}{2}\int\int \frac{\rho(x)\rho(y)}{|x-y|}\,dy\,dx. \tag{1.4}$$

Here

$$Q(\rho) = \frac{1}{\gamma - 1}P, \qquad J(r) = \int_0^r s\Omega^2(s)\,ds. \tag{1.5}$$

In this paper, we assume $J(r)$ is nonnegative, continuous, and bounded on $[0,+\infty)$. $P$ is nonnegative, continuous, and strictly increasing for $s > 0$, and satisfies:

$\mathrm{P}_1$: $\lim_{\rho\to 0} P(\rho)\rho^{-1} = 0$, $\lim_{\rho\to+\infty} P(\rho)\rho^{-\frac{4}{3}} = +\infty$.

In Section 2, first we prove the existence of a minimizer of the energy functional $F$ in $\mathcal{A}_M$. Then we give the properties of minimizers, they are stationary solutions of (1.1) with finite mass and compact support. The main difficulty in the proof is the loss of compactness due to the unboundedness of $\mathbb{R}^3$. To prevent mass from running off to spatial infinity along a minimizing sequence, our variational approach is related to the concentration-compactness principle due to Qiao [5]. Many other interesting results exist; see [6–9].

Throughout this paper, for simplicity of presentation, we use $\int$ to denote $\int_{\mathbb{R}^3}$, and we use $\|\cdot\|_p$ to denote $\|\cdot\|_{L^p(\mathbb{R}^3)}$. Define

$$\begin{aligned} &B_R(x) := \{y \in \mathbb{R}^3 \mid |y-x| \le R\}, \qquad B_{R,K}(x) := \{y \in \mathbb{R}^3 \mid R \le |y-x| \le K\}, \\ &F_{\mathrm{pot}}(\rho) := -\frac{g}{2}\int\int \frac{\rho(x)\rho(y)}{|x-y|}\,dy\,dx = -\frac{1}{8\pi g}\int |\nabla\Phi_\rho|^2\,dx < 0. \end{aligned} \tag{1.6}$$

Let $C$ denote a generic positive constant. $\chi$ is the indicator function.

## 2 Minimizer of the energy

In this section, we present some properties of the functional $F$, and we prove the existence of minimizer. It is easy to verify that the function $F$ is invariant under any vertical shift. That is, if $\rho \in \mathcal{A}_M$, then $T\rho(x) := \rho(x + ae_3) \in \mathcal{A}_M$ and $F(T\rho) = F(\rho)$ for any $a \in \mathbb{R}$. Here $e_3 = (0,0,1)$. Therefore, if $(\rho_n)$ is a minimizing sequence of $F$ in $\mathcal{A}_M$, then $(T\rho_n)$ is a minimizing sequence of $F$ in $\mathcal{A}_M$ too. First we give some estimates.

**Lemma 2.1** *Assume $\rho \in L^1 \cap L^\gamma(\mathbb{R}^3)$. If $1 \le \gamma \le \frac{3}{2}$, then $\Phi \in L^r(\mathbb{R}^3)$ for $3 < r < \frac{3\gamma}{3-2\gamma}$, and*

$$\|\Phi\|_r \le C\big(\|\rho\|_1^\alpha \|\rho\|_\gamma^{1-\alpha} + \|\rho\|_1^\beta \|\rho\|_\gamma^{1-\beta}\big), \tag{2.1}$$

*where $0 < \alpha, \beta < 1$. If $\gamma > \frac{3}{2}$, then $\Phi$ is bounded and continuous and satisfies (2.1) with $r = +\infty$.*

*Proof* The proof can be found in [1]. □

**Lemma 2.2** *Assume $\rho \in L^1 \cap L^{\frac{4}{3}}(\mathbb{R}^3)$, then $\nabla\Phi \in L^2(\mathbb{R}^3)$.*

*Proof* The interpolation inequality implies

$$\|\rho\|_{\frac{6}{5}} \le \|\rho\|_1^{1/3}\|\rho\|_{4/3}^{2/3}.$$

By Sobolev's theorem, $\|\Phi\|_6 \le C\|\rho\|_{\frac{6}{5}}$. So

$$\|\nabla\Phi\|_2^2 = 4\pi g\|\rho\Phi\|_1 \le C\|\rho\|_{\frac{6}{5}}\|\Phi\|_6 \le C\|\rho\|_{\frac{6}{5}}^2.$$

From the above estimates we can complete our proof. □

**Lemma 2.3** *Assume* $P_1$ *hold, then there exists a nonnegative constant* $C$*, which depends only on* $\frac{1}{|x|}$*,* $M$*, and* $J(r)$ *such that* $F \ge -C$.

