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The generalized viscosity implicit rules of nonexpansive mappings in Hilbert spaces

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Abstract
The generalized viscosity implicit rules of nonexpansive mappings in Hilbert spaces are established. The strong convergence theorems of the rules are proved under certain assumptions imposed on the sequences of parameters. The results presented in this paper extend and improve the main results of Refs. (Moudafi in J. Math. Anal. Appl. 241:46-55, 2000; Xu *et al.* in Fixed Point Theory Appl. 2015:41, 2015). Moreover, applications to a more general system of variational inequalities, the constrained convex minimization problem and K-mapping are included.

MSC: 47H09

Keywords: viscosity; generalized implicit rule; nonexpansive mapping; variational inequality; constrained convex minimization problem; K-mapping

1 Introduction

In this paper, we assume that H is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$, and C is a nonempty closed convex subset of H. Let $T: H \to H$ be a mapping and $F(T)$ be the set of fixed points of the mapping T, *i.e.*, $F(T) = \{x \in H : Tx = x\}$. A mapping $T: H \to H$ is called nonexpansive, if

$$\|Tx - Ty\| \le \|x - y\|$$

for all $x, y \in H$. A mapping $f: H \to H$ is called a contraction, if

$$\|f(x) - f(y)\| \le \theta\|x - y\|$$

for all $x, y \in H$ and some $\theta \in [0, 1)$.

In 2000, Moudafi [1] proved the following strong convergence theorem for nonexpansive mappings in real Hilbert spaces.

Theorem 1.1 [1] *Let C be a nonempty closed convex subset of the real Hilbert space H. Let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty. Let f be a contraction of C into itself with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n) + \frac{1}{1+\varepsilon_n} T(x_n), \quad n \ge 0,$$

where $\{\varepsilon_n\} \in (0,1)$ *satisfies*

(1) $\lim_{n\to\infty} \varepsilon_n = 0$;

(2) $\sum_{n=0}^{\infty} \varepsilon_n = \infty$;

(3) $\lim_{n\to\infty} |\frac{1}{\varepsilon_{n+1}} - \frac{1}{\varepsilon_n}| = 0$.

Then $\{x_n\}$ *converges strongly to a fixed point* x^* *of the nonexpansive mapping* T, *which is also the unique solution of the variational inequality* (*VI*)

$$\langle (I-f)x, y-x\rangle \geq 0, \quad \forall y \in F(T). \tag{1.1}$$

In other words, x^* *is the unique fixed point of the contraction* $P_{F(T)}f$, *that is,* $P_{F(T)}f(x^*) = x^*$.

Such a method for approximation of fixed points is called the viscosity approximation method. In 2015, Xu *et al.* [2] applied the viscosity technique to the implicit midpoint rule for nonexpansive mappings and proposed the following viscosity implicit midpoint rule (VIMR):

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right), \quad \forall n \geq 0.$$

The idea was to use contractions to regularize the implicit midpoint rule for nonexpansive mappings. They also proved that VIMR converges strongly to a fixed point of T, which also solved VI (1.1).

In this paper, motivated and inspired by Xu *et al.* [2], we give the following generalized viscosity implicit rules:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\bigl(s_n x_n + (1-s_n)x_{n+1}\bigr) \tag{1.2}$$

and

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n T\bigl(s_n x_n + (1-s_n)x_{n+1}\bigr) \tag{1.3}$$

for $n \geq 0$. We will prove that the generalized viscosity implicit rules (1.2) and (1.3) converge strongly to a fixed point of T under certain assumptions imposed on the sequences of parameters, which also solve VI (1.1).

The organization of this paper is as follows. In Section 2, we recall the notion of the metric projection, the demiclosedness principle of nonexpansive mappings and a convergence lemma. In Section 3, the strong convergence theorems of the generalized viscosity implicit rules (1.2) and (1.3) are proved under some conditions, respectively. Applications to a more general system of variational inequalities, the constrained convex minimization problem, and the K-mapping are presented in Section 4.

2 Preliminaries

Firstly, we recall the notion and some properties of the metric projection.

Definition 2.1 $P_C : H \to C$ is called a metric projection if for every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

Lemma 2.1 *Let C be a nonempty closed convex subset of the real Hilbert space H and $P_C : H \to C$ be a metric projection. Then*

(1) $\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y\rangle$, $\forall x, y \in H$;
(2) *P_C is a nonexpansive mapping, i.e.*, $\|P_C x - P_C y\| \leq \|x - y\|$, $\forall x, y \in H$;
(3) $\langle x - P_C x, y - P_C x\rangle \leq 0$, $\forall x \in H, y \in C$.

In order to prove our results, we need the demiclosedness principle of nonexpansive mappings, which is quite helpful in verifying the weak convergence of an algorithm to a fixed point of a nonexpansive mapping.

Lemma 2.2 (The demiclosedness principle) *Let C be a nonempty closed convex subset of the real Hilbert space H and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C such that*

$$x_n \rightharpoonup x^* \in C \quad \textit{and} \quad (I - T)x_n \to 0 \quad \textit{imply} \quad x^* = Tx^*,$$

where $\to$ (resp. $\rightharpoonup$) denotes strong (resp. weak) convergence.

In addition, we also need the following convergence lemma.

Lemma 2.3 [2] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that:

(1) $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

Theorem 3.1 *Let C be a nonempty closed convex subset of the real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : C \to C$ be a contraction with coefficient $\theta \in [0,1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T\big(s_n x_n + (1 - s_n)x_{n+1}\big), \tag{3.1}$$

where $\{\alpha_n\}, \{s_n\} \subset (0,1)$, satisfying the following conditions:

(1) $\lim_{n\to\infty} \alpha_n = 0$;
(2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(4) $0 < \varepsilon \leq s_n \leq s_{n+1} < 1$ *for all* $n \geq 0$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of the nonexpansive mapping T, which is also the unique solution of the variational inequality

$$\big\langle (I - f)x, y - x\big\rangle \geq 0, \quad \forall y \in F(T).$$

In other words, x^* is the unique fixed point of the contraction $P_{F(T)}f$, that is, $P_{F(T)}f(x^*) = x^*$.

Proof We divide the proof into five steps.

Step 1. Firstly, we show that $\{x_n\}$ is bounded.

Indeed, take $p \in F(T)$ arbitrarily, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \bigl\|\alpha_n f(x_n) + (1-\alpha_n)T\bigl(s_n x_n + (1-s_n)x_{n+1}\bigr) - p\bigr\| \\ &\le \alpha_n \bigl\|f(x_n) - p\bigr\| + (1-\alpha_n)\bigl\|T\bigl(s_n x_n + (1-s_n)x_{n+1}\bigr) - p\bigr\| \\ &\le \alpha_n \bigl\|f(x_n) - f(p)\bigr\| + \alpha_n \bigl\|f(p) - p\bigr\| + (1-\alpha_n)\bigl\|s_n x_n + (1-s_n)x_{n+1} - p\bigr\| \\ &\le \theta\alpha_n \|x_n - p\| + \alpha_n \bigl\|f(p) - p\bigr\| + (1-\alpha_n)\bigl[s_n\|x_n - p\| + (1-s_n)\|x_{n+1} - p\|\bigr] \\ &= \bigl[\theta\alpha_n + (1-\alpha_n)s_n\bigr]\|x_n - p\| + (1-\alpha_n)(1-s_n)\|x_{n+1} - p\| + \alpha_n\bigl\|f(p) - p\bigr\|. \end{aligned}$$

It follows that

$$\bigl[1 - (1-\alpha_n)(1-s_n)\bigr]\|x_{n+1} - p\| \le \bigl[\theta\alpha_n + (1-\alpha_n)s_n\bigr]\|x_n - p\| + \alpha_n\bigl\|f(p) - p\bigr\|. \tag{3.2}$$

Since $\alpha_n, s_n \in (0,1)$, $1 - (1-\alpha_n)(1-s_n) > 0$. Moreover, by (3.2), we get

$$\begin{aligned} \|x_{n+1} - p\| &\le \frac{\theta\alpha_n + (1-\alpha_n)s_n}{1 - (1-\alpha_n)(1-s_n)}\|x_n - p\| + \frac{\alpha_n}{1 - (1-\alpha_n)(1-s_n)}\bigl\|f(p) - p\bigr\| \\ &= \biggl[1 - \frac{\alpha_n(1-\theta)}{1 - (1-\alpha_n)(1-s_n)}\biggr]\|x_n - p\| + \frac{\alpha_n}{1 - (1-\alpha_n)(1-s_n)}\bigl\|f(p) - p\bigr\| \\ &= \biggl[1 - \frac{\alpha_n(1-\theta)}{1 - (1-\alpha_n)(1-s_n)}\biggr]\|x_n - p\| \\ &\quad + \frac{\alpha_n(1-\theta)}{1 - (1-\alpha_n)(1-s_n)}\biggl(\frac{1}{1-\theta}\bigl\|f(p) - p\bigr\|\biggr). \end{aligned}$$

Thus, we have

$$\|x_{n+1} - p\| \le \max\biggl\{\|x_n - p\|, \frac{1}{1-\theta}\bigl\|f(p) - p\bigr\|\biggr\}.$$

By induction, we obtain

$$\|x_n - p\| \le \max\biggl\{\|x_0 - p\|, \frac{1}{1-\theta}\bigl\|f(p) - p\bigr\|\biggr\}, \quad \forall n \ge 0.$$

Hence, it turns out that $\{x_n\}$ is bounded. Consequently, we deduce immediately that $\{f(x_n)\}$, $\{T(s_n x_x + (1-s_n)x_{n+1})\}$ are bounded.

