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# Further results on the generalized Mittag-Leffler function operator

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**Abstract**

The present paper deals with the study of a generalized Mittag-Leffler function operator. This paper is based on the generalized Mittag-Leffler function introduced and studied by Saxena and Nishimoto (J. Fract. Calc. 37:43-52, 2010). Laplace and Mellin transforms of this new operator are investigated. The results are useful where the Mittag-Leffler function occurs naturally. The boundedness and composition properties of this operator are established. The importance of the derived results further lies in the fact that the results of the generalized Mittag-Leffler function defined by Prabhakar (Yokohama Math. J. 19:7-15, 1971), Shukla and Prajapati (J. Math. Anal. Appl. 336:797-811, 2007), and the multiindex Mittag-Leffler function due to Kiryakova (Fract. Calc. Appl. Anal. 2:445-462, 1999; J. Comput. Appl. Math. 118:214-259, 2000; J. Fract. Calc. 40:29-41, 2011) readily follow as a special case of our findings. Further the results obtained are of general nature and include the results given earlier by Prajapati *et al.* (J. Inequal. Appl. 2013:33, 2013) and Srivastava and Tomovski (Appl. Math. Comput. 211:198-210, 2009). Some special cases of the established results are also given as corollaries.

**MSC:** 33E12; 44A10; 26A33

**Keywords:** generalized Mittag-Leffler function; Laplace transform; Mellin transform; *H*-function; Mellin-Barnes type integrals; Riemann-Liouville fractional integral; Hilfer derivative

## 1 Introduction

The Swedish mathematician Gösta Mittag-Leffler in the year 1903 introduced the function [1, 2]

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \quad \bigl(\alpha \in C; \operatorname{Re}(\alpha) > 0\bigr). \tag{1}$$

In the year 1905, Wiman [3] introduced a generalization of (1) in the form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \quad \bigl(\alpha, \beta \in C; \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0\bigr). \tag{2}$$

In connection with the solution of an integral equation, Prabhakar [4] in the year 1971 introduced a very interesting useful generalization of (2) in the form

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n + \beta)}, \tag{3}$$

where $\alpha, \beta, \gamma \in C$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $(\gamma)_0 = 1$, and

$$(\gamma)_n = \gamma(\gamma+1)\cdots(\gamma+n-1) = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}; \quad \gamma \neq 0. \tag{4}$$

In the year 2007, Shukla and Prajapati [5] introduced the following generalization of (3):

$$E_{\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n + \beta)}, \tag{5}$$

where $\alpha, \beta, \gamma \in C$; $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$ ($q \in (0,1) \cup \mathrm{N}$).

The above generalization is studied by Shukla and Prajapati [5], Prajapati *et al.* [6] and established relation with Wright function [7–10]. In the year 2009, a generalization of (5) was introduced by Srivastava and Tomovski [11],

$$E_{\alpha,\beta}^{\gamma,\kappa}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n} z^n}{\Gamma(\alpha n + \beta)}, \tag{6}$$

where $\alpha, \beta, \gamma \in C$; $\mathrm{Re}(\alpha) > \max[0, \mathrm{Re}(\kappa) - 1]$, $\min\{\mathrm{Re}(\beta), \mathrm{Re}(\kappa)\} > 0$.

The Pochhammer symbol is defined by

$$(\lambda)_\mu = \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)} = \begin{cases} 1, & \mu = 0, \lambda, \mu \in C\backslash(0), \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \mu = n \in \mathrm{N}; \lambda \in C. \end{cases} \tag{7}$$

A further generalization of the Mittag-Leffler function defined by Shukla and Prajapati [5] was given by Saxena *et al.* [12] in the year 2010 as

$$E_{\gamma,q}\bigl[(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m);z\bigr] = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\prod_{j=1}^{m} \Gamma(n\alpha_j + \beta_j)(n)!}, \tag{8}$$

where $\alpha_j, \beta_j, \gamma \in C$, $\mathrm{Re}(\alpha_j) > 0$, $\mathrm{Re}(\beta_j) > 0$ ($j = 1, \ldots, m$), $\mathrm{Re}(\sum_{j=1}^{m} \alpha_j) > \max[0, q-1]$ ($q \in (0,1) \cup \mathrm{N}$).

This was further generalized by Saxena and Nishimoto [13, 14] in the year 2010 in the following form:

$$E_{\gamma,\kappa}\bigl[(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m)z\bigr] = \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n} z^n}{\prod_{j=1}^{m} \Gamma(n\alpha_j + \beta_j)(n)!}, \tag{9}$$

where $\alpha_j, \beta_j, \gamma, \kappa \in C$, $\mathrm{Re}(\alpha_j) > 0$, $\mathrm{Re}(\beta_j) > 0$ ($j = 1, \ldots, m$); $\mathrm{Re}(\kappa) > 0$, $\mathrm{Re}(\sum_{j=1}^{m} \alpha_j) > \max[0, \kappa - 1]$.

If we set $\gamma = \kappa = 1$ in (9), it reduces to the following multiindex Mittag-Leffler function studied by Kiryakova [15–17]:

$$E_{1,1}\bigl[(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m)z\bigr] = \sum_{n=0}^{\infty} \frac{z^n}{\prod_{j=1}^{m} \Gamma(n\alpha_j + \beta_j)}, \tag{10}$$

where $\alpha_j, \beta_j \in C$, $\mathrm{Re}(\alpha_j) > 0$, $\mathrm{Re}(\beta_j) > 0$ ($j = 1, \ldots, m$).

For $\kappa = q$ ($q \in (0,1) \cup \mathrm{N}$), (9) reduces to the one studied by Saxena *et al.* [12].

