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New strong convergence theorems for split variational inclusion problems in Hilbert spaces

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Abstract

The split variational inclusion problem is an important problem, and it is a generalization of the split feasibility problem. In this paper, we present a descent-conjugate gradient algorithm for the split variational inclusion problems in Hilbert spaces. Next, a strong convergence theorem of the proposed algorithm is proved under suitable conditions. As an application, we give a new strong convergence theorem for the split feasibility problem in Hilbert spaces. Finally, we give numerical results for split variational inclusion problems to demonstrate the efficiency of the proposed algorithm.

Keywords: split variational inclusion problem; maximal monotone mapping; split feasibility problem; resolvent mapping; conjugate gradient method

1 Introduction

Let H be a real Hilbert space, and $B : H \multimap H$ be a set-valued mapping with domain $\mathcal{D}(B) := \{x \in H : B(x) \neq \emptyset\}$. Recall that B is called monotone if $\langle u - v, x - y\rangle \geq 0$ for any $u \in Bx$ and $v \in By$; B is maximal monotone if its graph $\{(x, y) : x \in \mathcal{D}(B), y \in Bx\}$ is not properly contained in the graph of any other monotone mapping. An important problem for set-valued monotone mappings is to find $\bar{x} \in H$ such that $0 \in B\bar{x}$. Here, $\bar{x}$ is called a zero point of B. A well-known method for approximating a zero point of a maximal monotone mapping defined in a real Hilbert space is the proximal point algorithm first introduced by Martinet [1] and generated by Rockafellar [2]. This is an iterative procedure which generates $\{x_n\}$ by $x_1 = x \in H$ and

$$x_{n+1} = J^B_{\beta_n} x_n, \quad n \in \mathbb{N}, \tag{1.1}$$

where $\{\beta_n\} \subseteq (0, \infty)$, B is a maximal monotone mapping in a real Hilbert space, and J^B_r is the resolvent mapping of B defined by $J^B_r = (I + rB)^{-1}$ for each $r > 0$. In 1976, Rockafellar [2] proved the following in the Hilbert space setting: If the solution set $B^{-1}(0)$ is nonempty and $\liminf_{n\to\infty} \beta_n > 0$, then the sequence $\{x_n\}$ in (1.1) converges weakly to an element of $B^{-1}(0)$. In particular, if B is the subdifferential ∂f of a proper lower semicontinuous and convex function $f : H \to \mathbb{R}$, then (1.1) is reduced to

$$x_{n+1} = \arg\min_{y\in H}\left\{f(y) + \frac{1}{2\beta_n}\|y - x_n\|^2\right\}, \quad n \in \mathbb{N}. \tag{1.2}$$

In this case, $\{x_n\}$ converges weakly to a minimizer of f.

Let H_1 and H_2 be two real Hilbert spaces, $B_1 : H_1 \multimap H_1$ and $B_2 : H_2 \multimap H_2$ be two set-valued maximal monotone mappings, $A : H_1 \to H_2$ be a linear and bounded operator, and A^* be the adjoint of A. Let $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be two proper lower semicontinuous, and convex functions. In 2011, Moudafi [3] presented the following general split variational inclusion problem:

$$\text{Find } \bar{x} \in H_1 \text{ such that } 0 \in f(\bar{x}) + B_1(\bar{x}) \text{ and } 0 \in g(A\bar{x}) + B_2(A\bar{x}). \tag{GSFVIP}$$

Clearly, we know that split variational inclusion problem (SFVIP) is a generalization of variational inclusion problems and a generalization of split feasibility problem. Hence, it is important to study the split variational inclusion problems in Hilbert spaces.

For problem (GSFVIP), Moudafi [3] gave the following algorithm and a weak convergence theorem under suitable conditions:

$$x_{n+1} := J_\lambda^{B_1}(I - \lambda f)\bigl(x_n + \gamma A^*\bigl(J_\lambda^{B_2}(I - \lambda g) - I\bigr)Ax_n\bigr).$$

It is worth noting that λ and γ are fixed numbers. Hence, it is important to establish generalized iteration processes and strong convergence theorems for problem (SFVIP).

In this paper, we consider the following split variational inclusion problems in Hilbert spaces:

$$\text{Find } \bar{x} \in H_1 \text{ such that } 0 \in B_1(\bar{x}) \text{ and } 0 \in B_2(A\bar{x}). \tag{SFVIP}$$

In 2011, Byrne *et al.* [4] gave the following two convergence theorems for split variational inclusion problems.

First, from the idea of the algorithms for fixed point theorem, the algorithm given in Theorem 1.1 can be seen as a Picard iteration method.

