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Structural stability for a Brinkman-Forchheimer type model with temperature-dependent solubility

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Abstract

We study the structural stability for the Brinkman-Forchheimer equations with temperature-dependent solubility. We prove both the convergence and continuous dependence results for the Forchheimer coefficient λ. We also demonstrate how to get the same results for the Forchheimer equations.

MSC: 35B30; 35K55; 35Q35

Keywords: structural stability; Forchheimer coefficient; convergence result; continuous dependence

1 Introduction

In the last few years, some researchers have studied the question of the continuous dependence or convergence of solutions of problems in partial differential equations on the coefficients in the equations. It is called the question of structural stability. For one thing, when we study continuous dependence or convergence, the notion of structural stability is on changes in the model itself instead of the original data. The majority of references to work of this nature are given in the monograph of Ames and Straughan [1], which studies the structural stability about changes in the model itself. Hence, we tend to know that changes in the coefficients in the partial differential equations may be reflected physically by changes in the constitutive parameters. If we deeply study these equations by mathematical analysis, it is certainly giving us a helping hand to indicate their applicability in physics. For another, because of some inevitable errors, which may have occurred, continuous dependence or convergence results are significant. It is relevant to know the magnitude of the effect of such errors in the solutions. Consequently, we think it is valuable for us to study the subject of structural stability.

We tend to find a wide range of papers in the literature coping with the structural stability for varieties equations. Most of them focus on the Brinkman, Darcym, and Forchheimer models. These equations are discussed in the books of Nield and Bejan [2] and Straughan [3, 4]. In addition, several papers have dealt with Saint-Venant type spatial decay results for the Brinkman, Darcy, Forchheimer, and other equations for porous media. More recent work on the stability and continuous dependence questions in porous media problems has been carried out by Ames and Payne [5], Franchi and Straughan [6], Kaloni and Qin

[7], Kaloni and Guo [8], Payne and Straughan [9–11], Payne *et al.* [12], Lin and Payne [13–15], Li and Lin [16], Celebi *et al.* [17, 18], Straughan [19], Scott [20], Scott and Straughan [21], and Harfash [22–24]. The fundamental model we study is based upon the equations of balance of momentum, balance of mass, conservation of energy, and conservation of salt concentration, adopting a Forchheimer approximation in the body force term in the momentum equation (see [3, 25]),

$$\begin{cases} \frac{\partial u_i}{\partial t} + \lambda|u|u_i = -p_{,i} + \Delta u_i + g_i T - h_i C, \quad (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial u_i}{\partial x_i} = 0, \quad (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} = \Delta T, \quad (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial C}{\partial t} + u_i \frac{\partial C}{\partial x_i} = \Delta C + Lf(T) - kC, \quad (x,t) \in \Omega \times [0,\tau], \end{cases} \tag{1.1}$$

where u_i is the velocity, p denotes the pressure, T is the temperature, and C is the salt concentration. Here $g_i(x)$, $h_i(x)$ are gravity fields. Here also Δ is the Laplacian operator. a, b, L, and k are positive constants. Equations (1.1) follow in practice by employing a Forchheimer approximation which accounts for the variable C allowing the incompressibility condition to hold (see Fife [26]). The function f is at least C^1.

Equations (1.1) hold in the region $\Omega \times [0,\tau]$, where Ω is a bounded, simply connected, and star-shaped domain with boundary $\partial\Omega$ in R^3, and τ is a given number satisfying $0 \leq \tau < \infty$. Associated with (1.1), we impose the boundary conditions

$$u_i = 0, \qquad \frac{\partial T}{\partial n} = 0, \qquad \frac{\partial C}{\partial n} = 0, \quad (x,t) \in \partial\Omega \times [0,\tau], \tag{1.2}$$

and additionally the concentration is given at $t = 0$, *i.e.*,

$$u_i(x,0) = u_{i0}(x), \tag{1.3}$$

$$T(x,0) = T_0(x), \qquad C(x,0) = C_0(x), \quad x \in \Omega. \tag{1.4}$$

We will derive both the convergence result and continuous result on the Forchheimer coefficient λ. In the present paper, a comma is used to indicate partial differentiation and the differentiation with respect to the direction x_k is denoted as $,k$, thus $u_{,i}$ denotes $\frac{\partial u}{\partial x_i}$. The usual summation convection is employed with repeated Latin subscripts summed from 1 to 3. Hence, $u_{i,i} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i}$, $\|\cdot\|$ denotes the norm of L^2, and $\|\cdot\|_p$ denotes the norm of L^p.

2 *A priori* bounds for the temperature T and the salt concentration C

Now we want an *a priori* bound or a maximum principle for T. Therefore multiplying $(1.1)_3$ by T^{2p-1} and integrating by parts, we can obtain

$$\int_\Omega T^{2p}\,dx - \int_\Omega T_0^{2p}\,dx = -\frac{2(2p-1)}{p}\int_0^t \int_\Omega \left(T^p\right)_{,i}\left(T^p\right)_{,i}\,dx\,d\eta \leq 0. \tag{2.1}$$

Inequality (2.1) is now integrated and then we take the $\frac{1}{2p}$ power to find

$$\left(\int_0^t \int_\Omega T^{2p}\,dx\,d\eta\right)^{\frac{1}{2p}} \leq \left(\int_0^t \int_\Omega T_0^{2p}\,dx\,d\eta\right)^{\frac{1}{2p}}. \tag{2.2}$$

We let $p \to \infty$ and then (2.2) leads to

$$\sup_{[0,t]} \|T\|_\infty \le \|T_0\|_\infty = T^M. \tag{2.3}$$

Since f is a C^1 function, and T is bounded, we can easily see that there exists a constant d such that

$$f(T) \le d. \tag{2.4}$$

For some $\xi \in (T, T^*)$, we easily get

$$\left|f'(\xi)\right| \le k_1, \tag{2.5}$$

where k_1 is a positive constant.