The following lemma was proved by Saxena and Nishimoto [14].

**Lemma** [18] *Let* $\alpha_j, \beta_j, \gamma, \kappa \in C, \operatorname{Re}(\alpha_j) > 0, \operatorname{Re}(\beta_j) > 0$ $(j = 1, \ldots, m)$; $\operatorname{Re}(\kappa) > 0, \operatorname{Re}(\sum_{j=1}^m \alpha_j) > \max[0, \kappa - 1]$, *then the function defined by* (10) *is represented by the Mellin-Barnes type integral as follows*:

$$E_{\gamma,\kappa}\big[(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m);z\big] = \frac{1}{2\pi i\Gamma(\gamma)}\int_{-i\infty}^{+i\infty}\frac{\Gamma(\xi)\Gamma(\gamma-\kappa\xi)(-z)^{-\xi}\,d\xi}{\prod_{j=1}^m\Gamma(\beta_j-\xi\alpha_j)}, \tag{11}$$

*where the contour of integration starts at* $-i\infty$ *and ends at* $+i\infty$ *and separates the poles at the point* $\xi = -n$ $(n \in \mathrm{N}_0)$ *to the left and all the poles of* $\Gamma(\gamma - \kappa\xi)$ *at the point* $\xi = \frac{\gamma}{\kappa} + n$ $(n \in \mathrm{N}_0)$ *to the right.*

The Riemann-Liouville fractional integral operator of $f(t)$ is defined as [14, 19–22]

$$\big(I_{a+}^{\alpha}f\big)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt \quad \big(\alpha \in C; \operatorname{Re}(\alpha) > 0\big). \tag{12}$$

The Riemann-Liouville fractional derivative operator of $f(t)$ is defined by [14, 19–23]

$$\big(D_{a+}^{\mu}f\big)(x) = \begin{cases} \frac{1}{\Gamma(-\mu)}\int_a^x (x-t)^{-\mu-1}f(t)\,dt & (\operatorname{Re}(\mu) < 0), \\ \frac{d^m}{dx^m}D_{a+}^{\mu-m}f(x) & (m-1 \le \operatorname{Re}(\mu) < m)\ (m \in \mathrm{N}). \end{cases} \tag{13}$$

The Caputo fractional derivative was defined by Caputo [24] in the form

$$\begin{aligned} {}_0^CD_t^{\alpha}f(t) &= \frac{\partial^m f(t)}{\partial t^m}\frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f^m(\tau)\,d\tau}{(t-\tau)^{\alpha+1-m}} \quad \big(m-1<\alpha<m; \operatorname{Re}(\alpha)>0, m \in \mathrm{N}\big) \\ &= \frac{\partial^m f(t)}{\partial t^m} \quad \text{if } \alpha = m, \end{aligned} \tag{14}$$

where $\frac{\partial^m f(t)}{\partial t^m}$ is the $m$th partial derivative of $f(t)$ with respect to $t$.

The Fox's $H$-function [20, 25] is defined in terms of the Mellin-Barnes type integral in the form

$$\begin{aligned} H_{p,q}^{m,n}(z) &= H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix}(b_q,B_q)\\(a_p,A_p)\end{matrix}\right] = H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix}(b_1,B_1),\ldots,(b_q,B_q)\\(a_1,A_1),\ldots,(a_p,A_p)\end{matrix}\right] \\ &= \frac{1}{2\pi i}\int_{\Omega}\Theta(\xi)z^{-\xi}\,d\xi, \end{aligned} \tag{15}$$

where

$$\Theta(\xi) = \frac{[\prod_{i=1}^m\Gamma(b_i+B_i\xi)][\prod_{j=1}^n\Gamma(1-a_j-A_j\xi)]}{[\prod_{i=m+1}^q\Gamma(1-b_i-B_i\xi)][\prod_{j=n+1}^p\Gamma(a_j+A_j\xi)]}. \tag{16}$$

A detailed definition, properties, asymptotic expansion and a comprehensive account of the $H$-function is available from the monographs written by Mathai *et al.* [20].

The generalized Wright function ${}_p\Psi_q$ $(p,q \in \mathrm{N}_0)$ is defined as ([26], p.183)

$${}_p\Psi_q\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p);\\(b_1,B_1),\ldots,(b_q,B_q);\end{matrix}\, z\right] = \sum_{n=0}^{\infty}\prod_{n=0}^{\infty}\frac{\Gamma(a_1+nA_1)\cdots\Gamma(a_p+nA_p)}{\Gamma(b_1+nB_1)\cdots\Gamma(b_q+nB_q)}\frac{z^n}{n!}, \tag{17}$$

$$
{}_p\Psi_q*\left[\begin{matrix}(a_1,A_1),\dots,(a_p,A_p);\\(b_1,B_1),\dots,(b_q,B_q);\end{matrix}z\right]=\sum_{n=0}^{\infty}\frac{(a_1)_{A_1n}\cdots(a_p)_{A_pn}}{(b)_{B_1n}\cdots(b_q)_{B_qn}}\frac{z^n}{n!}
$$

$$
=\frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}{}_p\Psi_q\left[\begin{matrix}(a_1,A_1),\dots,(a_p,A_p);\\(b_1,B_1),\dots,(b_q,B_q);\end{matrix}z\right],\tag{18}
$$

where

$$
A_j\in R\ (j=1,\dots,p),B_j\in R\ (j=1,\dots,q);\quad 1+\sum_{j=0}^{q}B_j-\sum_{j=0}^{p}A_j\geq 0,\tag{19}
$$

and the equality in the above condition holds true for suitably bounded values of $|z|$ given by $|z|<\nabla:=(\prod_{j=1}^{p}A_j^{-A_j})(\prod_{j=1}^{q}B_j^{B_j})$.