Step 2. Next, we prove that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

To see this, we apply (3.1) to get

$$\begin{aligned} \|x_{n+1} - x_n\| &= \bigl\|\alpha_n f(x_n) + (1-\alpha_n)T\bigl(s_n x_n + (1-s_n)x_{n+1}\bigr) \\ &\quad - \bigl[\alpha_{n-1} f(x_{n-1}) + (1-\alpha_{n-1})T\bigl(s_{n-1}x_{n-1} + (1-s_{n-1})x_n\bigr)\bigr]\bigr\| \\ &= \bigl\|\alpha_n\bigl[f(x_n) - f(x_{n-1})\bigr] + (\alpha_n - \alpha_{n-1})f(x_{n-1}) \\ &\quad + (1-\alpha_n)\bigl[T\bigl(s_n x_n + (1-s_n)x_{n+1}\bigr) - T\bigl(s_{n-1}x_{n-1} + (1-s_{n-1})x_n\bigr)\bigr] \end{aligned}$$

$$\begin{aligned}
&- (\alpha_n - \alpha_{n-1})T\bigl(s_{n-1}x_{n-1} + (1 - s_{n-1})x_n\bigr)\bigr\| \\
&\leq \alpha_n \bigl\|f(x_n) - f(x_{n-1})\bigr\| + |\alpha_n - \alpha_{n-1}| \cdot \bigl\|f(x_{n-1}) - T\bigl(s_{n-1}x_{n-1} + (1 - s_{n-1})x_n\bigr)\bigr\| \\
&\quad + (1 - \alpha_n)\bigl\|T\bigl(s_n x_n + (1 - s_n)x_{n+1}\bigr) - T\bigl(s_{n-1}x_{n-1} + (1 - s_{n-1})x_n\bigr)\bigr\| \\
&\leq \theta\alpha_n\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_1 \\
&\quad + (1 - \alpha_n)\bigl\|\bigl[s_n x_n + (1 - s_n)x_{n+1}\bigr] - \bigl[s_{n-1}x_{n-1} + (1 - s_{n-1})x_n\bigr]\bigr\| \\
&= \theta\alpha_n\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_1 \\
&\quad + (1 - \alpha_n)\bigl\|(1 - s_n)(x_{n+1} - x_n) + s_{n-1}(x_n - x_{n-1})\bigr\| \\
&\leq \theta\alpha_n\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_1 + (1 - \alpha_n)(1 - s_n)\|x_{n+1} - x_n\| \\
&\quad + (1 - \alpha_n)s_{n-1}\|x_n - x_{n-1}\| \\
&= (1 - \alpha_n)(1 - s_n)\|x_{n+1} - x_n\| \\
&\quad + \bigl[\theta\alpha_n + (1 - \alpha_n)s_{n-1}\bigr]\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_1,
\end{aligned}$$

where $M_1 > 0$ is a constant such that

$$M_1 \geq \sup_{n\geq 0}\bigl\|f(x_n) - T\bigl(s_n x_n + (1 - s_n)x_{n+1}\bigr)\bigr\|.$$

It turns out that

$$\bigl[1 - (1 - \alpha_n)(1 - s_n)\bigr]\|x_{n+1} - x_n\| \leq \bigl[\theta\alpha_n + (1 - \alpha_n)s_{n-1}\bigr]\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_1,$$

that is,

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \frac{\theta\alpha_n + (1 - \alpha_n)s_{n-1}}{1 - (1 - \alpha_n)(1 - s_n)}\|x_n - x_{n-1}\| + \frac{M_1}{1 - (1 - \alpha_n)(1 - s_n)}|\alpha_n - \alpha_{n-1}| \\
&= \biggl[1 - \frac{\alpha_n(1 - \theta) + (1 - \alpha_n)(s_n - s_{n-1})}{1 - (1 - \alpha_n)(1 - s_n)}\biggr]\|x_n - x_{n-1}\| \\
&\quad + \frac{M_1}{1 - (1 - \alpha_n)(1 - s_n)}|\alpha_n - \alpha_{n-1}|.
\end{aligned}$$

Note that $0 < \varepsilon \leq s_{n-1} \leq s_n < 1$, we have

$$0 < \varepsilon \leq s_n < 1 - (1 - \alpha_n)(1 - s_n) < 1$$

and

$$\frac{\alpha_n(1 - \theta) + (1 - \alpha_n)(s_n - s_{n-1})}{1 - (1 - \alpha_n)(1 - s_n)} \geq \alpha_n(1 - \theta).$$

Thus,

$$\|x_{n+1} - x_n\| \leq \bigl[1 - \alpha_n(1 - \theta)\bigr]\|x_n - x_{n-1}\| + \frac{M_1}{\varepsilon}|\alpha_n - \alpha_{n-1}|.$$

Since $\sum_{n=0}^{\infty}\alpha_n = \infty$ and $\sum_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, by Lemma 2.3, we can get $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

Step 3. Now, we prove that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

In fact, we can see that

$$\begin{aligned}
\|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \big\|x_{n+1} - T\big(s_n x_n + (1-s_n)x_{n+1}\big)\big\| \\
&\quad + \big\|T\big(s_n x_n + (1-s_n)x_{n+1}\big) - Tx_n\big\| \\
&\le \|x_n - x_{n+1}\| + \big\|\alpha_n\big[f(x_n) - T\big(s_n x_n + (1-s_n)x_{n+1}\big)\big]\big\| \\
&\quad + \big\|\big(s_n x_n + (1-s_n)x_{n+1}\big) - x_n\big\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n M_1 + (1-s_n)\|x_{n+1} - x_n\| \\
&\le (2-s_n)\|x_n - x_{n+1}\| + \alpha_n M_1 \\
&\le 2\|x_n - x_{n+1}\| + \alpha_n M_1.
\end{aligned}$$

Then, by $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty}\alpha_n = 0$, we get $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. Moreover, we have

$$\begin{aligned}
&\big\|T\big(s_n x_n + (1-s_n)x_{n+1}\big) - x_n\big\| \\
&\quad\le \big\|T\big(s_n x_n + (1-s_n)x_{n+1}\big) - Tx_n\big\| + \|Tx_n - x_n\| \\
&\quad\le \big\|\big(s_n x_n + (1-s_n)x_{n+1}\big) - x_n\big\| + \|Tx_n - x_n\| \\
&\quad= (1-s_n)\|x_{n+1} - x_n\| + \|Tx_n - x_n\| \\
&\quad\le \|x_{n+1} - x_n\| + \|Tx_n - x_n\| \to 0 \quad (\text{as } n \to \infty).
\end{aligned} \tag{3.3}$$

Step 4. In this step, we claim that $\limsup_{n\to\infty}\langle x^* - f(x^*), x^* - x_n\rangle \le 0$, where $x^* = P_{F(T)}f(x^*)$.

Indeed, take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\big\langle x^* - f\big(x^*\big), x^* - x_n\big\rangle = \lim_{n\to\infty}\big\langle x^* - f\big(x^*\big), x^* - x_{n_i}\big\rangle.$$

Since $\{x_n\}$ is bounded, there exists a subsequence of $\{x_n\}$ which converges weakly to p. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup p$. From $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and Lemma 2.2 we have $p = Tp$, that is, $p \in F(T)$. This together with the property of the metric projection implies that

$$\limsup_{n\to\infty}\big\langle x^* - f\big(x^*\big), x^* - x_n\big\rangle = \lim_{n\to\infty}\big\langle x^* - f\big(x^*\big), x^* - x_{n_i}\big\rangle = \big\langle x^* - f\big(x^*\big), x^* - p\big\rangle \le 0.$$

Step 5. Finally, we show that $x_n \to x^*$ as $n \to \infty$. Here again $x^* \in F(T)$ is the unique fixed point of the contraction $P_{F(T)}f$ or in other words, $x^* = P_{F(T)}f(x^*)$.