*Proof* For $\rho \in \mathcal{A}_M$, since $P_1$ holds, similar to [2], we know that there exists a constant $S_1 > 0$, such that

$$\begin{aligned} F(\rho) &\ge \int_{\rho<S_1} Q(\rho) + \int_{\rho\ge S_1} Q(\rho) - M\|J\|_\infty - CM^{2/3}\int \rho^{4/3} \\ &\ge \int_{\rho<S_1} Q(\rho) + \frac{1}{2}\int_{\rho\ge S_1} Q(\rho) - M\|J\|_\infty - CM^{2/3}\int_{\rho<S_1} \rho^{4/3} \\ &\ge \frac{1}{2}\int Q(\rho) - M\|J\|_\infty - CM^{5/3}S_1^{1/3}. \end{aligned}$$

So $F \ge -C_1$, here $C_1 = M\|J\|_\infty - CM^{5/3}S_1^{1/3}$. □

Let $h_M = \inf_{\mathcal{A}_M} F$, a simple scaling argument shows that $h_M < 0$: let $\overline{\rho}(x) = \varepsilon^3\rho(\varepsilon x)$, then $\int\overline{\rho} = \int\rho$. Since $\lim_{\rho\to 0} Q(\rho)\rho^{-1} = 0$, it is easy to see that, for $\varepsilon$ small enough, $\int Q(\overline{\rho}) = \int \varepsilon^{-3}Q(\varepsilon^3\rho) \to 0$. Therefore $h_M < 0$.

**Lemma 2.4** *Assume* $P_1$ *holds, then, for every* $0 < \widetilde{M} \le M$*, we have* $h_{\widetilde{M}} \ge (\frac{\widetilde{M}}{M})^{\frac{5}{3}} h_M$.

*Proof* Let $\widetilde{\rho}(x) = \rho(ax), \widetilde{J}(r) = J(ax)$ here $a = (M/\overline{M})^{1/3} \ge 1$. So, for any $\rho \in \mathcal{A}_M$ and $\widetilde{\rho} \in \mathcal{A}_{\overline{M}}$, we have

$$F(\widetilde{\rho}) = \int Q(\widetilde{\rho}) - \int \widetilde{\rho}\widetilde{J} + F_{\mathrm{pot}}(\widetilde{\rho}) \ge b^{-3}F(\rho), \tag{2.2}$$

the mappings $\mathcal{A}_M \to \mathcal{A}_{\widetilde{M}}$, $\rho \to \widetilde{\rho}$, $J \to \widetilde{J}$ are all one to one and onto; this completes our proof. □

From Lemma 2.3 we immediately find that any minimizing sequence $(\rho_n)_{n=1}^\infty \in \mathcal{A}_M$ of $F$ satisfies

$$\int \rho_n^{4/3} = \int_{\rho_n<S_1} \rho_n^{4/3} + \int_{\rho_n\ge S_1} \rho_n^{4/3} < MS_1^{1/3} + \int cQ(\rho_n) < 2cF(\rho_n) + C + MS_1^{1/3}.$$

**Lemma 2.5** *Let* $(\rho_n)_{n=1}^\infty$ *be bounded in* $L^{4/3}(\mathbb{R}^3)$ *and* $\rho_n \rightharpoonup \rho_0$ *weakly in* $L^{4/3}(\mathbb{R}^3)$*, then, for any* $R > 0$,

$$\int |\nabla\Phi_{\chi_{B_R}\rho_n}|^2\,dx \to \int |\nabla\Phi_{\chi_{B_R}\rho_0}|^2\,dx.$$

*Proof* By the Sobolev theorem and Lemma 2.1 we can complete the proof. □

**Lemma 2.6** *Assume* $\mathrm{P}_1$ *holds, let* $(\rho_n)_{n=1}^\infty \subset \mathcal{A}_M$ *be a minimizing sequence of* $F(\rho)$. *Then there exist a sequence* $(a_n)_{n=1}^\infty \subset \mathbb{R}^3$ *and* $\delta_0 > 0$, $R_0 > 0$ *such that*

$$\int_{a_n+B_R} \rho_n(x)\,dx \ge \delta_0, \quad R \ge R_0,$$

*for all sufficient large* $n \in \mathbb{N}$.