In particular, when $A_j=B_k=1$ ($j=1,\dots,p$ and $k=1,\dots,q$), the following relationship holds true:

$$
{}_p\Psi_q*\left[\begin{matrix}(a_1,1),\dots,(a_p,1);\\(b_1,1),\dots,(b_q,1);\end{matrix}z\right]={}_pF_q\left[\begin{matrix}a_1,\dots,a_p;\\b_1,\dots,b_q;\end{matrix}z\right]=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b)_n\cdots(b_q)_n}\frac{z^n}{n!}
$$

$$
=\frac{\Gamma(b_1)\dots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}{}_p\Psi_q\left[\begin{matrix}(a_1,1),\dots,(a_p,1);\\(b_1,1),\dots,(b_q,1);\end{matrix}z\right],\tag{20}
$$

in terms of the generalized hypergeometric function ${}_pF_q$ ($p,q\in\mathrm{N}_0$).

Recently, Górska *et al.* [27] studied operator solution for fractional Fokkar-Planck equations and Babusci *et al.* [28] investigated Mittag-leffler function and associated polynomials.

A generalization of Riemann-Liouville fractional derivative operator (13) as well as Caputo fractional derivative operator (14) was given by Hilfer [18] by introducing a right-sided fractional derivative operator of two parameters of order $0<\mu<1$ and $0\leq\nu\leq 1$ in the form

$$
D_{a+}^{\mu,\nu}N(x,t)=\left(I_{a+}^{\nu(1-\mu)}\frac{\partial}{\partial x}\left(I_{a+}^{(1-\nu)(1-\mu)}N(x,t)\right)\right),\tag{21}
$$

where the initial value term $(I_{0+}^{(1-\nu)(1-\mu)}f)(0+)$ involves the Riemann-Liouville fractional integral of order $(1-\nu)(1-\mu)$ evaluated in the limit as $t\to 0+$ provided that the Laplace transform exists.

The object of this paper is to derive certain properties including the Laplace transform and the Mellin transform of the integral operator associated with the generalized Mittag-Leffler function defined by

$$
\left(E_{\gamma,\kappa;\beta_1,\dots,\beta_m;\omega,a+}^{\alpha_1,\dots,\alpha_m}\varphi\right)(x)=\int_a^x(x-t)^{\beta_1-1}E_{\gamma,\kappa;\beta_1,\dots,\beta_m}^{\alpha_1,\dots,\alpha_m}\left[\omega(x-t)^{\alpha_1}\right]\varphi(t)\,dt\quad(x>a),\tag{22}
$$

where $\alpha_j,\beta_j,\gamma,\kappa,\omega\in C$, $\mathrm{Re}(\alpha_j)>0$, $\mathrm{Re}(\beta_j)>0$ ($j=1,\dots,m$); $\mathrm{Re}(\sum_{j=1}^{m}\alpha_j)>\max[0,\kappa-1]$, $\mathrm{Re}(\gamma)>0$, $\mathrm{Re}(\kappa)>0$.

This operator includes, as special cases, the operators defined and studied by Kilbas *et al.* [29], Shukla and Prajapati [5], Srivastava and Tomovski [11] and Kiryakova [16]. If

$\kappa = m = 1$ in (22) we obtain the following fractional integration operator defined by Kilbas *et al.* [29]:

$$\left(E^{\alpha}_{\gamma,\beta;\omega,a+}\varphi\right)(x) = \int_a^x (x-t)^{\beta-1} E^{\alpha}_{\gamma,\beta}\left[\omega(x-t)^{\alpha}\right]\varphi(t)\,dt \quad (x > a), \tag{23}$$

where $\alpha, \beta, \gamma, \omega \in C$; $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$.

As $\gamma$ tends to zero, then by virtue of the limit formula

$$E^{\alpha}_{\gamma,\beta}(z) = \frac{1}{\Gamma(\beta)}, \tag{24}$$

equation (22) reduces to the familiar Riemann-Liouville fractional integral defined by (12).

If we set $m = 1$ in (22), it gives the integral operator defined by Srivastava and Tomovski [11]

$$\left(E^{\alpha}_{\gamma,\kappa,\beta;\omega,a+}\varphi\right)(x) = \int_a^x (x-t)^{\beta_1-1} E^{\gamma,\kappa}_{\alpha,\beta}\left[\omega(x-t)^{\alpha}\right]\varphi(t)\,dt \quad (x > a), \tag{25}$$

where $\alpha, \beta, \gamma, \omega \in C$; $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\kappa) > 0$, $\mathrm{Re}(\alpha) = \mathrm{Re}(\kappa) - 1 > 0$, which for $\kappa = q$ reduces to the one given by Shukla and Prajapati [5].

If, however, we take $\gamma = \kappa = 1$, (22) yields the following integral operator associated with the multiindex Mittag-Leffler function defined by Kiryakova [16]:

$$\left(E^{\alpha_1,\ldots,\alpha_m}_{\gamma,\kappa;\beta_1,\ldots,\beta_m;\omega,a+}\varphi\right)(x) = \int_0^x (x-t)^{\beta_1-1} E^{\alpha_1,\ldots,\alpha_m}_{\gamma,\kappa;\beta_1,\ldots,\beta_m}\left[\omega(x-t)^{\alpha_1}\right]\varphi(t)\,dt \quad (x > a), \tag{26}$$

where $\alpha_j, \beta_j, \omega \in C$, $\mathrm{Re}(\alpha_j) > 0$, $\mathrm{Re}(\beta_j) > 0$ $(j = 1, \ldots, m)$.