In fact, we have

$$\begin{aligned}
\big\|x_{n+1} - x^*\big\|^2 &= \big\|\alpha_n f(x_n) + (1-\alpha_n)T\big(s_n x_n + (1-s_n)x_{n+1}\big) - x^*\big\|^2 \\
&= \big\|\alpha_n\big[f(x_n) - x^*\big] + (1-\alpha_n)\big[T\big(s_n x_n + (1-s_n)x_{n+1}\big) - x^*\big]\big\|^2 \\
&= \alpha_n^2\big\|f(x_n) - x^*\big\|^2 + (1-\alpha_n)^2\big\|T\big(s_n x_n + (1-s_n)x_{n+1}\big) - x^*\big\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\big\langle f(x_n) - x^*, T\big(s_n x_n + (1-s_n)x_{n+1}\big) - x^*\big\rangle
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n^2 \|f(x_n) - x^*\|^2 + (1-\alpha_n)^2 \|s_n x_n + (1-s_n)x_{n+1} - x^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x_n) - f(x^*), T(s_n x_n + (1-s_n)x_{n+1}) - x^*\rangle \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x^*) - x^*, T(s_n x_n + (1-s_n)x_{n+1}) - x^*\rangle \\
&\leq (1-\alpha_n)^2 \|s_n x_n + (1-s_n)x_{n+1} - x^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\|f(x_n) - f(x^*)\| \cdot \|T(s_n x_n + (1-s_n)x_{n+1}) - x^*\| + L_n \\
&\leq (1-\alpha_n)^2 \|s_n x_n + (1-s_n)x_{n+1} - x^*\|^2 \\
&\quad + 2\theta\alpha_n(1-\alpha_n)\|x_n - x^*\| \cdot \|s_n x_n + (1-s_n)x_{n+1} - x^*\| + L_n,
\end{aligned}$$

where

$$L_n := \alpha_n^2 \|f(x_n) - x^*\|^2 + 2\alpha_n(1-\alpha_n)\langle f(x^*) - x^*, T(s_n x_n + (1-s_n)x_{n+1}) - x^*\rangle.$$

It turns out that

$$\begin{aligned}
&(1-\alpha_n)^2 \|s_n x_n + (1-s_n)x_{n+1} - x^*\|^2 \\
&\quad + 2\theta\alpha_n(1-\alpha_n)\|x_n - x^*\| \cdot \|s_n x_n + (1-s_n)x_{n+1} - x^*\| + L_n - \|x_{n+1} - x^*\|^2 \geq 0.
\end{aligned}$$

Solving this quadratic inequality for $\|s_n x_n + (1-s_n)x_{n+1} - x^*\|$ yields

$$\begin{aligned}
&\|s_n x_n + (1-s_n)x_{n+1} - x^*\| \\
&\quad \geq \frac{1}{2(1-\alpha_n)^2}\Big\{-2\theta\alpha_n(1-\alpha_n)\|x_n - x^*\| \\
&\qquad + \sqrt{4\theta^2\alpha_n^2(1-\alpha_n)^2\|x_n - x^*\|^2 - 4(1-\alpha_n)^2(L_n - \|x_{n+1} - x^*\|^2)}\Big\} \\
&\quad = \frac{-\theta\alpha_n\|x_n - x^*\| + \sqrt{\theta^2\alpha_n^2\|x_n - x^*\|^2 - L_n + \|x_{n+1} - x^*\|^2}}{1-\alpha_n}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&s_n\|x_n - x^*\| + (1-s_n)\|x_{n+1} - x^*\| \\
&\quad \geq \frac{-\theta\alpha_n\|x_n - x^*\| + \sqrt{\theta^2\alpha_n^2\|x_n - x^*\|^2 - L_n + \|x_{n+1} - x^*\|^2}}{1-\alpha_n},
\end{aligned}$$

namely,

$$\begin{aligned}
&(s_n - s_n\alpha_n + \theta\alpha_n)\|x_n - x^*\| + (1-s_n)(1-\alpha_n)\|x_{n+1} - x^*\| \\
&\quad \geq \sqrt{\theta^2\alpha_n^2\|x_n - x^*\|^2 - L_n + \|x_{n+1} - x^*\|^2}.
\end{aligned}$$

Then

$$\begin{aligned}
&\theta^2\alpha_n^2\|x_n - x^*\|^2 - L_n + \|x_{n+1} - x^*\|^2 \\
&\quad \leq (s_n - s_n\alpha_n + \theta\alpha_n)^2\|x_n - x^*\|^2 + (1-s_n)^2(1-\alpha_n)^2\|x_{n+1} - x^*\|^2
\end{aligned}$$

$$
\begin{aligned}
&+ 2(s_n - s_n\alpha_n + \theta\alpha_n)(1-s_n)(1-\alpha_n)\|x_n - x^*\| \cdot \|x_{n+1} - x^*\| \\
&\le (s_n - s_n\alpha_n + \theta\alpha_n)^2\|x_n - x^*\|^2 + (1-s_n)^2(1-\alpha_n)^2\|x_{n+1} - x^*\|^2 \\
&+ (s_n - s_n\alpha_n + \theta\alpha_n)(1-s_n)(1-\alpha_n)\big[\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\big],
\end{aligned}
$$

which is reduced to the inequality

$$
\begin{aligned}
&\big[1 - (1-s_n)^2(1-\alpha_n)^2 - (s_n - s_n\alpha_n + \theta\alpha_n)(1-s_n)(1-\alpha_n)\big]\|x_{n+1} - x^*\|^2 \\
&\le \big[(s_n - s_n\alpha_n + \theta\alpha_n)^2 + (s_n - s_n\alpha_n + \theta\alpha_n)(1-s_n)(1-\alpha_n) - \theta^2\alpha_n^2\big]\|x_n - x^*\|^2 + L_n,
\end{aligned}
$$

that is,

$$
\begin{aligned}
&\big[1 - (1-s_n)(1-\alpha_n)\big(1 + (\theta-1)\alpha_n\big)\big]\|x_{n+1} - x^*\|^2 \\
&\le \big[(s_n - s_n\alpha_n + \theta\alpha_n)\big(1 + (\theta-1)\alpha_n\big) - \theta^2\alpha_n^2\big]\|x_n - x^*\|^2 + L_n.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 \le{}& \frac{(s_n - s_n\alpha_n + \theta\alpha_n)(1 + (\theta-1)\alpha_n) - \theta^2\alpha_n^2}{1 - (1-s_n)(1-\alpha_n)(1 + (\theta-1)\alpha_n)}\|x_n - x^*\|^2 \\
&+ \frac{L_n}{1 - (1-s_n)(1-\alpha_n)(1 + (\theta-1)\alpha_n)}.
\end{aligned}
\tag{3.4}
$$

Let

$$
\begin{aligned}
w_n :={}& \frac{1}{\alpha_n}\left\{1 - \frac{(s_n - s_n\alpha_n + \theta\alpha_n)(1 + (\theta-1)\alpha_n) - \theta^2\alpha_n^2}{1 - (1-s_n)(1-\alpha_n)(1 + (\theta-1)\alpha_n)}\right\} \\
={}& \frac{2(1-\theta) + (2\theta-1)\alpha_n}{1 - (1-s_n)(1-\alpha_n)(1 + (\theta-1)\alpha_n)}.
\end{aligned}
$$

Since the sequence $\{s_n\}$ satisfies $0 < \varepsilon \le s_n \le s_{n+1} < 1$ for all $n \ge 0$, $\lim_{n\to\infty} s_n$ exists; assume that

$$
\lim_{n\to\infty} s_n = s^* > 0.
$$

Then

$$
\lim_{n\to\infty} w_n = \frac{2(1-\theta)}{s^*} > 0.
$$

Let ρ_1 satisfy

$$
0 < \rho_1 < \frac{2(1-\theta)}{s^*},
$$

then there exists an integer N_1 big enough such that $w_n > \rho_1$ for all $n \ge N_1$. Hence, we have

$$
\frac{(s_n - s_n\alpha_n + \theta\alpha_n)(1 + (\theta-1)\alpha_n) - \theta^2\alpha_n^2}{1 - (1-s_n)(1-\alpha_n)(1 + (\theta-1)\alpha_n)} \le 1 - \rho_1\alpha_n
$$

for all $n \geq N_1$. It turns out from (3.4) that, for all $n \geq N_1$,

$$\|x_{n+1} - x^*\|^2 \leq (1 - \rho_1\alpha_n)\|x_n - x^*\|^2 + \frac{L_n}{1 - (1 - s_n)(1 - \alpha_n)(1 + (\theta - 1)\alpha_n)}. \tag{3.5}$$