Theorem 1.1 [4] *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a linear and bounded operator, and let A^* denote the adjoint of A. Let $B_1 : H_1 \multimap H_1$ and $B_2 : H_2 \multimap H_2$ be two set-valued maximal monotone mappings. Let $\beta > 0$ and $\rho \in (0, \frac{2}{\|A\|^2})$. Let Ω be the solution set of* (SFVIP) *and suppose that $\Omega \neq \emptyset$. Let $\{x_n\}$ be defined by*

$$x_{n+1} := J_\beta^{B_1}\bigl[x_n - \rho A^*\bigl(I - J_\beta^{B_2}\bigr)Ax_n\bigr]$$

for each $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to an element $\bar{x} \in \Omega$.

Next, from the idea of the algorithms for fixed point theorem, the algorithm given in Theorem 1.2 can be seen as Halpern's iteration method.

Theorem 1.2 [4] *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a linear and bounded operator, and let A^* denote the adjoint of A. Let $B_1 : H_1 \multimap H_1$ and $B_2 : H_2 \multimap H_2$*

be two set-valued maximal monotone mappings. Let $\{a_n\}$ *be a sequence of real numbers in* $[0,1]$ *and let* $\beta > 0$. *Let* $u \in H$ *be fixed and let* $\rho \in (0, \frac{2}{\|A\|^2})$. *Let* Ω *be the solution set of* (SFVIP) *and suppose that* $\Omega \neq \emptyset$. *Let* $\{x_n\}$ *be defined by*

$$x_{n+1} := a_n u + (1-a_n) J_\beta^{B_1}\bigl[x_n - \rho A^*\bigl(I - J_\beta^{B_2}\bigr)Ax_n\bigr]$$

for each $n \in \mathbb{N}$. *Assume that* $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^\infty a_n = \infty$. *Then* $\lim_{n\to\infty} x_n = \bar{x}$ *for some* $\bar{x} \in \Omega$.

Remark 1.1 In Theorems 1.1 and 1.2, we know that β and ρ are fixed numbers.

In 2013, Chuang [5] gave the following two convergent theorems for problem (SFVIP). Indeed, from the idea of the algorithms for fixed point theorem, the algorithm given in Theorem 1.3 can be seen as Halpern-Mann type iteration method.

Theorem 1.3 [5] *Let* H_1 *and* H_2 *be two real Hilbert spaces,* $A : H_1 \to H_2$ *be a linear and bounded operator, and let* A^* *denote the adjoint of* A. *Let* $B_1 : H_1 \multimap H_1$ *and* $B_2 : H_2 \multimap H_2$ *be two set-valued maximal monotone mappings. Let* $\{a_n\}$, $\{b_n\}$, *and* $\{c_n\}$ *be sequences of real numbers in* $[0,1]$ *with* $a_n + b_n + c_n = 1$ *and* $0 < a_n < 1$ *for each* $n \in \mathbb{N}$. *Let* $\{\beta_n\}$ *be a sequence in* $(0,\infty)$. *Let* $u \in H$ *be fixed. Let* $\{\rho_n\}$ *be a sequence in* $(0, \frac{2}{\|A\|^2+1})$. *Let* Ω *be the solution set of* (SFVIP) *and suppose that* $\Omega \neq \emptyset$. *Let* $\{x_n\}$ *be defined by*

$$x_{n+1} := a_n u + b_n x_n + c_n J_{\beta_n}^{B_1}\bigl[x_n - \rho_n A^*\bigl(I - J_{\beta_n}^{B_2}\bigr)Ax_n\bigr]$$

for each $n \in \mathbb{N}$. *Assume that* $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^\infty a_n = \infty$, $\liminf_{n\to\infty} c_n\rho_n > 0$, $\liminf_{n\to\infty} b_n c_n > 0$, *and* $\liminf_{n\to\infty} \beta_n > 0$. *Then* $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_\Omega u$.

Besides, the algorithm in Theorem 1.4 comes from the optimization theorem and Tikhonov regularization method.