Similarly we have

$$\left(\int_0^t \int_\Omega C^{2p}\,dx\,d\eta\right)^{\frac{1}{2p}} \le \left(\int_0^t \int_\Omega e^{(2p-1)(t-\eta)}\left(k(p) + \int_\Omega C_0^{2p}\,dx\right)dx\,d\eta\right)^{\frac{1}{2p}}, \tag{2.6}$$

where $k(p) = \int_0^\tau \int_\Omega (Lf(T))^{2p}\,dx\,dt$.

We let $p \to \infty$, (2.6) leads to

$$\sup_{[0,t]} \|C\|_\infty \le C^M, \tag{2.7}$$

where $C^M = \max\{e^\tau \|C_0\|_\infty, L\,de^\tau\}$.

3 Convergence and continuous dependence results for the Forchheimer coefficient λ

Now, let (u_i, T, C, p) be a solution to the boundary initial-value problem for the Brinkman-Forchheimer model,

$$\begin{cases} \frac{\partial u_i}{\partial t} + \lambda|u|u_i = -p_{,i} + \Delta u_i + g_i T - h_i C, & (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial u_i}{\partial x_i} = 0, & (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} = \Delta T, & (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial C}{\partial t} + u_i \frac{\partial C}{\partial x_i} = \Delta C + Lf(T) - kC, & (x,t) \in \Omega \times [0,\tau], \end{cases} \tag{3.1}$$

$$u_i = 0, \qquad \frac{\partial T}{\partial n} = 0, \qquad \frac{\partial C}{\partial n} = 0, \quad (x,t) \in \partial\Omega \times [0,\tau], \tag{3.2}$$

$$u_i(x,0) = u_{i0}(x), \tag{3.3}$$

$$T(x,0) = T_0(x), \qquad C(x,0) = C_0(x), \quad x \in \Omega. \tag{3.4}$$

Moreover, let (u_i^*, T^*, C^*, p^*) be a solution to the corresponding model with $\lambda = 0$,

$$\begin{cases} \frac{\partial u_i^*}{\partial t} = -p_{,i}^* + \Delta u_i^* + g_i T^* - h_i C^*, & (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial u_i^*}{\partial x_i} = 0, & (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial T^*}{\partial t} + u_i^* \frac{\partial T^*}{\partial x_i} = \Delta T^*, & (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial C^*}{\partial t} + u_i^* \frac{\partial C^*}{\partial x_i} = \Delta C^* + Lf(T^*) - kC^*, & (x,t) \in \Omega \times [0,\tau], \end{cases} \tag{3.5}$$

$$u_i^* = 0, \qquad \frac{\partial T^*}{\partial n} = 0, \qquad \frac{\partial C^*}{\partial n} = 0, \quad (x,t) \in \partial\Omega \times [0,\tau], \tag{3.6}$$

$$u_i^*(x,0) = u_{i0}(x), \tag{3.7}$$

$$T^*(x,0) = T_0(x), \qquad C^*(x,0) = C_0(x), \quad x \in \Omega. \tag{3.8}$$

The object of this section is to demonstrate that the solution of (3.1) converges to the solution of (3.5) as $\lambda \to 0$. Now, we define the difference variables ω_i, π, θ, and S by

$$\omega_i = u_i - u_i^*, \qquad \pi = p - p^*, \qquad \theta = T - T^*, \qquad S = C - C^*. \tag{3.9}$$

Then $(\omega_i, \theta, S, \pi)$ is a solution of the problem

$$\begin{cases} \frac{\partial \omega_i}{\partial t} + \lambda |u| u_i = -\pi_{,i} + \Delta\omega_i + g_i\theta - h_i S, \quad (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial \omega_i}{\partial x_i} = 0, \quad (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial \theta}{\partial t} + \omega_i \frac{\partial T}{\partial x_i} + u_i^* \frac{\partial \theta}{\partial x_i} = \Delta\theta, \quad (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial S}{\partial t} + \omega_i \frac{\partial C}{\partial x_i} + u_i^* S_{,i} = \Delta S + L(f(T) - f(T^*)) - kS, \quad (x,t) \in \Omega \times [0,\tau], \end{cases} \tag{3.10}$$

in $\Omega \times [0,\tau]$, subject to the boundary and initial conditions

$$\omega_i = 0, \qquad \frac{\partial \theta}{\partial n} = 0, \qquad \frac{\partial S}{\partial n} = 0, \quad (x,t) \in \partial\Omega \times [0,\tau], \tag{3.11}$$

$$\omega_i(x,0) = 0, \tag{3.12}$$

$$\theta(x,0) = 0, \qquad S(x,0) = 0, \quad x \in \Omega. \tag{3.13}$$

We will obtain the following result.

Theorem 1 *Let (u_i, T, C, p) be the classical solution to the initial-boundary problem* (3.1)-(3.4), *(u_i^*, T^*, C^*, p^*) be the classical solution to the initial-boundary problem* (3.5)-(3.8) *in $\Omega \times [0,\tau]$, and (w_i, θ, S, π) be the difference of (u_i, T, C, p) and (u_i^*, T^*, C^*, p^*), then the solution (u_i, T, C, p) converges to the solution (u_i^*, T^*, C^*, p^*) as the Boussinesq coefficient λ tends to* 0. *The difference (w_i, θ, S, π) satisfies*

$$\|\omega\|^2 + \|\theta\|^2 + \|S\|^2 \le \frac{2\lambda}{3M_1}\left(\frac{3}{4}k_2^2 k_3^{\frac{1}{2}} k_4^{\frac{3}{2}} + \frac{1}{4}|\Omega|\right) e^{M_1\tau}. \tag{3.14}$$