By $\lim_{n\to\infty} \alpha_n = 0$, (3.3), and Step 4, we have

$$\begin{aligned} &\limsup_{n\to\infty} \frac{L_n}{\rho_1\alpha_n[1 - (1 - s_n)(1 - \alpha_n)(1 + (\theta - 1)\alpha_n)]} \\ &\quad = \limsup_{n\to\infty} \frac{\alpha_n\|f(x_n) - x^*\|^2 + 2(1 - \alpha_n)\langle f(x^*) - x^*, T(s_n x_n + (1 - s_n)x_{n+1}) - x^*\rangle}{\rho_1[1 - (1 - s_n)(1 - \alpha_n)(1 + (\theta - 1)\alpha_n)]} \\ &\quad \leq 0. \end{aligned} \tag{3.6}$$

From (3.5), (3.6), and Lemma 2.2, we can obtain

$$\lim_{n\to\infty} \|x_{n+1} - x^*\|^2 = 0,$$

namely, $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

Theorem 3.2 *Let C be a nonempty closed convex subset of the real Hilbert space H. Let $T: C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f: C \to C$ be a contraction with coefficient $\theta \in [0,1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n T(s_n x_n + (1 - s_n)x_{n+1}), \tag{3.7}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{s_n\} \subset (0,1)$, satisfying the following conditions:

(1) $\alpha_n + \beta_n + \gamma_n = 1$ *and* $\lim_{n\to\infty} \gamma_n = 1$;
(2) $\sum_{n=0}^{\infty} \beta_n = \infty$;
(3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
(4) $0 < \varepsilon \leq s_n \leq s_{n+1} < 1$ *for all* $n \geq 0$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of the nonexpansive mapping T, which is also the unique solution of the variational inequality

$$\langle (I - f)x, y - x\rangle \geq 0, \quad \forall y \in F(T).$$

In other words, x^* is the unique fixed point of the contraction $P_{F(T)}f$, that is, $P_{F(T)}f(x^*) = x^*$.

Proof We divide the proof into five steps.

Step 1. Firstly, we show that $\{x_n\}$ is bounded.

Indeed, take $p \in F(T)$ arbitrarily, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n x_n + \beta_n f(x_n) + \gamma_n T(s_n x_n + (1 - s_n)x_{n+1}) - p\| \\ &\leq \alpha_n\|x_n - p\| + \beta_n\|f(x_n) - p\| + \gamma_n\|T(s_n x_n + (1 - s_n)x_{n+1}) - p\| \\ &\leq \alpha_n\|x_n - p\| + \beta_n\|f(x_n) - f(p)\| + \beta_n\|f(p) - p\| \\ &\quad + \gamma_n\|s_n x_n + (1 - s_n)x_{n+1} - p\| \end{aligned}$$

$$\begin{aligned}&\le (\alpha_n + \theta\beta_n)\|x_n - p\| + \beta_n\big\|f(p) - p\big\| + \gamma_n\big[s_n\|x_n - p\| + (1 - s_n)\|x_{n+1} - p\|\big] \\ &= (\alpha_n + \theta\beta_n + \gamma_n s_n)\|x_n - p\| + \gamma_n(1 - s_n)\|x_{n+1} - p\| + \beta_n\big\|f(p) - p\big\|.\end{aligned}$$

It follows that

$$\big[1 - \gamma_n(1 - s_n)\big]\|x_{n+1} - p\| \le (\alpha_n + \theta\beta_n + \gamma_n s_n)\|x_n - p\| + \beta_n\big\|f(p) - p\big\|. \tag{3.8}$$

Since $\gamma_n, s_n \in (0,1)$, $1 - \gamma_n(1 - s_n) > 0$. Moreover, by (3.8) and $\alpha_n + \beta_n + \gamma_n = 1$, we get

$$\begin{aligned}\|x_{n+1} - p\| &\le \frac{\alpha_n + \theta\beta_n + \gamma_n s_n}{1 - \gamma_n(1 - s_n)}\|x_n - p\| + \frac{\beta_n}{1 - \gamma_n(1 - s_n)}\big\|f(p) - p\big\| \\ &= \left[1 - \frac{1 - \alpha_n - \gamma_n - \theta\beta_n}{1 - \gamma_n(1 - s_n)}\right]\|x_n - p\| + \frac{\beta_n}{1 - \gamma_n(1 - s_n)}\big\|f(p) - p\big\| \\ &= \left[1 - \frac{\beta_n - \theta\beta_n}{1 - \gamma_n(1 - s_n)}\right]\|x_n - p\| + \frac{\beta_n}{1 - \gamma_n(1 - s_n)}\big\|f(p) - p\big\| \\ &= \left[1 - \frac{\beta_n(1 - \theta)}{1 - \gamma_n(1 - s_n)}\right]\|x_n - p\| + \frac{\beta_n(1 - \theta)}{1 - \gamma_n(1 - s_n)}\left(\frac{1}{1 - \theta}\big\|f(p) - p\big\|\right).\end{aligned}$$

Thus, we have

$$\|x_{n+1} - p\| \le \max\left\{\|x_n - p\|, \frac{1}{1 - \theta}\big\|f(p) - p\big\|\right\}.$$

By induction, we obtain

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{1}{1 - \theta}\big\|f(p) - p\big\|\right\}, \quad \forall n \ge 0.$$

Hence, it turns out that $\{x_n\}$ is bounded. Consequently, we deduce immediately that $\{f(x_n)\}$, $\{T(s_n x_x + (1 - s_n)x_{n+1})\}$ are bounded.

Step 2. Next, we prove that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

To see this, we apply (3.7) to get

$$\begin{aligned}\|x_{n+1} - x_n\| &= \big\|\alpha_n x_n + \beta_n f(x_n) + \gamma_n T\big(s_n x_n + (1 - s_n)x_{n+1}\big) \\ &\quad - \big[\alpha_{n-1}x_{n-1} + \beta_{n-1}f(x_{n-1}) + \gamma_{n-1}T\big(s_{n-1}x_{n-1} + (1 - s_{n-1})x_n\big)\big]\big\| \\ &= \big\|\alpha_n(x_n - x_{n-1}) + (\alpha_n - \alpha_{n-1})x_{n-1} + \beta_n\big[f(x_n) - f(x_{n-1})\big] \\ &\quad + (\beta_n - \beta_{n-1})f(x_{n-1}) \\ &\quad + \gamma_n\big[T\big(s_n x_n + (1 - s_n)x_{n+1}\big) - T\big(s_{n-1}x_{n-1} + (1 - s_{n-1})x_n\big)\big] \\ &\quad + (\gamma_n - \gamma_{n-1})T\big(s_{n-1}x_{n-1} + (1 - s_{n-1})x_n\big)\big\| \\ &= \big\|\alpha_n(x_n - x_{n-1}) + (\alpha_n - \alpha_{n-1})x_{n-1} + \beta_n\big[f(x_n) - f(x_{n-1})\big] \\ &\quad + (\beta_n - \beta_{n-1})f(x_{n-1}) \\ &\quad + \gamma_n\big[T\big(s_n x_n + (1 - s_n)x_{n+1}\big) - T\big(s_{n-1}x_{n-1} + (1 - s_{n-1})x_n\big)\big] \\ &\quad - \big[(\alpha_n - \alpha_{n-1}) + (\beta_n - \beta_{n-1})\big]T\big(s_{n-1}x_{n-1} + (1 - s_{n-1})x_n\big)\big\|\end{aligned}$$