Theorem 1.4 [5] *Let* H_1 *and* H_2 *be two real Hilbert spaces,* $A : H_1 \to H_2$ *be a linear and bounded operator, and let* A^* *denote the adjoint of* A. *Let* $B_1 : H_1 \multimap H_1$ *and* $B_2 : H_2 \multimap H_2$ *be two set-valued maximal monotone mappings. Let* $\{\beta_n\}$ *be a sequence in* $(0,\infty)$, $\{a_n\}$ *be a sequence in* $(0,1)$, *and* $\{\rho_n\}$ *be a sequence in* $(0, 2/(\|A\|^2+2))$. *Let* Ω *be the solution set of* (SFVIP) *and suppose that* $\Omega \neq \emptyset$. *Let* $\{x_n\}$ *be defined by*

$$x_{n+1} := J_{\beta_n}^{B_1}\bigl[(1-a_n\rho_n)x_n - \rho_n A^*\bigl(I - J_{\beta_n}^{B_2}\bigr)Ax_n\bigr]$$

for each $n \in \mathbb{N}$. *Assume that* $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^\infty a_n\rho_n = \infty$, $\liminf_{n\to\infty} \rho_n > 0$ *and* $\liminf_{n\to\infty} \beta_n > 0$. *Then* $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_\Omega 0$, *i.e.,* $\bar{x}$ *is the minimal norm solution of* (SFVIP).

Further, we also observed that Bnouhachem *et al.* [6] proposed the following descent-projection algorithm to study the split feasibility problem.

Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a bounded linear operator, and A^* be the adjoint of A. Let $f : \mathbb{R}^n \to (-\infty,\infty]$ and $g : \mathbb{R}^m \to (-\infty,\infty]$ be two proper, lower semicontinuous, and convex functions. Let $\{\rho_k\}$ be a sequence of positive real numbers.

Algorithm 1.1 For given $x_k \in \mathbb{R}^n$, find the approximate solution by the following iterative process.

Step 1. For $k \in \mathbb{N}$, let C_k and Q_k be

$$\begin{cases} C_k := \{u \in \mathbb{R}^n : f(x_k) + \langle u_k, u - x_k\rangle \leq 0\}, \\ Q_k := \{v \in \mathbb{R}^m : g(Ax_k) + \langle v_k, v - Ax_k\rangle \leq 0\}, \end{cases}$$

where $u_k \in \partial f(x_k)$ and $v_k \in \partial g(Ax_k)$.

Step 2. $y_k = P_{C_k}[x_k - \rho_k A^\top(I - P_{Q_k})Ax_k]$, where $\rho_k > 0$ satisfies

$$\rho_k \| A^\top(I - P_{Q_k})Ax_n - A^\top(I - P_{Q_k})Ay_k \| \leq \delta \|x_k - y_k\|, \quad 0 < \delta < 1.$$

Step 3. If $y_k = x_k$, then stop. Otherwise, go to Step 4.

Step 4. The new iterative x_{k+1} is defined by

$$x_{k+1} = P_{C_k}\bigl[x_k - \alpha_k d(x_k, \rho_k)\bigr],$$

where

$$\begin{cases} d(x_k, \rho_k) := x_k - y_k + \rho_k A^\top(I - P_{Q_k})Ay_k, \\ \varepsilon_k := \rho_k[A^\top(I - P_{Q_k})Ay_k - A^\top(I - P_{Q_k})Ax_k], \\ D(x_k, \rho_k) := x_k - y_k + \varepsilon_k, \\ \phi(x_k, \rho_k) := \langle x_k - y_k, D(x_k, \rho_k)\rangle, \\ \alpha_k := \frac{\phi(x_k, \rho_k)}{\|d(x_k, \rho_k)\|^2}. \end{cases}$$

Let H_1 and H_2 be infinite dimensional Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator, and A^* be the adjoint of A. Let $B_1 : H_1 \multimap H_1$ and $B_2 : H_2 \multimap H_2$ be set-valued maximal monotone mappings. Let $\{a_n\}$, $\{\eta_n\}$, $\{\gamma_n\}$, and $\{\rho_n\}$ be real sequences. Let δ be a fixed real numbers. Let Ω be the solution set of problem (SFVIP). In this paper, motivated by the above works and related results, we present the following algorithm with conjugate gradient method for the split variational inclusion problems in Hilbert spaces.

Motivated by Algorithm 1.1 and the above results, we want to give a strong convergence theorem in infinite dimensional real Hilbert spaces. (Indeed, for computers and program language, we can only give examples for a finite dimensional space.) Next, we want that the convergent rate of the given algorithm are faster than the above algorithms. Hence, we give the following algorithm with conjugate method. In our numerical results, we know that this algorithm is very fast under some conditions.

Algorithm 1.2

Step 0. Choose $x_1 \in H_1$ arbitrarily, set $r_1 \in (0,1)$ and $d_0 = 0$.

Step 1. $d_n := -A^*(I - J_{\beta_n}^{B_2})Ax_n + \eta_n d_{n-1}$.