Proof We multiply $(3.10)_1$ by ω_i and integrate over Ω to find

$$\begin{aligned} \frac{d}{dt}\|\omega\|^2 &= -2\lambda \int_\Omega |u| u_i \omega_i \, dx - 2\|\nabla\omega\|^2 + 2\int_\Omega g_i \theta \omega_i \, dx - 2\int_\Omega h_i S \omega_i \, dx \\ &\le -2\lambda \int_\Omega |u| u_i u_i \, dx + 2\lambda \int_\Omega |u| u_i u_i^* \, dx + 2\|\omega\|^2 + h^2\|S\|^2 + g^2\|\theta\|^2 \\ &\le \frac{2}{3}\lambda \int_\Omega |u^*| u_i^* u_i^* \, dx + 2\|\omega\|^2 + h^2\|S\|^2 + g^2\|\theta\|^2. \end{aligned} \tag{3.15}$$

We will use the following inequality in three dimensions:

$$\|f\|_4 \le k_2^{\frac{1}{2}} \|f\|^{\frac{1}{4}} \|\nabla f\|^{\frac{3}{4}}, \tag{3.16}$$

where k_2 is a positive constant.

From the above inequality and the Young inequality, we can get

$$\begin{aligned}\int_\Omega |u^*|u_i^*u_i^*\,dx &\le \frac{3}{4}\int_\Omega \left(u_i^*u_i^*\right)^2\,dx + \int_\Omega \frac{1}{4}\,dx \\ &\le \frac{3}{4}k_2^2\left\|u^*\right\|\left\|\nabla u^*\right\|^3 + \frac{1}{4}|\Omega|,\end{aligned} \tag{3.17}$$

where $|\Omega|$ is the measure of Ω.

We multiply $(3.5)_1$ by u_i^* and integrate over Ω to find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|u^*\right\|^2 + \left\|\nabla u^*\right\|^2 &= 7 - \int_\Omega p_{,i}^*u_i^*\,dx + \int_\Omega g_iT^*u_i^*\,dx - \int_\Omega h_iC^*u_i^*\,dx \\ &\le \left(g^2\int_\Omega u_i^*u_i^*\,dx\int_\Omega \left(T^*\right)^2\,dx\right)^{\frac{1}{2}} \\ &\quad + \left(h^2\int_\Omega u_i^*u_i^*\,dx\int_\Omega \left(C^*\right)^2\,dx\right)^{\frac{1}{2}} \\ &\le \left\|u^*\right\|^2 + \frac{1}{2}|\Omega|\left[g^2\left(T^M\right)^2 + h^2\left(C^M\right)^2\right],\end{aligned} \tag{3.18}$$

where $g^2 = \max_\Omega g_ig_i$, $h^2 = \max_\Omega h_ih_i$.

Integration of (3.18) leads to

$$\left\|u^*\right\|^2 \le \|u_0\|^2e^{2\tau} + \frac{1}{2}\left[g^2\left(T^M\right)^2 + h^2\left(C^M\right)^2\right]|\Omega|\left(e^{2\tau} - 1\right) = k_3. \tag{3.19}$$

We now want to give a bound for $\|\nabla u^*\|^2$. Multiplying $(3.5)_1$ by $u_{i,t}^*$ and integrating over Ω, we have

$$\int_\Omega u_{i,t}^*u_{i,t}^*\,dx - \int_\Omega u_{i,t}^*u_{i,jj}^*\,dx = \int_\Omega g_iu_{i,t}^*T^*\,dx - \int_\Omega h_iu_{i,t}^*C^*\,dx. \tag{3.20}$$

We can obtain

$$\begin{aligned}&\int_0^t\int_\Omega u_{i,t}^*u_{i,t}^*\,dx\,d\eta + \frac{1}{2}\int_\Omega u_{i,j}^*u_{i,j}^*\,dx|_{\eta=t} \\ &\quad\le \int_0^t\int_\Omega u_{i,t}^*u_{i,t}^*\,dx\,d\eta + \frac{1}{2}\int_0^t\int_\Omega \left(T^*\right)^2\,dx\,d\eta \\ &\qquad + \frac{1}{2}\int_0^t\int_\Omega \left(C^*\right)^2\,dx\,d\eta + \frac{1}{2}\int_\Omega u_{i0,j}u_{i0,j}\,dx\end{aligned} \tag{3.21}$$

or

$$\int_\Omega u_{i,j}^*u_{i,j}^*\,dx|_{\eta=t} \le \left(T^M\right)^2|\Omega|\tau + \left(C^M\right)^2|\Omega|\tau + \int_\Omega u_{i0,j}u_{i0,j}\,dx = k_4. \tag{3.22}$$

Combining (3.15), (3.17), (3.19), and (3.22) gives

$$\frac{d}{dt}\|\omega\|^2 \le \frac{2}{3}\lambda\left(\frac{3}{4}k_2^2k_3^{\frac{1}{2}}k_4^{\frac{3}{2}} + \frac{1}{4}|\Omega|\right) + 2\|\omega\|^2 + h^2\|S\|^2 + g^2\|\theta\|^2. \tag{3.23}$$

Multiplying $(3.10)_3$ by θ and integrating over Ω, we derive

$$\begin{aligned}\frac{d}{dt}\|\theta\|^2 + 2\int_\Omega \theta_{,i}\theta_{,i}\,dx &= -2\int_\Omega \theta\omega_i T_{,i}\,dx = 2\int_\Omega \theta_{,i}\omega_i T\,dx \\ &\leq 2\int_\Omega \theta_{,i}\theta_{,i}\,dx + \frac{(T^M)^2}{2}\|\omega\|^2. \end{aligned}\tag{3.24}$$