$$\begin{aligned}
&= \alpha_n\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \cdot \|x_{n-1} - T(s_{n-1}x_{n-1} + (1-s_{n-1})x_n)\| \\
&\quad + \beta_n\|f(x_n) - f(x_{n-1})\| + |\beta_n - \beta_{n-1}| \\
&\quad \cdot \|f(x_{n-1}) - T(s_{n-1}x_{n-1} + (1-s_{n-1})x_n)\| \\
&\quad + \gamma_n\|T(s_nx_n + (1-s_n)x_{n+1}) - T(s_{n-1}x_{n-1} + (1-s_{n-1})x_n)\| \\
&\leq \alpha_n\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_2 + \theta\beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|M_2 \\
&\quad + \gamma_n\|[s_nx_n + (1-s_n)x_{n+1}] - [s_{n-1}x_{n-1} + (1-s_{n-1})x_n]\| \\
&= \alpha_n\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_2 + \theta\beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|M_2 \\
&\quad + \gamma_n\|(1-s_n)(x_{n+1} - x_n) + s_{n-1}(x_n - x_{n-1})\| \\
&\leq \alpha_n\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_2 + \theta\beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|M_2 \\
&\quad + \gamma_n(1-s_n)\|x_{n+1} - x_n\| + \gamma_ns_{n-1}\|x_n - x_{n-1}\| \\
&= \gamma_n(1-s_n)\|x_{n+1} - x_n\| + (\alpha_n + \theta\beta_n + \gamma_ns_{n-1})\|x_n - x_{n-1}\| \\
&\quad + (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|)M_2,
\end{aligned}$$

where $M_2 > 0$ is a constant such that

$$M_2 \geq \max\left\{\sup_{n\geq 0}\|x_n - T(s_nx_n + (1-s_n)x_{n+1})\|, \sup_{n\geq 0}\|f(x_n) - T(s_nx_n + (1-s_n)x_{n+1})\|\right\}.$$

It turns out that

$$\begin{aligned}
&[1 - \gamma_n(1-s_n)]\|x_{n+1} - x_n\| \\
&\quad \leq (\alpha_n + \theta\beta_n + \gamma_ns_{n-1})\|x_n - x_{n-1}\| + (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|)M_2,
\end{aligned}$$

that is,

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \frac{\alpha_n + \theta\beta_n + \gamma_ns_{n-1}}{1 - \gamma_n(1-s_n)}\|x_n - x_{n-1}\| \\
&\quad + \frac{M_2}{1 - \gamma_n(1-s_n)}(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|) \\
&= \left[1 - \frac{\beta_n(1-\theta) + \gamma_n(s_n - s_{n-1})}{1 - \gamma_n(1-s_n)}\right]\|x_n - x_{n-1}\| \\
&\quad + \frac{M_2}{1 - \gamma_n(1-s_n)}(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|).
\end{aligned}$$

Note that $0 < \varepsilon \leq s_{n-1} \leq s_n < 1$, we have

$$0 < \varepsilon \leq s_n < 1 - \gamma_n(1-s_n) < 1$$

and

$$\frac{\beta_n(1-\theta) + \gamma_n(s_n - s_{n-1})}{1 - \gamma_n(1-s_n)} \geq \beta_n(1-\theta).$$

Thus,

$$\|x_{n+1} - x_n\| \le \big[1 - \beta_n(1-\theta)\big]\|x_n - x_{n-1}\| + \frac{M_2}{\varepsilon}\big(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|\big).$$

Since $\sum_{n=0}^{\infty}\beta_n = \infty$, $\sum_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, and $\sum_{n=0}^{\infty}|\beta_{n+1} - \beta_n| < \infty$, by Lemma 2.3, we can get $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

Step 3. Now, we prove that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

In fact, it can see that

$$\begin{aligned}
\|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \big\|x_{n+1} - T\big(s_nx_n + (1-s_n)x_{n+1}\big)\big\| \\
&\quad + \big\|T\big(s_nx_n + (1-s_n)x_{n+1}\big) - Tx_n\big\| \\
&\le \|x_n - x_{n+1}\| + \big\|\alpha_n\big[x_n - T\big(s_nx_n + (1-s_n)x_{n+1}\big)\big] \\
&\quad + \beta_n\big[f(x_n) - T\big(s_nx_n + (1-s_n)x_{n+1}\big)\big]\big\| + \big\|\big(s_nx_n + (1-s_n)x_{n+1}\big) - x_n\big\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\big\|x_n - T\big(s_nx_n + (1-s_n)x_{n+1}\big)\big\| \\
&\quad + \beta_n\big\|f(x_n) - T\big(s_nx_n + (1-s_n)x_{n+1}\big)\big\| + (1-s_n)\|x_{n+1} - x_n\| \\
&\le (2-s_n)\|x_n - x_{n+1}\| + (\alpha_n + \beta_n)M_2 \\
&\le 2\|x_n - x_{n+1}\| + (1-\gamma_n)M_2.
\end{aligned}$$

Then, by $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty}\gamma_n = 1$, we get $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. Similarly to (3.3), we also have

$$\big\|T\big(s_nx_n + (1-s_n)x_{n+1}\big) - x_n\big\| \to 0 \quad (\text{as } n \to \infty). \tag{3.9}$$

Step 4. In this step, we claim that $\limsup_{n\to\infty}\langle x^* - f(x^*), x^* - x_n\rangle \le 0$, where $x^* = P_{F(T)}f(x^*)$.

The proof is the same as Step 4 in Theorem 3.1, here we omit it.

Step 5. Finally, we show that $x_n \to x^*$ as $n \to \infty$. Here again $x^* \in F(T)$ is the unique fixed point of the contraction $P_{F(T)}f$ or in other words, $x^* = P_{F(T)}f(x^*)$.

In fact, we have

$$\begin{aligned}
\big\|x_{n+1} - x^*\big\|^2 &= \big\|\alpha_nx_n + \beta_nf(x_n) + \gamma_nT\big(s_nx_n + (1-s_n)x_{n+1}\big) - x^*\big\|^2 \\
&= \big\|\alpha_n\big[x_n - x^*\big] + \beta_n\big[f(x_n) - x^*\big] + \gamma_n\big[T\big(s_nx_n + (1-s_n)x_{n+1}\big) - x^*\big]\big\|^2 \\
&= \alpha_n^2\big\|x_n - x^*\big\|^2 + \beta_n^2\big\|f(x_n) - x^*\big\|^2 + \gamma_n^2\big\|T\big(s_nx_n + (1-s_n)x_{n+1}\big) - x^*\big\|^2 \\
&\quad + 2\alpha_n\beta_n\big\langle x_n - x^*, f(x_n) - x^*\big\rangle \\
&\quad + 2\alpha_n\gamma_n\big\langle x_n - x^*, T\big(s_nx_n + (1-s_n)x_{n+1}\big) - x^*\big\rangle \\
&\quad + 2\beta_n\gamma_n\big\langle f(x_n) - x^*, T\big(s_nx_n + (1-s_n)x_{n+1}\big) - x^*\big\rangle \\
&\le \alpha_n^2\big\|x_n - x^*\big\|^2 + \beta_n^2\big\|f(x_n) - x^*\big\|^2 + \gamma_n^2\big\|s_nx_n + (1-s_n)x_{n+1} - x^*\big\|^2 \\
&\quad + 2\alpha_n\beta_n\big\langle x_n - x^*, f(x_n) - x^*\big\rangle \\
&\quad + 2\alpha_n\gamma_n\big\|x_n - x^*\big\| \cdot \big\|T\big(s_nx_n + (1-s_n)x_{n+1}\big) - x^*\big\|
\end{aligned}$$

$$\begin{aligned}
&+ 2\beta_n\gamma_n\langle f(x_n) - f(x^*), T(s_nx_n + (1-s_n)x_{n+1}) - x^*\rangle \\
&+ 2\beta_n\gamma_n\langle f(x^*) - x^*, T(s_nx_n + (1-s_n)x_{n+1}) - x^*\rangle \\
\le{}& \alpha_n^2\|x_n - x^*\|^2 + \gamma_n^2\|s_nx_n + (1-s_n)x_{n+1} - x^*\|^2 \\
&+ 2\alpha_n\gamma_n\|x_n - x^*\| \cdot \|s_nx_n + (1-s_n)x_{n+1} - x^*\| \\
&+ 2\beta_n\gamma_n\|f(x_n) - f(x^*)\| \cdot \|T(s_nx_n + (1-s_n)x_{n+1}) - x^*\| + K_n \\
\le{}& \alpha_n^2\|x_n - x^*\|^2 + \gamma_n^2\|s_nx_n + (1-s_n)x_{n+1} - x^*\|^2 \\
&+ 2\alpha_n\gamma_n\|x_n - x^*\| \cdot \|s_nx_n + (1-s_n)x_{n+1} - x^*\| \\
&+ 2\theta\beta_n\gamma_n\|x_n - x^*\| \cdot \|s_nx_n + (1-s_n)x_{n+1} - x^*\| + K_n \\
={}& \alpha_n^2\|x_n - x^*\|^2 + \gamma_n^2\|s_nx_n + (1-s_n)x_{n+1} - x^*\|^2 \\
&+ 2\gamma_n(\alpha_n + \theta\beta_n)\|x_n - x^*\| \cdot \|s_nx_n + (1-s_n)x_{n+1} - x^*\| + K_n,
\end{aligned}$$

where

$$\begin{aligned}
K_n :={}& \beta_n^2\|f(x_n) - x^*\|^2 + 2\alpha_n\beta_n\langle x_n - x^*, f(x_n) - x^*\rangle \\
&+ 2\beta_n\gamma_n\langle f(x^*) - x^*, T(s_nx_n + (1-s_n)x_{n+1}) - x^*\rangle.
\end{aligned}$$