Step 2. For $n \in \mathbb{N}$, set y_n as

$$y_n = J_{\beta_n}^{B_1}\bigl[(1 - a_n\rho_n)x_n - \rho_n A^*\bigl(I - J_{\beta_n}^{B_2}\bigr)Ax_n + \gamma_n d_n\bigr], \tag{1.3}$$

where $\rho_n > 0$ satisfies

$$\rho_n \bigl\| A^*\bigl(I - J_{\beta_n}^{B_2}\bigr)Ax_n - A^*\bigl(I - J_{\beta_n}^{B_2}\bigr)Ay_n \bigr\| \leq \delta \|x_n - y_n\|, \quad 0 < \delta < 1. \tag{1.4}$$

Step 3. If $x_n = y_n$, then set $n := n + 1$ and go to Step 1. Otherwise, go to Step 3.
Step 4. The new iterative x_{n+1} is defined by

$$x_{n+1} = J_{\beta_n}^{B_1}\big[x_n - \alpha_n D(x_n, \rho_n)\big], \tag{1.5}$$

where

$$D(x_n, \rho_n) := x_n - y_n + \rho_n\big[A^*\big(I - J_{\beta_n}^{B_2}\big)Ay_n - A^*\big(I - J_{\beta_n}^{B_2}\big)Ax_n\big], \tag{1.6}$$

$$\alpha_n := \frac{\langle x_n - y_n, D(x_n, \rho_n)\rangle}{\|D(x_n, \rho_n)\|^2}. \tag{1.7}$$

Then update $n := n + 1$ and go to Step 1.

Remark 1.2

(1) It is worth noting that d_n is defined by using the idea of the so-called conjugate gradient direction ([7], Chapter 5). Further, it is natural to assume that $\{x_n\}$ is a bounded sequence for the convergence theorems with the conjugate gradient direction method.

(2) If we set

$$\varepsilon_n := \rho_n\big[A^*\big(I - J_{\beta_n}^{B_2}\big)Ay_n - A^*\big(I - J_{\beta_n}^{B_2}\big)Ax_n\big], \tag{1.8}$$

then it follows from (1.4) and (1.8) that

$$\big|\langle x_n - y_n, \varepsilon_n\rangle\big| \le \|x_n - y_n\| \cdot \|\varepsilon_n\| \le \delta\|x_n - y_n\| \cdot \|x_n - y_n\| = \delta\|x_n - y_n\|^2. \tag{1.9}$$

(3) If we choose ρ_n such that $0 < \rho_n \le \frac{\delta}{\|A\|\cdot\|A^*\|} = \frac{\delta}{\|A\|^2}$, then (1.4) holds.

(4) In our convergence theorem, we may assume that $x_n \neq y_n$ for each $n \in \mathbb{N}$ by the assumptions on the sequence $\{a_n\}$.

Next, a strong convergence theorem of the proposed algorithm is proved under suitable conditions. As an application, we give a descent-projection-conjugate gradient algorithm and a strong convergence theorem for the split feasibility problem. Finally, we give numerical results to demonstrate the efficiency of the proposed algorithm.

2 Preliminaries

Let H be a (real) Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. We denote the strongly convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. From [8, 9], for each $x, y, u, v \in H$ and $\lambda \in [0,1]$, we have

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle; \tag{2.1}$$

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2; \tag{2.2}$$

$$2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.3}$$

Let C be a nonempty, closed, and convex subset of a real Hilbert space H, and let $T : C \to H$ be a mapping. Let $\mathrm{Fix}(T) := \{x \in C : Tx = x\}$. Then T is said to be a nonexpansive mapping if $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$. T is said to be a quasi-nonexpansive mapping if $\mathrm{Fix}(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for every $x \in C$ and $y \in \mathrm{Fix}(T)$. It is easy to see that $\mathrm{Fix}(T)$ is a closed convex subset of C if T is a quasi-nonexpansive mapping. Besides, T is said to be a firmly nonexpansive mapping if $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle$ for every $x, y \in C$, that is, $\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2$ for every $x, y \in C$.

Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Then for each $x \in H$, there is a unique element $\bar{x} \in C$ such that

$$\|x - \bar{x}\| = \min_{y \in C} \|x - y\|.$$

Here, set $P_C x = \bar{x}$, and P_C is called the metric projection from H onto C.