We now proceed to derive the Mellin transform of the integral operator defined by (22).

## 2 Mellin transform of the integral operator defined by (22)

**Theorem 2.1** *It will be shown here that*

$$\begin{aligned} &M\left[\left(E^{\alpha_1,\ldots,\alpha_m}_{\gamma,\kappa;\beta_1,\ldots,\beta_m;\omega,a+}\varphi\right)(x);s\right] \\ &\quad = \frac{1}{\Gamma(\gamma)\Gamma(1-s)} H^{2,1}_{1,m+1}\left[-\omega t^{\alpha} \,\middle|\, \begin{matrix} (1-\gamma,\kappa) \\ (0,1),(1-s-\beta_1,\alpha_1),(1-s-\beta_j,\alpha_j)_{2,m} \end{matrix}\right] \\ &\qquad \times M\left[t^{\beta_1}\varphi(t)\right], \end{aligned} \tag{27}$$

*where* $\alpha_j, \beta_j, \gamma, \kappa, \omega \in C$, $\mathrm{Re}(\alpha_j) > 0$, $\mathrm{Re}(\beta_j) > 0$ $(j = 1, \ldots, m)$; $\mathrm{Re}(\sum_{j=1}^m \alpha_j) > \max[0, \kappa - 1]$, $\mathrm{Re}(\gamma) > 0$, $\mathrm{Re}(\kappa) > 0$, $\mathrm{Re}(1 - s - \beta_1) > 0$ *and* $H^{2,1}_{1,m+1}(\cdot)$ *is the H-function defined by* (15).

*Proof* The Mellin transform of a function $f(t)$ is defined by

$$M\left[\varphi(x);s\right] = \int_0^{\infty} x^{s-1}\varphi(x)\,dx. \tag{28}$$

Therefore we have

$$M\left[\left(E^{\alpha_1,\ldots,\alpha_m}_{\gamma,\kappa;\beta_1,\ldots,\beta_m;\omega,a+}\varphi\right)(x);s\right] = \int_0^{\infty} x^{s-1} \int_0^x (x-t)^{\beta_1-1} E^{\alpha_1,\ldots,\alpha_m}_{\gamma,\kappa;\beta_1,\ldots,\beta_m}\left[\omega(x-t)^{\alpha_1}\right]\varphi(t)\,dt\,dx,$$

where $x > 0$ and interchanging the order of integration, which is permissible under the conditions stated in Theorem 2.1, we find that

$$M\big[\big(E^{\alpha_1,\dots,\alpha_m}_{\gamma,\kappa;\beta_1,\dots,\beta_m;\omega,a+}\varphi\big)(x);s\big] = \int_0^\infty \varphi(t)\int_t^\infty x^{s-1}(x-t)^{\beta_1-1}E^{\alpha_1,\dots,\alpha_m}_{\gamma,\kappa;\beta_1,\dots,\beta_m}\big[\omega(x-t)^{\alpha_1}\big]\,dx\,dt. \tag{29}$$

If we set $x = t + u$ on the right-hand side of (29), it gives

$$M\big[\big(E^{\alpha_1,\dots,\alpha_m}_{\gamma,\kappa;\beta_1,\dots,\beta_m;\omega,a+}\varphi\big)(x);s\big] = \int_0^\infty \varphi(t)\int_0^\infty (t+u)^{s-1}u^{\beta_1-1}E^{\alpha_1,\dots,\alpha_m}_{\gamma,\kappa;\beta_1,\dots,\beta_m}\big[\omega u^{\alpha_1}\big]\,du\,dt. \tag{30}$$

To evaluate the $u$-integral, we express the generalized Mittag-Leffler function in terms of its Mellin-Barnes contour integral by means of formula (11), then the above integral transforms into the form

$$\begin{aligned} &M\big[\big(E^{\alpha_1,\dots,\alpha_m}_{\gamma,\kappa;\beta_1,\dots,\beta_m;\omega,a+}\varphi\big)(x);s\big] \\ &\quad= \int_0^\infty \varphi(t)\,dt\frac{1}{2\pi i\Gamma(\gamma)}\int_{-i\infty}^{+i\infty}\frac{\Gamma(-\xi)\Gamma(\gamma+\kappa\xi)(-\omega u^{\alpha})^{\xi}\,d\xi}{\prod_{j=1}^m\Gamma(\beta_j-\xi\alpha_j)} \\ &\qquad\times\int_0^\infty (t+u)^{s-1}u^{\beta_1-1}\,du. \end{aligned} \tag{31}$$

Now we evaluate the $u$-integral with the help of the formula [30]

$$\int_0^\infty t^{\nu-1}(t+a)^{-\rho}\,dt = \frac{\Gamma(\nu)\Gamma(\rho-\nu)}{\Gamma(\rho)}a^{\nu-\rho}, \tag{32}$$

where $\mathrm{Re}(\nu) > 0$, $\mathrm{Re}(\rho-\nu) > |\arg(-a)| < \pi$; we observe that the right-hand side of the above equation (32) and the definition of $H$-function (15) yields the desired result (27). □

**Corollary 2.1** *For $\kappa = q$, Theorem 2.1 reduces to the following result:*

$$\begin{aligned} &M\big[\big(E^{\alpha_1,\dots,\alpha_m}_{\gamma,q;\beta_1,\dots,\beta_m;\omega,a+\varphi}\big)(x);s\big] \\ &\quad= \frac{1}{\Gamma(\gamma)\Gamma(1-s)}\times H^{2,1}_{1,m+1}\left[-\omega t^{\alpha}\,\middle|\,\begin{matrix}(1-\gamma,q)\\(0,1),(1-s-\beta_1,\alpha_1),(1-s-\beta_j,\alpha_j)_{2,m}\end{matrix}\right] \\ &\qquad\times M\big[t^{\beta_1}\varphi(t)\big], \end{aligned} \tag{33}$$

*which holds under the conditions as given with (27) with $\kappa = q$ ($q \in (0,1)\cup\mathrm{N}$).*

When $m = 1$, Corollary 2.1 reduces to the following one given by Prajapati *et al.* [6].