It turns out that

$$\begin{aligned}
&\gamma_n^2\|s_nx_n + (1-s_n)x_{n+1} - x^*\|^2 \\
&\quad+ 2\gamma_n(\alpha_n + \theta\beta_n)\|x_n - x^*\| \cdot \|s_nx_n + (1-s_n)x_{n+1} - x^*\| \\
&\quad+ K_n + \alpha_n^2\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \ge 0.
\end{aligned}$$

Solving this quadratic inequality for $\|s_nx_n + (1-s_n)x_{n+1} - x^*\|$ yields

$$\begin{aligned}
&\|s_nx_n + (1-s_n)x_{n+1} - x^*\| \\
&\quad\ge \frac{1}{2\gamma_n^2}\Big\{-2\gamma_n(\alpha_n + \theta\beta_n)\|x_n - x^*\| \\
&\qquad+ \sqrt{4\gamma_n^2(\alpha_n + \theta\beta_n)^2\|x_n - x^*\|^2 - 4\gamma_n^2(K_n + \alpha_n^2\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2)}\Big\} \\
&\quad= \frac{1}{\gamma_n}\Big[-(\alpha_n + \theta\beta_n)\|x_n - x^*\| \\
&\qquad+ \sqrt{(\alpha_n + \theta\beta_n)^2\|x_n - x^*\|^2 - K_n - \alpha_n^2\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2}\Big].
\end{aligned}$$

This implies that

$$\begin{aligned}
&s_n\|x_n - x^*\| + (1-s_n)\|x_{n+1} - x^*\| \\
&\quad\ge \frac{1}{\gamma_n}\Big[-(\alpha_n + \theta\beta_n)\|x_n - x^*\| \\
&\qquad+ \sqrt{(\alpha_n + \theta\beta_n)^2\|x_n - x^*\|^2 - K_n - \alpha_n^2\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2}\Big],
\end{aligned}$$

namely,

$$
\begin{aligned}
&(s_n\gamma_n + \alpha_n + \theta\beta_n)\|x_n - x^*\| + (1-s_n)\gamma_n\|x_{n+1} - x^*\| \\
&\quad\geq \sqrt{(\alpha_n + \theta\beta_n)^2\|x_n - x^*\|^2 - K_n - \alpha_n^2\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&(\alpha_n + \theta\beta_n)^2\|x_n - x^*\|^2 - K_n - \alpha_n^2\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2 \\
&\quad\leq (s_n\gamma_n + \alpha_n + \theta\beta_n)^2\|x_n - x^*\|^2 + (1-s_n)^2\gamma_n^2\|x_{n+1} - x^*\|^2 \\
&\qquad + 2(s_n\gamma_n + \alpha_n + \theta\beta_n)(1-s_n)\gamma_n\|x_n - x^*\| \cdot \|x_{n+1} - x^*\| \\
&\quad\leq (s_n\gamma_n + \alpha_n + \theta\beta_n)^2\|x_n - x^*\|^2 + (1-s_n)^2\gamma_n^2\|x_{n+1} - x^*\|^2 \\
&\qquad + (s_n\gamma_n + \alpha_n + \theta\beta_n)(1-s_n)\gamma_n\bigl[\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\bigr],
\end{aligned}
$$

which is reduced to the inequality

$$
\begin{aligned}
&\bigl[1 - (1-s_n)^2\gamma_n^2 - (s_n\gamma_n + \alpha_n + \theta\beta_n)(1-s_n)\gamma_n\bigr]\|x_{n+1} - x^*\|^2 \\
&\quad\leq \bigl[(s_n\gamma_n + \alpha_n + \theta\beta_n)^2 + (s_n\gamma_n + \alpha_n + \theta\beta_n)(1-s_n)\gamma_n + \alpha_n^2 - (\alpha_n + \theta\beta_n)^2\bigr] \\
&\qquad \times \|x_n - x^*\|^2 + K_n,
\end{aligned}
$$

that is,

$$
\begin{aligned}
&\bigl[1 - (1-s_n)\gamma_n\bigl(1 + (\theta-1)\beta_n\bigr)\bigr]\|x_{n+1} - x^*\|^2 \\
&\quad\leq \bigl[(s_n\gamma_n + \alpha_n + \theta\beta_n)\bigl(1 + (\theta-1)\beta_n\bigr) - 2\theta\alpha_n\beta_n - \theta^2\beta_n^2\bigr]\|x_n - x^*\|^2 + K_n.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 \leq{}& \frac{(s_n\gamma_n + \alpha_n + \theta\beta_n)(1 + (\theta-1)\beta_n) - 2\theta\alpha_n\beta_n - \theta^2\beta_n^2}{1 - (1-s_n)\gamma_n(1 + (\theta-1)\beta_n)}\|x_n - x^*\|^2 \\
&+ \frac{K_n}{1 - (1-s_n)\gamma_n(1 + (\theta-1)\beta_n)}.
\end{aligned}
\tag{3.10}
$$

Let

$$
\begin{aligned}
y_n &:= \frac{1}{\beta_n}\left\{1 - \frac{(s_n\gamma_n + \alpha_n + \theta\beta_n)(1 + (\theta-1)\beta_n) - 2\theta\alpha_n\beta_n - \theta^2\beta_n^2}{1 - (1-s_n)\gamma_n(1 + (\theta-1)\beta_n)}\right\} \\
&= \frac{2 + 2\theta\alpha_n - \beta_n}{1 - (1-s_n)\gamma_n(1 + (\theta-1)\beta_n)}.
\end{aligned}
$$

Since the sequence $\{s_n\}$ satisfies $0 < \varepsilon \leq s_n \leq s_{n+1} < 1$ for all $n \geq 0$, $\lim_{n\to\infty} s_n$ exists; assume that

$$
\lim_{n\to\infty} s_n = s^* > 0.
$$

Then

$$\lim_{n\to\infty} y_n = \frac{2}{s^*} > 0.$$

Let ρ_2 satisfy

$$0 < \rho_2 < \frac{2}{s^*},$$

then there exists an integer N_2 big enough such that $y_n > \rho_2$ for all $n \geq N_2$. Hence, we have

$$\frac{(s_n\gamma_n + \alpha_n + \theta\beta_n)(1 + (\theta - 1)\beta_n) - 2\theta\alpha_n\beta_n - \theta^2\beta_n^2}{1 - (1 - s_n)\gamma_n(1 + (\theta - 1)\beta_n)} \leq 1 - \rho_2\beta_n$$

for all $n \geq N_2$. It turns out from (3.10) that, for all $n \geq N_2$,

$$\left\|x_{n+1} - x^*\right\|^2 \leq (1 - \rho_2\beta_n)\left\|x_n - x^*\right\|^2 + \frac{K_n}{1 - (1 - s_n)\gamma_n(1 + (\theta - 1)\beta_n)}. \tag{3.11}$$

By $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \gamma_n = 1$, (3.9), and Step 4, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{K_n}{\rho_2\beta_n[1 - (1 - s_n)\gamma_n(1 + (\theta - 1)\beta_n)]} \\
&\quad = \limsup_{n\to\infty}\left(\frac{\beta_n\|f(x_n) - x^*\|^2 + 2\alpha_n\langle x_n - x^*, f(x_n) - x^*\rangle}{\rho_2[1 - (1 - s_n)\gamma_n(1 + (\theta - 1)\beta_n)]}\right. \\
&\qquad \left. + \frac{2\gamma_n\langle f(x^*) - x^*, T(s_nx_n + (1 - s_n)x_{n+1}) - x^*\rangle}{\rho_2[1 - (1 - s_n)\gamma_n(1 + (\theta - 1)\beta_n)]}\right) \\
&\quad \leq 0.
\end{aligned} \tag{3.12}$$