The Lagrange theorem says that

$$f(T) - f(T^*) = \theta f'(\xi). \tag{3.25}$$

Multiplying $(3.10)_4$ by S and integrating over Ω and using (2.5), we find

$$\begin{aligned}\frac{d}{dt}\|S\|^2 + 2\int_\Omega S_{,i}S_{,i}\,dx &= -2\int_\Omega S\omega_i C_{,i}\,dx + 2L\int_\Omega S\big(f(T) - f(T^*)\big)\,dx - 2k\int_\Omega S^2\,dx \\ &\leq 2\int_\Omega S_{,i}\omega_i C\,dx + 2Lk_1\int_\Omega S\theta\,dx \\ &\leq 2\int_\Omega S_{,i}S_{,i}\,dx + \frac{(C^M)^2}{2}\|\omega\|^2 + L^2k_1^2\|S\|^2 + \|\theta\|^2. \end{aligned}\tag{3.26}$$

From (3.23), (3.24), and (3.26), we get

$$\begin{aligned}\frac{d}{dt}\big(\|\omega\|^2 + \|\theta\|^2 + \|S\|^2\big) &\leq \frac{2}{3}\lambda\left(\frac{3}{4}k_2^2k_3^{\frac{1}{2}}k_4^{\frac{3}{2}} + \frac{1}{4}|\Omega|\right) \\ &\quad + \left(2 + \frac{(T^M)^2}{2} + \frac{(C^M)^2}{2}\right)\|\omega\|^2 \\ &\quad + \big(L^2k_1^2 + h^2\big)\|S\|^2 + \big(1 + g^2\big)\|\theta\|^2. \end{aligned}\tag{3.27}$$

We set

$$F_1 = \|\omega\|^2 + \|\theta\|^2 + \|S\|^2.$$

Then we have

$$\frac{dF_1}{dt} \leq \frac{2}{3}\lambda\left(\frac{3}{4}k_2^2k_3^{\frac{1}{2}}k_4^{\frac{3}{2}} + \frac{1}{4}|\Omega|\right) + M_1F_1, \tag{3.28}$$

where $M_1 = \max\{2 + \frac{(T^M)^2}{2} + \frac{(C^M)^2}{2}, L^2k_1^2 + h^2, 1 + g^2\}$.

We easily see that

$$\|\omega\|^2 + \|\theta\|^2 + \|S\|^2 \leq \frac{2\lambda}{3M_1}\left(\frac{3}{4}k_2^2k_3^{\frac{1}{2}}k_4^{\frac{3}{2}} + \frac{1}{4}|\Omega|\right)e^{M_1t}. \tag{3.29}$$

Inequality (3.29) demonstrates the convergence of u_i to u_i^*, T to T^*, and C to C^* as $\lambda \to 0$ in the indicated measure. □

Next, we will discuss the continuous dependence on the Forchheimer coefficient λ. Let (u_i, p, T, C) be a solution of the boundary initial-value problem for the thermal convection model,

$$\begin{cases} \frac{\partial u_i}{\partial t} + \lambda_1 |u| u_i = -p_{,i} + \Delta u_i + g_i T - h_i C, & (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial u_i}{\partial x_i} = 0, & (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} = \Delta T, & (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial C}{\partial t} + u_i \frac{\partial C}{\partial x_i} = \Delta C + Lf(T) - kC, & (x,t) \in \Omega \times [0,\tau], \end{cases} \tag{3.30}$$

$$u_i = 0, \qquad w\frac{\partial T}{\partial n} = 0, \qquad \frac{\partial C}{\partial n} = 0, \quad (x,t) \in \partial\Omega \times [0,\tau], \tag{3.31}$$

$$u_i(x,0) = u_{i0}(x), \tag{3.32}$$

$$T(x,0) = T_0(x), \qquad C(x,0) = C_0(x), \quad x \in \Omega. \tag{3.33}$$

Furthermore, let (u_i^*, p^*, T^*, C^*) be a solution to the following boundary initial-value problem:

$$\begin{cases} \frac{\partial u_i^*}{\partial t} + \lambda_2 |u^*| u_i^* = -p^*_{,i} + \Delta u_i^* + g_i T^* - h_i C^*, & (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial u_i^*}{\partial x_i} = 0, & (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial T^*}{\partial t} + u_i^* \frac{\partial T^*}{\partial x_i} = \Delta T^*, & (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial C^*}{\partial t} + u_i^* \frac{\partial C^*}{\partial x_i} = \Delta C^* + Lf(T^*) - kC^*, & (x,t) \in \Omega \times [0,\tau], \end{cases} \tag{3.34}$$

$$u_i^* = 0, \qquad \frac{\partial T^*}{\partial n} = 0, \qquad \frac{\partial C^*}{\partial n} = 0, \quad (x,t) \in \partial\Omega \times [0,\tau], \tag{3.35}$$

$$u_i^*(x,0) = u_{i0}(x), \tag{3.36}$$

$$T^*(x,0) = T_0(x), \qquad C^*(x,0) = C_0(x), \quad x \in \Omega. \tag{3.37}$$

In this section, we establish the continuous dependence on the coefficient. To do this, let (u_i, T, C, p) and (u_i^*, T^*, C^*, p^*) be solutions of (3.30) and (3.34) with the same boundary and initial conditions. Now, we define

$$\omega_i = u_i - u_i^*, \qquad \pi = p - p^*, \qquad \theta = T - T^*, \qquad S = C - C^*. \tag{3.38}$$

Then $(\omega_i, \theta, S, \pi)$ is a solution of the problem

$$\begin{cases} \frac{\partial \omega_i}{\partial t} + (\lambda_1 |u| u_i - \lambda_2 |u^*| u_i^*) = -\pi_{,i} + \Delta \omega_i + g_i \theta - h_i S, & (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial \omega_i}{\partial x_i} = 0, & (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial \theta}{\partial t} + \omega_i \frac{\partial T}{\partial x_i} + u_i^* \frac{\partial \theta}{\partial x_i} = \Delta \theta, & (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial S}{\partial t} + \omega_i \frac{\partial C}{\partial x_i} + u_i^* S_{,i} = \Delta S + L(f(T) - f(T^*)) - kS, & (x,t) \in \Omega \times [0,\tau], \end{cases} \tag{3.39}$$

in $\Omega \times t > 0$, subject to the boundary and initial conditions

$$\omega_i = 0, \qquad \frac{\partial \theta}{\partial n} = 0, \qquad \frac{\partial S}{\partial n} = 0, \quad (x,t) \in \partial\Omega \times [0,\tau], \tag{3.40}$$

$$\omega_i(x,0) = 0, \tag{3.41}$$

$$\theta(x,0) = 0, \qquad S(x,0) = 0, \quad x \in \Omega. \tag{3.42}$$

We will obtain the following result.