**Corollary 2.2** *There holds the following result:*

$$\begin{aligned} &M\big[\big(E^{\alpha}_{\gamma,q;\beta;\omega,0+}\varphi\big)(x);s\big] \\ &\quad= \frac{1}{\Gamma(\gamma)\Gamma(1-s)}\times H^{2,1}_{1,m+1}\left[-\omega t^{\alpha}\,\middle|\,\begin{matrix}(1-\gamma,q)\\(0,1),(1-s-\beta,\alpha)\end{matrix}\right]M\big[t^{\beta}\varphi(t)\big]. \end{aligned} \tag{34}$$

If we further take $q = 1$, Corollary 2.2 reduces to the following result [6].

**Corollary 2.3** *There holds the following result*:

$$\begin{aligned} &M\big[\big(E^{\alpha}_{\gamma;\beta;\omega,0+}\varphi\big)(x);s\big] \\ &\quad = \frac{1}{\Gamma(\gamma)\Gamma(1-s)} \times H^{2,1}_{1,2}\left[-\omega t^{\alpha} \,\middle|\, \begin{matrix} (1-\gamma,1) \\ (0,1),(1-s-\beta,\alpha) \end{matrix}\right] M\big[t^{\beta}\varphi(t)\big]. \end{aligned} \tag{35}$$

For $m = 1$, Theorem 2.1 gives the following result given in [6].

**Corollary 2.4** *There holds the formula*

$$\begin{aligned} &M\big[\big(E^{\alpha}_{\gamma,\kappa;\beta;\omega,0+}\varphi\big)(x);s\big] \\ &\quad = \frac{1}{\Gamma(\gamma)\Gamma(1-s)} \times H^{2,1}_{1,2}\left[-\omega t^{\alpha} \,\middle|\, \begin{matrix} (1-\gamma,\kappa) \\ (0,1),(1-s-\beta,\alpha) \end{matrix}\right] M\big[t^{\beta}\varphi(t)\big], \end{aligned} \tag{36}$$

*where* $\alpha, \beta, \gamma \in C$; $\mathrm{Re}(\alpha) > \max[0, \mathrm{Re}(\kappa) - 1]$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\kappa) > 0$.

When $\kappa = 1$, it yields the following result associated with the multiindex Mittag-Leffler function [15, 16].

**Corollary 2.5** *The following result holds*:

$$\begin{aligned} &M\big[\big(E^{\alpha_1,\ldots,\alpha_m}_{\beta_1,\ldots,\beta_m;\omega,0+}\varphi\big)(x);s\big] \\ &\quad = \frac{1}{\Gamma(\gamma)\Gamma(1-s)} \times H^{2,1}_{1,m+1}\left[-\omega t^{\alpha} \,\middle|\, \begin{matrix} (0,1) \\ (0,1),(1-s-\beta_1,\alpha_1),(1-s-\beta_j,\alpha_j)_{2,m} \end{matrix}\right] \\ &\qquad \times M\big[t^{\beta_1}\varphi(t)\big]. \end{aligned} \tag{37}$$

## 3 Laplace transform of the operator defined by (22)

**Theorem 3.1** *It will be shown here that*

$$L\big[\big(E^{\alpha_1,\ldots,\alpha_m}_{\gamma,\kappa;\beta_1,\ldots,\beta_m;\omega,0+}\varphi\big)(x);p\big] = \frac{1}{\Gamma(\gamma)} p^{-\beta_1} {}_1\Psi_{m-1}\left[\begin{matrix} (\gamma,\kappa); \\ (\beta_j,\alpha_j)_{2,m}; \end{matrix} \frac{\omega}{p^{\alpha_1}}\right] \tilde{\varphi}(p), \tag{38}$$

*where* $\alpha_j, \beta_j, \gamma, \kappa, \omega \in C$, $\mathrm{Re}(\alpha_j) > 0$, $\mathrm{Re}(\beta_j) > 0$ $(j = 1, \ldots, m)$; $\mathrm{Re}(\sum_{j=1}^{m} \alpha_j) > \max[0, \mathrm{Re}(\kappa) - 1]$, ${}_1\Psi_{m-1}$ *is the Fox-Wright function defined by*

$$L\big[\big(E^{\alpha}_{\gamma;\beta;\omega,0+}\varphi\big)(x);p\big] = \frac{1}{\Gamma(\gamma)} p^{-\beta}\left[1 - \frac{\omega}{p^{\alpha}}\right]^{-\gamma} \tilde{\varphi}(p), \tag{39}$$

*where* $\alpha, \beta, \gamma, \omega \in C$; $\mathrm{Re}(p) > 0$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > 0$ *and*

$$L\big[f(t);p\big] = \int_0^{\infty} e^{-pt} f(t)\, dt, \tag{40}$$

*where* $\mathrm{Re}(p) > 0$ *and the integral is convergent.*

*Proof* By virtue of the results (22) and (40), we find that

$$L\big[\big(E^{\alpha_1,\ldots,\alpha_m}_{\gamma,\kappa;\beta_1,\ldots,\beta_m;\omega,0+}\varphi\big)(x);p\big] = \int_0^\infty e^{-px}\int_0^x (x-t)^{\beta_1-1}E^{\alpha_1,\ldots,\alpha_m}_{\gamma,\kappa;\beta_1,\ldots,\beta_m}\big[\omega(x-t)^{\alpha_1}\big]\varphi(t)\,dt\,dx.$$