From (3.11), (3.12), and Lemma 2.2, we can obtain that

$$\lim_{n\to\infty}\left\|x_{n+1} - x^*\right\|^2 = 0,$$

namely, $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

4 Application

4.1 A more general system of variational inequalities

Let C be a nonempty closed convex subset of the real Hilbert space H and $\{A_i\}_{i=1}^N : C \to H$ be a family of mappings. In [3], Cai and Bu considered the problem of finding $(x_1^*, x_2^*, \dots, x_N^*) \in C \times C \times \cdots \times C$ such that

$$\begin{cases}
\langle \lambda_N A_N x_N^* + x_1^* - x_N^*, x - x_1^*\rangle \geq 0, \quad \forall x \in C, \\
\langle \lambda_{N-1} A_{N-1} x_{N-1}^* + x_N^* - x_{N-1}^*, x - x_N^*\rangle \geq 0, \quad \forall x \in C, \\
\dots, \\
\langle \lambda_2 A_2 x_2^* + x_3^* - x_2^*, x - x_3^*\rangle \geq 0, \quad \forall x \in C, \\
\langle \lambda_1 A_1 x_1^* + x_2^* - x_1^*, x - x_2^*\rangle \geq 0, \quad \forall x \in C.
\end{cases} \tag{4.1}$$

Equation (4.1) can be rewritten

$$\begin{cases} \langle x_1^* - (I - \lambda_N A_N)x_N^*, x - x_1^* \rangle \geq 0, & \forall x \in C, \\ \langle x_N^* - (I - \lambda_{N-1} A_{N-1})x_{N-1}^*, x - x_N^* \rangle \geq 0, & \forall x \in C, \\ \ldots, \\ \langle x_3^* - (I - \lambda_2 A_2)x_2^*, x - x_3^* \rangle \geq 0, & \forall x \in C, \\ \langle x_2^* - (I - \lambda_1 A_1)x_1^*, x - x_2^* \rangle \geq 0, & \forall x \in C, \end{cases}$$

which is called a more general system of variational inequalities in Hilbert spaces, where $\lambda_i > 0$ for all $i \in \{1, 2, \ldots, N\}$. We also have the following lemmas.

Lemma 4.1 [3] *Let C be a nonempty closed convex subset of the real Hilbert space H. For $i = 1, 2, \ldots, N$, let $A_i : C \to H$ be δ_i-inverse-strongly monotone for some positive real number δ_i, namely,*

$$\langle A_i x - A_i y, x - y \rangle \geq \delta_i \|A_i x - A_i y\|^2, \quad \forall x, y \in C.$$

Let $G : C \to C$ be a mapping defined by

$$G(x) = P_C(I - \lambda_N A_N)P_C(I - \lambda_{N-1} A_{N-1}) \cdots P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x, \quad \forall x \in C. \tag{4.2}$$

If $0 < \lambda_i \leq 2\delta_i$ for all $i \in \{1, 2, \ldots, N\}$, then G is nonexpansive.

Lemma 4.2 [4] *Let C be a nonempty closed convex subset of the real Hilbert space H. Let $A_i : C \to H$ be a nonlinear mapping, where $i = 1, 2, \ldots, N$. For given $x_i^* \in C$, $i = 1, 2, \ldots, N$, $(x_1^*, x_2^*, \ldots, x_N^*)$ is a solution of the problem* (4.1) *if and only if*

$$x_1^* = P_C(I - \lambda_N A_N)x_N^*, \qquad x_i^* = P_C(I - \lambda_{i-1} A_{i-1})x_{i-1}^*, \quad i = 2, 3, \ldots, N, \tag{4.3}$$

that is,

$$x_1^* = P_C(I - \lambda_N A_N)P_C(I - \lambda_{N-1} A_{N-1}) \cdots P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x_1^*.$$

From Lemma 4.2, we know that $x_1^* = G(x_1^*)$, that is, x_1^* is a fixed point of the mapping G, where G is defined by (4.2). Moreover, if we find the fixed point x_1^*, it is easy to get the other points by (4.3), in other words, we solve the problem (4.1). Applying Theorems 3.1 and 3.2, we get the results below.

Theorem 4.1 *Let C be a nonempty closed convex subset of the real Hilbert space H. For $i = 1, 2, \ldots, N$, let $A_i : C \to H$ be δ_i-inverse-strongly monotone for some positive real number δ_i with $F(G) \neq \emptyset$, where $G : C \to C$ is defined by*

$$G(x) = P_C(I - \lambda_N A_N)P_C(I - \lambda_{N-1} A_{N-1}) \cdots P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x, \quad \forall x \in C.$$

Let $f : C \to C$ be a contraction with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)G\big(s_n x_n + (1 - s_n)x_{n+1}\big),$$

where $\lambda_i \in (0, 2\delta_i)$, $i = 1, 2, \ldots, N$, $\{\alpha_n\}, \{s_n\} \subset (0,1)$, *satisfying the following conditions*:
(1) $\lim_{n\to\infty} \alpha_n = 0$;
(2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(4) $0 < \varepsilon \leq s_n \leq s_{n+1} < 1$ *for all* $n \geq 0$.
Then $\{x_n\}$ *converges strongly to a fixed point* x^* *of the nonexpansive mapping* G, *which is also the unique solution of the variational inequality*

$$\bigl\langle (I - f)x, y - x \bigr\rangle \geq 0, \quad \forall y \in F(G).$$

In other words, x^* *is the unique fixed point of the contraction* $P_{F(G)}f$, *that is*, $P_{F(G)}f(x^*) = x^*$.

Theorem 4.2 *Let* C *be a nonempty closed convex subset of the real Hilbert space* H. *For* $i = 1, 2, \ldots, N$, *let* $A_i : C \to H$ *be* δ_i-*inverse-strongly monotone for some positive real number* δ_i *with* $F(G) \neq \emptyset$, *where* $G : C \to C$ *is defined by*

$$G(x) = P_C(I - \lambda_N A_N)P_C(I - \lambda_{N-1}A_{N-1}) \cdots P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x, \quad \forall x \in C.$$

Let $f : C \to C$ *be a contraction with coefficient* $\theta \in [0,1)$. *Pick any* $x_0 \in C$, *let* $\{x_n\}$ *be a sequence generated by*

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n G\bigl(s_n x_n + (1 - s_n)x_{n+1}\bigr),$$

where $\lambda_i \in (0, 2\delta_i)$, $i = 1, 2, \ldots, N$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{s_n\} \subset (0,1)$, *satisfying the following conditions*:
(1) $\alpha_n + \beta_n + \gamma_n = 1$ *and* $\lim_{n\to\infty} \gamma_n = 1$;
(2) $\sum_{n=0}^{\infty} \beta_n = \infty$;
(3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
(4) $0 < \varepsilon \leq s_n \leq s_{n+1} < 1$ *for all* $n \geq 0$.
Then $\{x_n\}$ *converges strongly to a fixed point* x^* *of the nonexpansive mapping* G, *which is also the unique solution of the variational inequality*

$$\bigl\langle (I - f)x, y - x \bigr\rangle \geq 0, \quad \forall y \in F(G).$$

In other words, x^* *is the unique fixed point of the contraction* $P_{F(G)}f$, *that is*, $P_{F(G)}f(x^*) = x^*$.

4.2 The constrained convex minimization problem

Next, we consider the following constrained convex minimization problem:

$$\min_{x\in C} \varphi(x), \tag{4.4}$$

where $\varphi : C \to R$ is a real-valued convex function and assumes that the problem (4.4) is consistent (*i.e.*, its solution set is nonempty). Let Ω denote its solution set.

For the minimization problem (4.4), if φ is (Fréchet) differentiable, then we have the following lemma.

Lemma 4.3 (Optimality condition) [5] *A necessary condition of optimality for a point* $x^* \in C$ *to be a solution of the minimization problem* (4.4) *is that* x^* *solves the variational* *inequality*

$$\langle \nabla\varphi(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{4.5}$$

Equivalently, $x^* \in C$ *solves the fixed point equation*

$$x^* = P_C\bigl(x^* - \lambda\nabla\varphi(x^*)\bigr)$$

for every constant $\lambda > 0$. *If, in addition*, φ *is convex, then the optimality condition* (4.5) *is also sufficient*.

It is well known that the mapping $P_C(I - \lambda A)$ is nonexpansive when the mapping A is δ-inverse-strongly monotone and $0 < \lambda < 2\delta$. We therefore have the following results.