Theorem 2 *Let (u_i,T,C,p) be the classical solution to the initial-boundary problem* (3.30)-(3.33), *(u_i^*,T^*,C^*,p^*) be the classical solution to the initial-boundary problem* (3.34)-(3.37) *in $\Omega\times(0,\tau)$, and (w_i,θ,S,π) be the difference of (u_i,T,C,p) and (u_i^*, T^*,C^*,p^*), then the solution (u_i,T,C,p) converges to the solution (u_i^*,T^*,C^*,p^*) as the Forchheimer coefficient λ_1 tends to λ_2. If we suppose that $\int_\Omega u_{i,t}(x,0)u_{i,t}(x,0)\,dx + \int_\Omega T_{i,t}(x,0)T_{i,t}(x,0)\,dx+\int_\Omega C_{i,t}(x,0)C_{i,t}(x,0)\,dx\le R$, the difference (w_i,θ,S,π) satisfies*

$$\|\omega\|^2+\|\theta\|^2+\|S\|^2\le\lambda^2te^{-M_3t}k_2^2k_5^{\frac{1}{2}}k_6^{\frac{3}{2}}, \tag{3.43}$$

where $\lambda=\lambda_1-\lambda_2$.

Proof We first observe that

$$\frac{1}{2}\frac{d}{dt}\|\omega\|^2+\int_\Omega\bigl(\lambda_1|u|u_i-\lambda_2\bigl|u^*\bigr|u_i^*\bigr)\omega_i\,dx+\|\nabla\omega\|^2=\int_\Omega g_i\theta\omega_i\,dx-\int_\Omega h_iS\omega_i\,dx. \tag{3.44}$$

Moreover, we can get

$$\begin{aligned}\int_\Omega\bigl(\lambda_1|u|u_i-\lambda_2\bigl|u^*\bigr|u_i^*\bigr)\omega_i\,dx&\ge(\lambda+\lambda_2)\int_\Omega|u|u_i\omega_i\,dx-\lambda_2\int_\Omega\bigl|u^*\bigr|u_i^*\omega_i\,dx\\&\ge\lambda\int_\Omega|u|u_i\omega_i\,dx+\lambda_2\int_\Omega\bigl(|u|u_i\omega_i-\bigl|u^*\bigr|u_i^*\omega_i\bigr)\,dx.\end{aligned} \tag{3.45}$$

Since the operator $T(u)=|u|u$ is a monotonous operator, we get

$$\int_\Omega\bigl(|u|u-\bigl|u^*\bigr|u^*\bigr)\omega\,dx\ge0. \tag{3.46}$$

From the above discussion, we can get

$$\int_\Omega\bigl(\lambda_1|u|u_i-\lambda_2\bigl|u^*\bigr|u_i^*\bigr)\omega_i\,dx\ge\lambda\int_\Omega|u|u_i\omega_i\,dx. \tag{3.47}$$

Hence we get a similar inequality,

$$\frac{1}{2}\frac{d}{dt}\|\omega\|^2+\lambda\int_\Omega|u|u_i\omega_i\,dx+\|\nabla\omega\|^2\le\int_\Omega g_i\theta\omega_i\,dx-\int_\Omega h_iS\omega_i\,dx. \tag{3.48}$$

Nevertheless, we use another method to get the bound for $\int_\Omega|u|u_i\omega_i\,dx$. We can use a similar method to give the bound for $\|u\|^2$,

$$\|u\|^2\le\|u_0\|^2e^{2\tau}+\frac{1}{2}\bigl[g^2\bigl(T^M\bigr)^2+h^2\bigl(C^M\bigr)^2\bigr]|\Omega|\bigl(e^{2\tau}-1\bigr)=k_5. \tag{3.49}$$

The next step is to give a bound for $\|\nabla u\|^2$. In [27], Liu used the similar method. Multiplying $(1.1)_1$ by u_i and integrating over Ω, we have

$$\begin{aligned}\int_\Omega u_{i,j}u_{i,j}\,dx&\le-\int_\Omega u_{i,t}u_i\,dx+\int_\Omega g_iu_iT\,dx-\int_\Omega h_iu_iC\,dx\\&\le\Bigl(\int_\Omega u_{i,t}u_{i,t}\,dx\Bigr)^{\frac{1}{2}}\Bigl(\int_\Omega u_iu_i\,dx\Bigr)^{\frac{1}{2}}+\Bigl(\int_\Omega u_iu_i\,dx\Bigr)^{\frac{1}{2}}\Bigl(g^2\int_\Omega T^2\,dx\Bigr)^{\frac{1}{2}}\end{aligned}$$

$$+\left(\int_\Omega u_i u_i\,dx\right)^{\frac{1}{2}}\left(h^2\int_\Omega C^2\,dx\right)^{\frac{1}{2}}$$
$$\leq k_5^{\frac{1}{2}}\left[\left(\int_\Omega u_{i,t}u_{i,t}\,dx\right)^{\frac{1}{2}}+|\Omega|^{\frac{1}{2}}gT^M+|\Omega|^{\frac{1}{2}}hC^M\right]. \tag{3.50}$$