*Proof* Split the potential energy:

$$\begin{aligned} -\frac{2}{g} F_{\text{pot}} &:= \int\int_{|x-y|\le 1/R} \frac{\rho_n(x)\rho_n(y)}{|x-y|}\,dy\,dx + \int\int_{1/R<|x-y|<R} \cdots + \int\int_{|x-y|\ge R} \cdots \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

From Lemma 2.2 we easily know $I_1 \le \frac{C}{R}$. The estimates for $I_2$ and $I_3$ are straightforward:

$$\begin{aligned} I_2 &\le R \int\int_{|x-y|<R} \rho_n(x)\rho_n(y)\,dx\,dy \le MR \sup_{a\in\mathbb{R}^3} \int_{a+B_R} \rho_n(x)\,dx, \\ I_3 &= \int\int_{|x-y|\ge R} \frac{\rho(x)\rho(y)}{|x-y|}\,dy\,dx \le \frac{M^2}{R}. \end{aligned}$$

Therefore

$$\sup_{a\in\mathbb{R}^3} \int_{a+B_R} \rho_n(x)\,dx \ge \frac{1}{MR}\left(-\frac{2}{g}F_{\text{pot}} - \frac{M^2}{R} - \frac{C}{R}\right). \tag{2.3}$$

We know $F_{\text{pot}}(\rho_n) < 0$ from (1.6). Thus when $R$ is large enough, $-F_{\text{pot}} > 0$ dominates the sign of (2.3), so there exist $\delta_0 > 0$, $R_0 > 0$ as required. □

We are now ready to show the existence of a minimizer of $h_M$ as $\mathrm{P}_1$ holds.

**Theorem 2.1** *Assume* $\mathrm{P}_1$ *holds. Let* $(\rho_n)_{n=1}^\infty \in \mathcal{A}_M$ *be a minimizing sequence of* $F$. *Then there exist a subsequence, still denoted by* $(\rho_n)_{n=1}^\infty$, *and a sequence of translations* $T\rho_n := \rho_n(\cdot + a_n e_3)$, *where* $a_n$ *are constants and* $e_3 = (0,0,1)$, *such that*

$$F(\rho_0) = \inf_{\mathcal{A}_M} F(\rho) = h_M$$

*and* $T\rho_n \rightharpoonup \rho_0$ *weakly in* $L^{\frac{4}{3}}(\mathbb{R}^3)$. *For the induced potentials we have* $\nabla\Phi_{T\rho_n} \to \nabla\Phi_{\rho_0}$ *strongly in* $L^2(\mathbb{R}^3)$.

**Remark 2.1** Without admitting the spatial shifts, the assertion of the theorem is false: given a minimizer $\rho_0$ and a sequence of shift vectors $(a_n e_3) \in \mathbb{R}^3$, the functional $F$ is translation invariant, that is, $F(T\rho) = F(\rho)$. But if $|a_n e_3| \to \infty$ this minimizing sequence converges weakly to zero, which is not in $\mathcal{A}_M$.

*Proof* Split $\rho \in \mathcal{A}_M$ into three different parts:

$$\rho = \chi_{B_{R_1}}\rho + \chi_{B_{R_1,R_2}}\rho + \chi_{B_{R_2,\infty}}\rho := \rho_1 + \rho_2 + \rho_3$$

with

$$I_{lm} := \int\int \frac{\rho_l(x)\rho_m(y)}{|x-y|}\,dy\,dx, \quad l,m = 1,2,3,$$

thus

$$F(\rho) := F(\rho_1) + F(\rho_2) + F(\rho_3) - I_{12} - I_{13} - I_{23}.$$

If we choose $R_2 > 2R_1$, then

$$I_{13} \le 2\int_{B_{R_1}} \rho(x)\,dx \int_{B_{R_2,\infty}} |y|^{-1}\rho(y)\,dy \le \frac{C_1}{R_2}.$$

Next we estimate $I_{12}$ and $I_{23}$:

$$\begin{aligned} I_{12} + I_{23} &= -\int \rho_1\Phi_2\,dx - \int \rho_2\Phi_3\,dx = \frac{1}{4\pi g}\int \nabla(\Phi_1 + \Phi_3)\cdot\nabla\Phi_2\,dx \\ &\le C_2\|\rho_1 + \rho_3\|_{\frac{6}{5}}\|\nabla\Phi_2\|_2 \le C_3\|\nabla\Phi_2\|_2, \end{aligned}$$

where $\Phi_l = \Phi_{\rho_l}$.