Theorem 4.3 *Let* C *be a nonempty closed convex subset of the real Hilbert space* H. *For the minimization problem* (4.4), *assume that* φ *is* (*Fréchet*) *differentiable and the gradient* $\nabla\varphi$ *is a* δ-*inverse-strongly monotone mapping for some positive real number* δ. *Let* $f : C \to C$ *be a contraction with coefficient* $\theta \in [0,1)$. *Pick any* $x_0 \in C$, *let* $\{x_n\}$ *be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)P_C(I - \lambda\nabla\varphi)\bigl(s_n x_n + (1 - s_n)x_{n+1}\bigr),$$

where $\lambda \in (0, 2\delta)$, $\{\alpha_n\}, \{s_n\} \subset (0,1)$, *satisfying the following conditions*:
(1) $\lim_{n\to\infty} \alpha_n = 0$;
(2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(4) $0 < \varepsilon \leq s_n \leq s_{n+1} < 1$ *for all* $n \geq 0$.
Then $\{x_n\}$ *converges strongly to a solution* x^* *of the minimization problem* (4.4), *which is also the unique solution of the variational inequality*

$$\langle (I - f)x, y - x \rangle \geq 0, \quad \forall y \in \Omega.$$

In other words, x^* *is the unique fixed point of the contraction* $P_\Omega f$, *that is*, $P_\Omega f(x^*) = x^*$.

Theorem 4.4 *Let* C *be a nonempty closed convex subset of the real Hilbert space* H. *For the minimization problem* (4.4), *assume that* φ *is* (*Fréchet*) *differentiable and the gradient* $\nabla\varphi$ *is a* δ-*inverse-strongly monotone mapping for some positive real number* δ. *Let* $f : C \to C$ *be a contraction with coefficient* $\theta \in [0,1)$. *Pick any* $x_0 \in C$, *let* $\{x_n\}$ *be a sequence generated by*

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n P_C(I - \lambda\nabla\varphi)\bigl(s_n x_n + (1 - s_n)x_{n+1}\bigr),$$

where $\lambda \in (0, 2\delta)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{s_n\} \subset (0,1)$, *satisfying the following conditions*:
(1) $\alpha_n + \beta_n + \gamma_n = 1$ *and* $\lim_{n\to\infty} \gamma_n = 1$;
(2) $\sum_{n=0}^{\infty} \beta_n = \infty$;
(3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
(4) $0 < \varepsilon \leq s_n \leq s_{n+1} < 1$ *for all* $n \geq 0$.

Then $\{x_n\}$ *converges strongly to a solution* x^* *of the minimization problem* (4.4), *which is also the unique solution of the variational inequality*

$$\langle (I-f)x, y-x\rangle \geq 0, \quad \forall y \in \Omega.$$

In other words, x^* *is the unique fixed point of the contraction* $P_\Omega f$, *that is,* $P_\Omega f(x^*) = x^*$.

4.3 *K*-Mapping

In 2009, Kangtunyakarn and Suantai [6] gave K-mapping generated by $T_1, T_2, \dots, T_N$ and $\lambda_1, \lambda_2, \dots, \lambda_N$ as follows.

Definition 4.1 [6] Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of mappings of C into itself and let $\lambda_1, \lambda_2, \dots, \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i = 1, 2, \dots, N$. We define a mapping $K : C \to C$ as follows:

$$\begin{aligned}
&U_1 = \lambda_1 T_1 + (1-\lambda_1)I,\\
&U_2 = \lambda_2 T_2 U_1 + (1-\lambda_2)U_1,\\
&U_3 = \lambda_3 T_3 U_2 + (1-\lambda_3)U_2,\\
&\dots,\\
&U_{N-1} = \lambda_{N-1} T_{N-1} U_{N-2} + (1-\lambda_{N-1})U_{N-2},\\
&K = U_N = \lambda_N T_N U_{N-1} + (1-\lambda_N)U_{N-1}.
\end{aligned}$$

Such a mapping K is called the K-mapping generated by $T_1, T_2, \dots, T_N$ and $\lambda_1, \lambda_2, \dots, \lambda_N$.

In 2014, Suwannaut and Kangtunyakarn [7] established the following main result for the K-mapping generated by $T_1, T_2, \dots, T_N$ and $\lambda_1, \lambda_2, \dots, \lambda_N$.

Lemma 4.4 [7] *Let* C *be a nonempty closed convex subset of the real Hilbert space* H. *For* $i = 1, 2, \dots, N$, *let* $\{T_i\}_{i=1}^N$ *be a finite family of* κ_i*-strictly pseudo-contractive mapping of* C *into itself with* $\kappa_i \leq \omega_1$ *and* $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, *namely, there exist constants* $\kappa_i \in [0,1)$ *such that*

$$\|T_i x - T_i y\|^2 \leq \|x-y\|^2 + \kappa_i \big\|(I-T_i)x - (I-T_i)y\big\|^2, \quad \forall x, y \in C.$$

Let $\lambda_1, \lambda_2, \dots, \lambda_N$ *be real numbers with* $0 < \lambda_i < \omega_2$ *for all* $i = 1, 2, \dots, N$ *and* $\omega_1 + \omega_2 < 1$. *Let* K *be the* K*-mapping generated by* $T_1, T_2, \dots, T_N$ *and* $\lambda_1, \lambda_2, \dots, \lambda_N$. *Then the following properties hold*:

(1) $F(K) = \bigcap_{i=1}^N F(T_i)$;
(2) K *is a nonexpansive mapping.*

Based on Lemma 4.4, we have the following results.

Theorem 4.5 *Let* C *be a nonempty closed convex subset of the real Hilbert space* H. *For* $i = 1, 2, \dots, N$, *let* $\{T_i\}_{i=1}^N$ *be a finite family of* κ_i*-strictly pseudo-contractive mapping of* C *into itself with* $\kappa_i \leq \omega_1$ *and* $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. *Let* $\lambda_1, \lambda_2, \dots, \lambda_N$ *be real numbers with* $0 < \lambda_i < \omega_2$ *for all* $i = 1, 2, \dots, N$ *and* $\omega_1 + \omega_2 < 1$. *Let* K *be the* K*-mapping generated by* $T_1, T_2, \dots, T_N$ *and*

$\lambda_1, \lambda_2, \ldots, \lambda_N$. Let $f : C \to C$ be a contraction with coefficient $\theta \in [0,1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)K\bigl(s_n x_n + (1-s_n)x_{n+1}\bigr),$$

where $\{\alpha_n\}, \{s_n\} \subset (0,1)$, satisfying the following conditions:

(1) $\lim_{n\to\infty} \alpha_n = 0$;
(2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(4) $0 < \varepsilon \leq s_n \leq s_{n+1} < 1$ *for all* $n \geq 0$.

Then $\{x_n\}$ converges strongly to a common fixed point x^* of the mappings $\{T_i\}_{i=1}^N$, which is also the unique solution of the variational inequality

$$\bigl\langle (I-f)x, y-x\bigr\rangle \geq 0, \quad \forall y \in F(K) = \bigcap_{i=1}^N F(T_i).$$

In other words, the point x^* is the unique fixed point of the contraction $P_{\bigcap_{i=1}^N F(T_i)} f$, that is, $P_{\bigcap_{i=1}^N F(T_i)} f(x^*) = x^*$.

Theorem 4.6 *Let C be a nonempty closed convex subset of the real Hilbert space H. For $i = 1, 2, \ldots, N$, let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strictly pseudo-contractive mapping of C into itself with $\kappa_i \leq \omega_1$ and $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers with $0 < \lambda_i < \omega_2$ for all $i = 1, 2, \ldots, N$ and $\omega_1 + \omega_2 < 1$. Let K be the K-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$. Let $f : C \to C$ be a contraction with coefficient $\theta \in [0,1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n G\bigl(s_n x_n + (1-s_n)x_{n+1}\bigr),$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{s_n\} \subset (0,1)$, satisfying the following conditions:

(1) $\alpha_n + \beta_n + \gamma_n = 1$ *and* $\lim_{n\to\infty} \gamma_n = 1$;
(2) $\sum_{n=0}^{\infty} \beta_n = \infty$;
(3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
(4) $0 < \varepsilon \leq s_n \leq s_{n+1} < 1$ *for all* $n \geq 0$.

Then $\{x_n\}$ converges strongly to a common fixed point x^* of the mappings $\{T_i\}_{i=1}^N$, which is also the unique solution of the variational inequality

$$\bigl\langle (I-f)x, y-x\bigr\rangle \geq 0, \quad \forall y \in F(K) = \bigcap_{i=1}^N F(T_i).$$

In other words, the point x^* is the unique fixed point of the contraction $P_{\bigcap_{i=1}^N F(T_i)} f$, that is, $P_{\bigcap_{i=1}^N F(T_i)} f(x^*) = x^*$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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