In order to have a bound for $\int_\Omega u_{i,j}u_{i,j}\,dx$, we need only give a bound for $\int_\Omega u_{i,t}u_{i,t}\,dx$. We can observe that

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_{i,t}u_{i,t}\,dx &= 2\int_\Omega u_{i,t}\big[-\lambda|u|u_i-p_{,i}+u_{i,jj}+g_iT-h_iC\big]_{,t}\,dx\\
&\leq -2\lambda\int_\Omega u_{i,t}|u|_{,t}u_i\,dx-2\int_\Omega u_{i,jt}u_{i,jt}\,dx+2\int_\Omega u_{i,t}(g_iT_{,t}-h_iC_{,t})\,dx\\
&\leq -2\lambda\int_\Omega u_{i,t}u_i\frac{u_ku_{k,t}}{|u|}\,dx+2\int_\Omega u_{i,t}u_{i,t}\,dx+g^2\int_\Omega T_{,t}T_{,t}\,dx\\
&\quad +h^2\int_\Omega C_{,t}C_{,t}\,dx\\
&\leq 2\int_\Omega u_{i,t}u_{i,t}\,dx+g^2\int_\Omega T_{,t}T_{,t}\,dx+h^2\int_\Omega C_{,t}C_{,t}\,dx.\end{aligned} \tag{3.51}$$

Hence we should give the bound for $\int_\Omega T_{,t}T_{,t}\,dx$ and $\int_\Omega C_{,t}C_{,t}\,dx$. We find that

$$\begin{aligned}\frac{d}{dt}\int_\Omega T_{,t}T_{,t}\,dx &= 2\int_\Omega T_{,t}(T_{,iit}-u_{i,t}T_{,i}-u_iT_{,it})\,dx=-2\int_\Omega T_{,it}T_{,it}\,dx+2\int_\Omega T_{,it}u_{i,t}T\,dx\\
&\leq \frac{(T^M)^2}{2}\int_\Omega u_{i,t}u_{i,t}\,dx.\end{aligned} \tag{3.52}$$

Similarly we can get

$$\begin{aligned}\frac{d}{dt}\int_\Omega C_{,t}C_{,t}\,dx &= 2\int_\Omega C_{,t}\big(C_{,iit}-u_{i,t}C_{,i}-u_iC_{,it}+L\big[f(T)\big]_{,t}-kC_{,t}\big)\,dx\\
&= -2\int_\Omega C_{,it}C_{,it}\,dx+2\int_\Omega C_{,it}u_{i,t}C\,dx\\
&\quad +2L\int_\Omega C_{,t}\big[f(T)\big]_{,t}-2k\int_\Omega C_{,t}C_{,t}\,dx\\
&\leq \frac{(C^M)^2}{2}\int_\Omega u_{i,t}u_{i,t}\,dx+\frac{L^2}{2k}\int_\Omega\big[f(T)\big]_{,t}\big[f(T)\big]_{,t}\,dx\\
&\leq \frac{(C^M)^2}{2}\int_\Omega u_{i,t}u_{i,t}\,dx+\frac{L^2k_1^2}{2k}\int_\Omega T_{,t}T_{,t}\,dx.\end{aligned} \tag{3.53}$$

We set

$$F_2=\int_\Omega u_{i,t}u_{i,t}\,dx+\int_\Omega T_{,t}T_{,t}\,dx+\int_\Omega C_{,t}C_{,t}\,dx.$$

Combining (3.51)-(3.53), we can get

$$\frac{dF_2}{dt}\leq M_2F_2, \tag{3.54}$$

where $M_2=\max\{2+\frac{(C^M)^2}{2}+\frac{(T^M)^2}{2},\frac{L^2k_1^2}{2k}+g^2,h^2\}$.

Hence, from our assumption, we can get

$$F_2 \leq \left(\int_\Omega u_{i,t}(x,0)u_{i,t}(x,0)\,dx + \int_\Omega T_{i,t}(x,0)T_{i,t}(x,0)\,dx + \int_\Omega C_{i,t}(x,0)C_{i,t}(x,0)\,dx\right)e^{M_2 t}$$
$$\leq Re^{M_2\tau}. \tag{3.55}$$

So we can draw the conclusion that

$$\int_\Omega u_{i,j}u_{i,j}\,dx \leq k_5^{\frac{1}{2}}\left(Re^{M_2\tau} + |\Omega|^{\frac{1}{2}}T^M g + |\Omega|^{\frac{1}{2}}C^M h\right) = k_6. \tag{3.56}$$

Using the inequalities (3.16), we multiply $(3.39)_1$ by ω_i and integrate over Ω to find

$$\begin{aligned}\frac{d}{dt}\|\omega\|^2 &= -2\lambda\int_\Omega |u|u_i\omega_i\,dx - 2\|\nabla\omega\|^2\\ &\quad + 2\int_\Omega g_i\theta\omega_i\,dx - 2\int_\Omega h_i S\omega_i\,dx\\ &\leq \lambda^2\int_\Omega |u|^4\,dx + 3\|\omega\|^2 + g^2\|\theta\|^2 + h^2\|S\|^2.\end{aligned} \tag{3.57}$$

From (3.24), (3.26), and (3.57), we get

$$\begin{aligned}\frac{d}{dt}\left(\|\omega\|^2 + \|\theta\|^2 + \|S\|^2\right) &\leq \lambda^2\int_\Omega |u|^4\,dx + \left(3 + \frac{(T^M)^2}{2} + \frac{(C^M)^2}{2}\right)\|\omega\|^2\\ &\quad + \left(L^2k_1^2 + h^2\right)\|S\|^2 + \left(1 + g^2\right)\|\theta\|^2.\end{aligned} \tag{3.58}$$

We set

$$F_3 = \|\omega\|^2 + \|\theta\|^2 + \|S\|^2.$$

Then we have

$$\frac{dF_3}{dt} \leq \lambda^2 k_2^2 k_5^{\frac{1}{2}} k_6^{\frac{3}{2}} + M_3 F_3, \tag{3.59}$$

where $M_3 = \max\{3 + \frac{(T^M)^2}{2} + \frac{(C^M)^2}{2}, L^2k_1^2 + h^2, 1 + g^2\}$.