Define $M_l = \int \rho_l$, $l = 1,2,3$, then $M = M_1 + M_2 + M_3$. Using the above estimates and Lemma 2.4, we have

$$\begin{aligned} h_M - F(\rho) &\le \left(1 - \left(\frac{M_1}{M}\right)^{5/3} - \left(\frac{M_2}{M}\right)^{5/3} - \left(\frac{M_3}{M}\right)^{5/3}\right)h_M + \frac{C_1}{R_2} + C_3\|\nabla\Phi_2\|_2 \\ &\le C_4 h_M M_1 M_3 + C_5\left(\frac{1}{R_2} + \|\nabla\Phi_2\|_2\right), \end{aligned} \tag{2.4}$$

here $C_4$, $C_5$ are positive and depend on $M$ but not on $R_1$ or $R_2$. Let $(\rho_n) \in \mathcal{A}_M$ be a minimizing sequence and $(a_n e_3) \in \mathbb{R}^3$, such that Lemma 2.6 holds. Since $F$ is translation invariant, the sequence $(T\rho_n)$ is a minimizing sequence too. So $\|T\rho_n\|_1 \le M$. Thus there exists a subsequence, denoted by $(T\rho_n)$ again, such that $T\rho_n \rightharpoonup \rho_0$ weakly in $L^{\frac{4}{3}}(\mathbb{R}^3)$. By Mazur's lemma and Fatou's lemma

$$\int Q(\rho_0)\,dx \le \liminf_{n\to\infty}\int Q(T\rho_n)\,dx. \tag{2.5}$$

Now we want to show that

$$\nabla\Phi_{T\rho_n} \to \nabla\Phi_{\rho_0} \quad \text{strongly in } L^2(\mathbb{R}^3). \tag{2.6}$$

Due to Lemma 2.5, $\nabla\Phi_{T\rho_{n,1}+T\rho_{n,2}}$ converges strongly in $L^2(B_{R_2})$. Therefore we only need to show that, for any $\varepsilon > 0$,

$$\int |\nabla\Phi_{T\rho_{n,3}}|^2\,dx < \varepsilon.$$

By Lemmas 2.1 and 2.2, it suffices to prove

$$\int T\rho_{n,3}\,dx < \varepsilon. \tag{2.7}$$

Choose $R_0 < R_1$, we see that $M_{n,1} \geq \delta_0$ for $n$ large enough from Lemma 2.6. By (2.4), we have

$$\begin{aligned} -C_4 h_M \delta_0 M_{n,3} &\leq -C_4 h_M M_{n,1} M_{n,3} \\ &\leq \frac{C_5}{R_2} + C_5 \|\nabla\Phi_{0,2}\|_2 + C_5 \|\nabla\Phi_{n,2} - \nabla\Phi_{0,2}\|_2 + \bigl|F(T\rho_n) - h_M\bigr|, \end{aligned} \tag{2.8}$$

where $\Phi_{n,l}$ is the potential induced by $T\rho_{n,l}$, which in turn has mass $M_{n,l}$, $n \in \mathbb{N} \cup \{0\}$, and the index $l = 1, 2, 3$ refers to the splitting.

Given any $\varepsilon > 0$. By Lemma 2.6 we can increase $R_1 > R_0$ such that $C_5 \|\nabla\Phi_{0,2}\|_2 < \varepsilon/4$. Next choose $R_2 > 2R_1$ such that the first term in (2.8) is less than $\varepsilon/4$. Now that $R_1$ and $R_2$ are fixed, the third term converges to zero by Lemma 2.5. Since $(T\rho_n)$ is minimizing sequence, $|F(T\rho_n) - h_M| < \varepsilon/4$ for suitable $n$. If $n$ is sufficiently large, then we have

$$-C_4 h_M \delta_0 M_{n,3} \leq \varepsilon, \quad \textit{i.e.}\ M_{n,3} \leq \varepsilon,$$