Interchanging the order of integration, which is permissible under the conditions stated in Theorem 3.1, we find that

$$L\big[\big(E^{\alpha_1,\ldots,\alpha_m}_{\gamma,\kappa;\beta_1,\ldots,\beta_m;\omega,0+}\varphi\big)(x);p\big] = \int_0^\infty \varphi(t)\int_t^\infty e^{-px}(x-t)^{\beta_1-1}E^{\alpha_1,\ldots,\alpha_m}_{\gamma,\kappa;\beta_1,\ldots,\beta_m}\big[\omega(x-t)^{\alpha_1}\big]\,dx\,dt. \tag{41}$$

If we set $x = t + u$ on the right-hand side of (41), it becomes

$$L\big[\big(E^{\alpha_1,\ldots,\alpha_m}_{\gamma,\kappa;\beta_1,\ldots,\beta_m;\omega,a+}\varphi\big)(x);p\big] = \int_0^\infty \varphi(t)\int_0^\infty e^{-p(t+u)}u^{\beta_1-1}E^{\alpha_1,\ldots,\alpha_m}_{\gamma,\kappa;\beta_1,\ldots,\beta_m}\big[\omega u^{\alpha_1}\big]\,du\,dt. \tag{42}$$

On making use of the series definition of function (10), we see that

$$\begin{aligned}
&L\big[\big(E^{\alpha_1,\ldots,\alpha_m}_{\gamma,\kappa;\beta_1,\ldots,\beta_m;\omega,0+}\varphi\big)(x);p\big] \\
&\quad= \sum_{n=0}^\infty \frac{(\gamma)_{\kappa n}\omega^n}{\prod_{j=1}^m \Gamma(n\alpha_j+\beta_j)}\int_0^\infty e^{-pt}\varphi(t)\,dt\int_0^\infty e^{-pu}u^{\beta_1+\alpha_1 n-1}\,du \\
&\quad= \sum_{n=0}^\infty \frac{(\gamma)_{\kappa n}\omega^n}{p^{\beta_1+\alpha_1 n}\prod_{j=2}^m \Gamma(n\alpha_j+\beta_j)}\int_0^\infty e^{-pt}\varphi(t)\,dt \\
&\quad= \frac{1}{\Gamma(\gamma)}p^{-\beta_1}{}_1\Psi_{m-1}\left[\begin{matrix}(\gamma,\kappa); \\ (\beta_j,\alpha_j)_{2,m};\end{matrix}\ \frac{\omega}{p^{\alpha_1}}\right]\tilde{\varphi}(p),
\end{aligned}$$

where $\tilde{\varphi}(p)$ is the Laplace transform of $\varphi(t)$.

This completes the proof of Theorem 3.1. □

If we set $\kappa = q$, we obtain the following Corollary 3.1.

**Corollary 3.1** *The following result holds*:

$$L\big[\big(E^{\alpha_1,\ldots,\alpha_m}_{\gamma,q;\beta_1,\ldots,\beta_m;\omega,0+}\varphi\big)(x);p\big] = \frac{1}{\Gamma(\gamma)}p^{-\beta_1}{}_1\Psi_{m-1}\left[(\gamma,q);(\beta_j,\alpha_j)_{2,m};\frac{\omega}{p^{\alpha_1}}\right]\tilde{\varphi}(p),$$

*where* $\alpha_j,\beta_j,\gamma,\omega\in C$, $\mathrm{Re}(\alpha_j)>0$, $\mathrm{Re}(\beta_j)>0$ $(j=1,\ldots,m)$; $\mathrm{Re}(\sum_{j=1}^m\alpha_j)>\max[0,q-1]$, $\mathrm{Re}(p)>0$, $\mathrm{Re}(\gamma)>0$.

When $m = 1$, the above result reduces to the following.

**Corollary 3.2** *The following result holds*:

$$L\big[\big(E^{\alpha}_{\gamma,q;\beta;\omega,0+}\varphi\big)(x);p\big] = \frac{1}{\Gamma(\gamma)}p^{-\beta}{}_1\Psi_{m-1}\left[(\gamma,q);-\ \frac{\omega}{p^{\alpha}}\right]\tilde{\varphi}(p),$$

*where* $\alpha, \beta, \gamma, \omega \in C$; $\mathrm{Re}(p) > 0$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > 0$, $(q \in (0,1) \cup \mathrm{N})$, *here* $\tilde{\varphi}(p)$ *is the Laplace transform of* $\varphi(t)$.

When $q = 1$, the above corollary yields the following result given in [6].

**Corollary 3.3** *There holds the formula*

$$L\big[\big(E^{\alpha}_{\gamma;\beta;\omega,0+}\varphi\big)(x);p\big] = \frac{1}{\Gamma(\gamma)} p^{-\beta}\left[1 - \frac{\omega}{p^{\alpha}}\right]^{-\gamma} \tilde{\varphi}(p), \tag{43}$$

*where* $\alpha, \beta, \gamma, \omega \in C$; $\mathrm{Re}(p) > 0$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > 0$.

For $\gamma = \kappa = 1$, we obtain the following result associated with the multiindex Mittag-Leffler function [16].