We easily see that

$$\|\omega\|^2 + \|\theta\|^2 + \|S\|^2 \leq \lambda^2 t e^{-M_3 t} k_2^2 k_5^{\frac{1}{2}} k_6^{\frac{3}{2}}. \tag{3.60}$$

Inequality (3.60) demonstrates the convergence of u_i to u_i^*, T to T^*, and C to C^* as $\lambda_1 \to \lambda_2$ in the indicated measure. □

4 The case for the Forchheimer equations

The above equations we discussed are of the Brinkman-Forchheimer equations type. If we consider the Forchheimer equations if Δu is deleted, we will demonstrate another theorem. Now, let (u_i, p, T, C) be a solution to the boundary initial-value problem for the

Forchheimer model,

$$\begin{cases} \frac{\partial u_i}{\partial t} + \lambda |u| u_i = -p_{,i} + g_i T - h_i C, & (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial u_i}{\partial x_i} = 0, & (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} = \Delta T, & (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial C}{\partial t} + u_i \frac{\partial C}{\partial x_i} = \Delta C + Lf(T) - kC, & (x,t) \in \Omega \times [0,\tau], \end{cases} \tag{4.1}$$

$$u_i = 0, \qquad \frac{\partial T}{\partial n} = 0, \qquad \frac{\partial C}{\partial n} = 0, \quad (x,t) \in \partial\Omega \times [0,\tau], \tag{4.2}$$

$$u_i(x,0) = u_{i0}(x), \tag{4.3}$$

$$T(x,0) = T_0(x), \qquad C(x,0) = C_0(x), \quad x \in \Omega. \tag{4.4}$$

Furthermore, let (u_i^*, p^*, T^*, C^*) be a solution to the following boundary initial-value problem:

$$\begin{cases} \frac{\partial u_i^*}{\partial t} = -p_{,i}^* + g_i T^* - h_i C^*, & (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial u_i^*}{\partial x_i} = 0, & (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial T^*}{\partial t} + u_i^* \frac{\partial T^*}{\partial x_i} = \Delta T^*, & (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial C^*}{\partial t} + u_i^* \frac{\partial C^*}{\partial x_i} = \Delta C^* + Lf(T^*) - kC^*, & (x,t) \in \Omega \times [0,\tau], \end{cases} \tag{4.5}$$

$$u_i^* = 0, \qquad \frac{\partial T^*}{\partial n} = 0, \qquad \frac{\partial C^*}{\partial n} = 0, \quad (x,t) \in \partial\Omega \times [0,\tau], \tag{4.6}$$

$$u_i^*(x,0) = u_{i0}(x), \tag{4.7}$$

$$T^*(x,0) = T_0(x), \qquad C^*(x,0) = C_0(x), \quad x \in \Omega. \tag{4.8}$$

In this section, we establish convergence on the coefficient λ. To do this, let (u_i, T, C, P) and (u_i^*, T^*, C^*, P^*) be solutions of (4.1) and (4.5) with the same boundary and initial conditions. Now we define

$$\omega_i = u_i - u_i^*, \qquad \pi = p - p^*, \qquad \theta = T - T^*, \qquad S = C - C^*. \tag{4.9}$$

Then $(\omega_i, \theta, S, \pi)$ is a solution of the problem

$$\begin{cases} \frac{\partial \omega_i}{\partial t} + \lambda |u| u_i = -\pi_{,i} + g_i \theta - h_i S, & (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial \omega_i}{\partial x_i} = 0, & (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial \theta}{\partial t} + \omega_i \frac{\partial T}{\partial x_i} + u_i^* \frac{\partial \theta}{\partial x_i} = \Delta \theta, & (x,t) \in \Omega \times [0,\tau], \\ \frac{\partial S}{\partial t} + \omega_i \frac{\partial C}{\partial x_i} + u_i^* S_{,i} = \Delta S + L(f(T) - f(T^*)) - kS, & (x,t) \in \Omega \times [0,\tau], \end{cases} \tag{4.10}$$

in $\Omega \times [0,\tau]$, subject to the boundary and initial conditions

$$\omega_i = 0, \qquad \frac{\partial \theta}{\partial n} = 0, \qquad \frac{\partial S}{\partial n} = 0, \quad (x,t) \in \partial\Omega \times [0,\tau], \tag{4.11}$$

$$\omega_i(x,0) = 0, \tag{4.12}$$

$$\theta(x,0) = 0, \qquad S(x,0) = 0, \quad x \in \Omega. \tag{4.13}$$

We will obtain the following result.

Theorem 3 *Let (u_i, T, C, p) be the classical solution to the initial-boundary problem* (4.1)-(4.4), *(u_i^*, T^*, C^*, p^*) be the classical solution to the initial-boundary problem* (4.5)-(4.8) *in $\Omega \times (0,\tau)$, and (w_i, θ, S, π) be the difference of (u_i, T, C, p) and (u_i^*, T^*, C^*, p^*), then the solution (u_i, T, C, p) converges to the solution (u_i^*, T^*, C^*, p^*) as the Forchheimer coefficient λ tends to* 0. *The difference (w_i, θ, S, π) satisfies*

$$\|\omega\|^2 + \|\theta\|^2 + \|S\|^2 \le \lambda^2 t e^{-M_3 t} k_2^2 k_5^{\frac{1}{2}} k_7^{\frac{3}{2}}. \tag{4.14}$$

We can also get the continuous dependence result for different Forchheimer coefficients $\lambda \to 0$.