thus (2.7) holds, (2.6) follows, and

$$M \geq \int_{a_n + B_{R_2}} T\rho_n = M - M_{n,3} \geq M - \varepsilon.$$

Since $T\rho_n \rightharpoonup \rho_0$ weakly in $L^1(\mathbb{R}^N)$, for any $\varepsilon > 0$, there exists $R > 0$ such that

$$M \geq \int_{B_R} \rho_0 \geq M - \varepsilon,$$

thus

$$\rho_0 \in L^1(\mathbb{R}^N) \quad \text{with} \int \rho_0\,dx = M,$$

so $\rho_0 \in \mathcal{A}_M$. Together with (2.5) we obtain

$$F(\rho_0) = \inf_{\mathcal{A}_M} F = h_M.$$

The proof is completed. □

Next we show that the minimizers obtained above are steady states of (1.1).

**Theorem 2.2** *Let $\rho_0 \in \mathcal{A}_M$ be a minimizer of $F(\rho)$ with induced potential $\Phi_0$. Then*

$$\Phi_0 + Q'(\rho_0) - J(r) = K_0 \quad \textit{on the support of } \rho_0,$$

*where $K_0$ is a constant. Furthermore, $\rho_0$ satisfies* (1.1).

*Proof* We will derive the Euler-Lagrange equation for the variational problem. Let $\rho_0 \in \mathcal{A}_M$ be a minimizer with induced potential $\Phi_0$. For any $\varepsilon > 0$, we define

$$V_\varepsilon := \left\{ x \in \mathbb{R}^3 \,\middle|\, \varepsilon \le \rho_0 \le \frac{1}{\varepsilon} \right\}.$$

For a test function $\omega \in L^\infty(\mathbb{R}^3)$ which has compact support and is nonnegative on $V_\varepsilon^c$, define

$$\rho_\tau := \rho_0 + \tau\omega - \tau \frac{\int \omega\,dy}{\operatorname{meas}(V_\varepsilon)} \chi_{V_\varepsilon},$$

where $\tau \ge 0$ is small, such that

$$\rho_\tau \ge 0, \qquad \int \rho_\tau = \int \rho_0 = M.$$

Therefore $\rho_\tau \in \mathcal{A}_M$. Since $\rho_0$ is a minimizer of $F(\rho)$, we have

$$\begin{aligned}
0 &\le F(\rho_\tau) - F(\rho_0) \\
&= \int Q(\rho_\tau) - Q(\rho_0)\,dx - \int J(r)(\rho_\tau - \rho_0) + \frac{1}{2}\int (\rho_\tau \Phi_\tau - \rho_0 \Phi_0)\,dx \\
&\le \int \bigl(Q'(\rho_0) - J(r)\bigr)(\rho_\tau - \rho_0)\,dx + \int (\rho_\tau \Phi_0 - \rho_0 \Phi_0)\,dx + o(\tau) \\
&= \tau \int \bigl(Q'(\rho_0) - J(r) + \Phi_0\bigr)\left(\omega - \frac{\int \omega\,dy}{\operatorname{meas}(V_\varepsilon)} \chi_{V_\varepsilon}\right) dx + o(\tau).
\end{aligned}$$

Hence

$$\int \left[ Q'(\rho_0) - J(r) + \Phi_0 - \frac{1}{\operatorname{meas}(V_\varepsilon)} \left( \int_{V_\varepsilon} Q'(\rho_0) - J(r) + \Phi_0\,dy \right) \right] \omega\,dx \ge 0.$$

This holds for all test functions $\omega$ positive and negative on $V_\varepsilon$ as specified above, hence, for all $\varepsilon > 0$ small enough,

$$Q'(\rho_0) - J(r) + \Phi_0 = K_\varepsilon \quad \text{on } V_\varepsilon \quad \text{and} \quad Q'(\rho_0) - J(r) + \Phi_0 \ge K_\varepsilon \quad \text{on } V_\varepsilon^c, \tag{2.9}$$

where $K_\varepsilon$ is constant. Taking $\varepsilon \to 0$, we get

$$Q'(\rho_0) - J(r) + \Phi_0 = K_0 \quad \text{on the support of } \rho_0. \tag{2.10}$$

By taking the gradient on both sides of (2.10), we can prove that $\rho_0$ satisfies the equilibrium equation (1.1). □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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