**Corollary 3.4** *There holds the formula*

$$L\big[\big(E^{\alpha_1,\ldots,\alpha_m}_{\gamma,q;\beta_1,\ldots,\beta_m;\omega,0+}\varphi\big)(x);p\big] = \frac{1}{\Gamma(\gamma)} p^{-\beta_1}{}_1\Psi_{m-1}s\left[\begin{matrix}(\gamma,q); & \dfrac{\omega}{}\\ (\beta_j,\alpha_j)_{2,m}; & p^{\alpha_1}\end{matrix}\right]\tilde{\varphi}(p),$$

*where* $\alpha_j, \beta_j, \gamma, \omega \in C$, $\mathrm{Re}(\alpha_j) > 0$, $\mathrm{Re}(\beta_j) > 0$ $(j = 1,\ldots,m)$, $\mathrm{Re}(p) > 0$, $\mathrm{Re}(\gamma) > 0$.

For $m = 1$, Theorem 3.1 yields the following result [6].

**Corollary 3.5** *There holds the formula*

$$L\big[\big(E^{\alpha}_{\gamma,\kappa;\beta;\omega,0+}\varphi\big)(x);p\big] = \frac{1}{\Gamma(\gamma)} p^{-\beta}{}_1\Psi_{m-1}\left[\begin{matrix}(\gamma,\kappa); & \dfrac{\omega}{p^{\alpha}}\\ - & \end{matrix}\right]\tilde{\varphi}(p),$$

*where* $\alpha, \beta, \gamma, \omega \in C$; $\mathrm{Re}(p) > 0$, $\mathrm{Re}(\alpha) > \max[0, \mathrm{Re}(\kappa) - 1]$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > 0$, *when* $\gamma$ *tends to zero and* $\beta$ *is replaced by* $\alpha$, *then by virtue of the limit formula* (24), *the result* (10) *reduces to the Riemann-Liouville fractional integral and we arrive at an important result given in Samko et al.* [17].

**Corollary 3.6** *There holds the following result*:

$$L\big[\big(I^{\alpha}_{0+}\varphi\big)(x);p\big] = p^{-\alpha}\tilde{\varphi}(p), \tag{44}$$

*where* $\alpha \in C$; $\mathrm{Re}(p) > 0$, $\mathrm{Re}(\alpha) > 0$ *and* $\tilde{\varphi}(p)$ *is the Laplace transform of* $\varphi(t)$.

## 4 Further properties of the operator defined by (22)

By using the technique analogous to that employed by Kilbas *et al.* [29], we obtain the composition properties of the Riemann-Liouville fractional integral operator $I^{\mu}_{a+}$ involving the operator defined by (22).

**Theorem 4.1** *Under the various constraints already stated with definition* (22), *let the function* $f$ *be the nth space* $L(a,b)$ *of the Lebesgue measurable functions on finite interval* $[a,b]$ $(b > a)$ *of the real line* $R$ *given by*

$$L(a,b) = \big\{g(x) : \|g_1\|_1\big\} = \int_a^b \big|g(x)\big|\,dx < \infty. \tag{45}$$

*The space $C(a,b)$ is of continuous functions on $[a,b]$, the integral defined by* (22) *is bounded on $L(a,b)$ and*

$$\left\| E^{\alpha_1,\dots,\alpha_m}_{\gamma,q;\beta_1,\dots,\beta_m;\omega,0+}\varphi \right\| \leq d\|\varphi\|_1, \tag{46}$$

*where the constant $d$ is given by*

$$d = (b-a)^{\operatorname{Re}(\alpha_1)} \sum_{n=0}^{\infty} \frac{|(\gamma)_{\kappa n}||\omega(b-a)^{\operatorname{Re}(\alpha_1)}|^n}{\prod_{j=1}^{m} \Gamma(n\alpha_j+\beta_j)|\operatorname{Re}(\alpha_1)n+\beta_1|(n)!}. \tag{47}$$

For $\kappa = m = 1$, (47) reduces to the one given by Kilbas *et al.* [29].

**Theorem 4.2** *Under the various constraints mentioned already with the parameters, there hold the following results*:

$$I^{\alpha}_{a+}E^{\alpha_1,\dots,\alpha_m}_{\gamma,\kappa;\beta_1,\dots,\beta_m;\omega,0+}\varphi = E^{\alpha_1,\dots,\alpha_m}_{\gamma,\kappa;\beta_1+\alpha,\dots,\beta_m;\omega,0+}\varphi = E^{\alpha_1,\dots,\alpha_m}_{\gamma,\kappa;\beta_1,\dots,\beta_m;\omega,0+}I^{\alpha}_{a+}\varphi \tag{48}$$

*and*

$$D^{\alpha}_{a+}E^{\alpha_1,\dots,\alpha_m}_{\gamma,\kappa;\beta_1,\dots,\beta_m;\omega,0+}\varphi = E^{\alpha_1,\dots,\alpha_m}_{\gamma,\kappa;\beta_1-\alpha,\dots,\beta_m;\omega,0+}\varphi = E^{\alpha_1,\dots,\alpha_m}_{\gamma,\kappa;\beta_1,\dots,\beta_m;\omega,0+}D^{\alpha}_{a+}\varphi. \tag{49}$$

For $m = 1$, (48) and (49) reduce to the result given by Srivastava and Tomovski [11].