Proof First of all, we may calculate $\int_0^t \int_\Omega C_{,i} C_{,i}\,dx\,d\eta$ and $\int_0^t \int_\Omega T_{,i} T_{,i}\,dx\,d\eta$,

$$\begin{aligned}\frac{\partial}{\partial t}\int_\Omega C^2\,dx &= 2\int_\Omega C\big(\Delta C + Lf(T) - kC - u_i C_{,i}\big)\,dx \\ &= -2\int_\Omega C_{,i}C_{,i}\,dx + 2L\int_\Omega Cf(T)\,dx - 2k\int_\Omega C^2\,dx \\ &\le -2\int_\Omega C_{,i}C_{,i}\,dx + \frac{L^2 d^2}{2k}|\Omega|.\end{aligned} \tag{4.15}$$

Integrating (4.15) over Ω, we find

$$2\int_0^t \int_\Omega C_{,i}C_{,i}\,dx\,d\eta \le \int_\Omega C_0^2\,dx + \frac{1}{2k}L^2 d^2 \tau |\Omega|. \tag{4.16}$$

Similarly, we can obtain

$$2\int_0^t \int_\Omega T_{,i}T_{,i}\,dx\,d\eta \le \int_\Omega T_0^2\,dx. \tag{4.17}$$

Then we want to give a bound for $\|\nabla u\|^2$. We know that

$$\int_\Omega u_{i,j}u_{i,j}\,dx = \int_\Omega u_{i,j}(u_{i,j} - u_{j,i})\,dx + \int_\Omega u_{i,j}u_{j,i}\,dx. \tag{4.18}$$

After some integration by parts, it implies

$$\begin{aligned}\int_\Omega u_{i,j}u_{j,i}\,dx &= \oint_{\partial\Omega} u_{i,j}u_j n_i\,ds - \int_\Omega u_{i,ij}u_j\,dx \\ &= -\oint_{\partial\Omega} u_{i,i}u_j n_j\,ds + \int_\Omega u_{i,i}u_{j,j}\,dx = 0.\end{aligned} \tag{4.19}$$

We set

$$J(t) = \int_\Omega u_{i,j}(u_{i,j} - u_{j,i})\,dx. \tag{4.20}$$

We note that

$$k_g^2 = \max_\Omega g_{i,j}g_{i,j}, \qquad k_h^2 = \max_\Omega h_{i,j}h_{i,j}. \tag{4.21}$$

We can obtain

$$\begin{aligned}\frac{dJ}{dt} &= 2\int_\Omega u_{i,jt}u_{i,j}\,dx - \int_\Omega u_{i,jt}u_{j,i}\,dx - \int_\Omega u_{j,it}u_{i,j}\,dx \\ &= 2\int_\Omega u_{i,jt}(u_{i,j}-u_{j,i})\,dx \\ &= 2\int_\Omega (u_{i,j}-u_{j,i})\big[-\lambda\big(|u|u_i\big)_{,j} - p_{i,j} + (g_iT)_{,j} - (h_iC)_{,j}\big]\,dx \\ &= -2\lambda\int_\Omega |u|u_{i,j}u_{i,j}\,dx - 2\lambda\int_\Omega u_{i,j}u_i\frac{u_ku_{k,j}}{|u|}\,dx + 2\lambda\oint_{\partial\Omega} u_{j,i}|u|u_in_j\,ds \\ &\quad + 2\int_\Omega (u_{i,j}-u_{j,i})(g_{i,j}T - h_{i,j}C)\,dx + 2\int_\Omega (u_{i,j}-u_{j,i})(g_iT_{,j} - h_iC_{,j})\,dx \\ &\le 2\int_\Omega (u_{i,j}-u_{j,i})(u_{i,j}-u_{j,i})\,dx + 2k_m^2\int_\Omega \big(T^2+C^2\big)\,dx \\ &\quad + 2k_n^2\int_\Omega (T_{,j}T_{,j} + C_{,j}C_{,j})\,dx, \end{aligned} \tag{4.22}$$

where $k_m^2 = \max\{k_g^2, k_h^2\}$, $k_n^2 = \max\{g^2, h^2\}$.

We know

$$\int_\Omega (u_{i,j}-u_{j,i})(u_{i,j}-u_{j,i})\,dx = 2\int_\Omega (u_{i,j}-u_{j,i})u_{i,j}\,dx = 2J(t). \tag{4.23}$$

From (4.22) and (4.23), we can get

$$\frac{dJ}{dt} \le 4J + 2k_m^2|\Omega|\big[\big(T^M\big)^2 + \big(C^M\big)^2\big] + 2k_n^2\int_\Omega (T_{,j}T_{,j} + C_{,j}C_{,j})\,dx. \tag{4.24}$$

Combining (4.16), (4.17), and (4.24), we have

$$\begin{aligned}\|\nabla u\|^2 &\le 2k_m^2e^{4\tau}|\Omega|\tau\big[\big(T^M\big)^2 + \big(C^M\big)^2\big] \\ &\quad + e^{4\tau}k_n^2\bigg(\int_\Omega T_0^2 + C_0^2\,dx + \frac{1}{2k}L^2d^2\tau|\Omega|\bigg) \\ &= k_7. \end{aligned} \tag{4.25}$$

Similarly we can obtain

$$\|u\|_4^4 \le k_2^2k_5^{\frac{1}{2}}k_7^{\frac{3}{2}}. \tag{4.26}$$

We can use a similar method to get the result that

$$\|\omega\|^2 + \|\theta\|^2 + \|S\|^2 \le \lambda^2te^{-M_3t}k_2^2k_5^{\frac{1}{2}}k_7^{\frac{3}{2}}. \tag{4.27}$$

In inequalities (4.27) we demonstrate the convergence of u_i to u_i^*, T to T^*, and C to C^* as $\lambda \to 0$ in the indicated measure. We can also get the continuous dependence result. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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