## 5 Applications

In this section, we present the solution of the following differential equation associated with a Hilfer derivative. The following result readily follows from paper [31]:

$$L\big[E_{\gamma,\kappa}\big[(\alpha_1,\beta_1),\dots,(\alpha_m,\beta_m);z\big];p\big] = \frac{1}{\Gamma(\gamma)}p^{-\beta_1}{}_1\Psi_{m-1}\left[\begin{matrix}(\gamma,\kappa); & \dfrac{\omega}{} \\ (\beta_j,\alpha_j)_{2,m}; & p^{\alpha_1}\end{matrix}\right], \tag{50}$$

where $\alpha_j,\beta_j,\gamma,\kappa \in C$, $\operatorname{Re}(\alpha_j) > 0$, $\operatorname{Re}(\beta_j) > 0$ $(j = 1,\dots,m)$; $\operatorname{Re}(\kappa) > 0$, $\operatorname{Re}(\sum_{j=1}^m \alpha_j) > \max[0,\kappa-1]$ and $\operatorname{Re}(p) > 0$.

**Theorem 5.1** *It will be shown here that the solution of the fractional differential equation*

$$\big(D^{\mu,\nu}_{0+}f\big)(x) = \lambda\big(E^{\alpha_1,\dots,\alpha_m}_{\gamma,\kappa;\beta_1,\dots,\beta_m;\omega,a+}\omega x^{\alpha_1}\big)(x) + g(x), \tag{51}$$

*where* $0 < \mu < 1$, $0 \leq \nu \leq 1$, $\omega \in C$, $\operatorname{Re}(\sum_{j=1}^m \alpha_j) > \max[0,\operatorname{Re}(\kappa)-1]$, $\operatorname{Re}(\beta_j) > 0$ $(j = 1,\dots,m)$, $\min\{\operatorname{Re}(\gamma),\operatorname{Re}(\kappa)\} > 0$,

$$\big(I^{(1-\nu)(1-\mu)}_{0+}f\big)(0+) = e, \tag{52}$$

*has the solution given by*

$$\begin{aligned} f(x) &= e\frac{x^{\mu-\nu(\mu-1)-1}}{\Gamma(\mu-\nu+\mu\nu)} + \frac{\lambda x^{\mu+\beta_1}}{\Gamma(\gamma)}E_{\gamma,\kappa}\big[(\alpha_1,\beta_1+\mu+1),\dots,(\alpha_m,\beta_m);\omega x^{\alpha_1}\big] \\ &\quad + \frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1}g(t)\,dt, \end{aligned} \tag{53}$$

*where $e$ is an arbitrary constant.*

*Proof* Applying the Laplace transform to each side of (53) and using (21) along with convolution theorem of the Laplace transform, we find that

$$\begin{aligned} p^{\mu}F(p) - ep^{\nu(1-\mu)} &= L\big[\big(E^{\alpha_1,\ldots,\alpha_m}_{\gamma,\kappa;\beta_1,\ldots,\beta_m;\omega,a+}\varphi\big);p\big]L(1;p) + G(p) \\ &= \frac{1}{\Gamma(\gamma)}p^{-\beta_{1-1}}{}_1\Psi_{m-1}\left[\begin{matrix}(\gamma,\kappa); & \dfrac{\omega}{p^{\alpha_1}} \\ (\beta_j,\alpha_j)_{2,m}; & \end{matrix}\right] + G(p), \end{aligned} \tag{54}$$

which readily gives

$$F(p) = ep^{\nu(1-\mu)-\mu} + \frac{\lambda}{\Gamma(\gamma)}p^{-\beta_1-\mu-1}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+\kappa n)(\frac{\omega}{p^{\alpha_1}})^n}{\prod_{j=1}^{m}\Gamma(n\operatorname{Re}(\alpha_j)+\beta_j)(n!)} + p^{-\mu}G(p). \tag{55}$$

Now, by taking the inverse Laplace transform of each side of (55), we obtain

$$\begin{aligned} F(x) &= eL^{-1}\big(p^{\nu(1-\mu)-\mu};x\big) + \frac{\lambda}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+\kappa n)(\frac{\omega}{p^{\alpha_1}})^n}{\prod_{j=1}^{m}\Gamma(n\operatorname{Re}(\alpha_j+\beta_j))(n!)}L^{-1}\big(p^{-\alpha_1 n-\beta_1-\mu-1};x\big) \\ &\quad + L^{-1}L\left(\frac{x^{\mu-1}}{\Gamma(\mu)} * g(x)\right) \\ &= e\frac{x^{\mu-\nu(1-\mu)-1}}{\Gamma(\mu-\nu(1-\mu))} + \frac{\lambda x^{\mu+\beta_1}}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+\kappa n)(\omega x^{\alpha_1})^n}{\prod_{j=1}^{m}\Gamma(n\operatorname{Re}(\alpha_j)+\beta_j)\Gamma(\alpha_1 n+\beta_1+\mu+1)(n)!} \\ &\quad + \frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1}g(t)\,dt, \end{aligned}$$

which yields

$$\begin{aligned} f(x) &= e\frac{x^{\mu-\nu(1-\mu)-1}}{\Gamma(\mu-\nu(1-\mu))} + \frac{\lambda x^{\mu+\beta_1}}{\Gamma(\gamma)}E_{\gamma,\kappa}\big[(\alpha_1,\beta_1+\mu+1),\ldots,(\alpha_m,\beta_m);\omega x^{\alpha_1}\big] \\ &\quad + \frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1}g(t)\,dt. \end{aligned}$$

This completes the proof of Theorem 5.1. □

It is interesting to observe that for $m = 1$, Theorem 5.1 reduces to the one given by Srivastava and Tomovski [11].

## 6 Conclusion

In this paper we have derived the Laplace transform and the Mellin transform and various properties of the operator defined by (21). These properties are useful in solving applied problems of science, engineering and technology such as fractional kinetic equations, fractional reaction-diffusion equation, *etc.*

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
The authors contributed equally and significantly in writing this article under the supervision of Prof. R. K. Saxena. The authors read and approved the final manuscript.

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