Lemma 2.1 [8] *Let C be a nonempty, closed, and convex subset of a real Hilbert space H, and let P_C be the metric projection from H onto C. Then $\langle x - P_C x, P_C x - y\rangle \geq 0$ for all $x \in H$ and $y \in C$.*

Lemma 2.2 *Let H be a real Hilbert space. Let $B : H \multimap H$ be a set-valued maximal monotone mapping, $\beta > 0$, and let J_β^B be defined by $J_\beta^B := (I + \beta B)^{-1}$ (J_β^B is called resolvent mapping). Then the following are satisfied:*

(i) *for each $\beta > 0$, J_β^B is a single-valued and firmly nonexpansive mapping;*
(ii) $\mathcal{D}(J_\beta^B) = H$ *and* $\mathrm{Fix}(J_\beta^B) = \{x \in \mathcal{D}(B) : 0 \in Bx\}$;
(iii) $\|x - J_\beta^B x\| \leq \|x - J_\gamma^B x\|$ *for all $0 < \beta \leq \gamma$ and for all $x \in H$;*
(iv) $(I - J_\beta^B)$ *is a firmly nonexpansive mapping for each $\beta > 0$;*
(v) *suppose that $B^{-1}(0) \neq \emptyset$, then $\|x - J_\beta^B x\|^2 + \|J_\beta^B x - \bar{x}\|^2 \leq \|x - \bar{x}\|^2$ for each $x \in H$, each $\bar{x} \in B^{-1}(0)$, and each $\beta > 0$;*
(vi) *suppose that $B^{-1}(0) \neq \emptyset$, then $\langle x - J_\beta^B x, J_\beta^B x - w\rangle \geq 0$ for each $x \in H$ and each $w \in B^{-1}(0)$, and each $\beta > 0$.*

Lemma 2.3 *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a linear operator, and A^* be the adjoint of A, and let $\beta > 0$ be fixed. Let $B : H_2 \multimap H_2$ be a set-valued maximal monotone mapping, and let J_β^B be a resolvent mapping of B. Let $T : H_1 \to H_1$ be defined by $Tx := A^*(I - J_\beta^B)Ax$ for each $x \in H_1$. Then*

(i) $\|(I - J_\beta^B)Ax - (I - J_\beta^B)Ay\|^2 \leq \langle Tx - Ty, x - y\rangle$ *for all $x, y \in H_1$;*
(ii) $\|A^*(I - J_\beta^B)Ax - A^*(I - J_\beta^B)Ay\|^2 \leq \|A\|^2 \cdot \langle Tx - Ty, x - y\rangle$ *for all $x, y \in H_1$.*

Lemma 2.4 *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a linear operator, and A^* be the adjoint of A, and let $\beta > 0$ be fixed, and let $\rho \in (0, \frac{2}{\|A\|^2})$. Let $B_2 : H_2 \multimap H_2$ be a set-valued maximal monotone mapping, and let $J_\beta^{B_2}$ be a resolvent mapping of B_2. Then*

$$\begin{aligned} &\bigl\|\bigl[x - \rho A^*\bigl(I - J_\beta^{B_2}\bigr)Ax\bigr] - \bigl[y - \rho A^*\bigl(I - J_\beta^{B_2}\bigr)Ay\bigr]\bigr\|^2 \\ &\quad \leq \|x - y\|^2 - \bigl(2\rho - \rho^2\|A\|^2\bigr)\bigl\|\bigl(I - J_\beta^{B_2}\bigr)Ax - \bigl(I - J_\beta^{B_2}\bigr)Ay\bigr\|^2 \end{aligned}$$

for all $x, y \in H_1$. Furthermore, $I - \rho A^*(I - J_\beta^{B_2})A$ is a nonexpansive mapping.

Lemma 2.5 [10] *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : C \to H$ be a nonexpansive mapping, and let $\{x_n\}$ be a sequence in C. If $x_n \rightharpoonup w$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $Tw = w$.*

Lemma 2.6 [11] *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subseteq \mathbb{N}$ such that $m_k \to \infty$, $a_{m_k} \leq a_{m_k+1}$, and $a_k \leq a_{m_k+1}$ are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$. In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.*

Lemma 2.7 [12] *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence of real numbers in $[0,1]$ with $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\{u_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} u_n < \infty$, $\{t_n\}$ a sequence of real numbers with $\limsup t_n \leq 0$. Suppose that $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n t_n + u_n$ for each $n \in \mathbb{N}$. Then $\lim_{n\to\infty} a_n = 0$.*

3 Strong convergence theorems for (SFVIP)

In Remark 1.2, we have said that it is natural to assume that $\{x_n\}$ is a bounded sequence in the following result. For example, ([13], Theorem 3.1) use the assumption: $\{\nabla f_2(z_n)\}$ is a bounded sequence; ([14], Assumption 3.2, Theorem 3.1) use the assumption: $\{y_n^i\}_{n\in\mathbb{N}}$ is a bounded sequence; ([15], Assumption 2, Proposition 2.7) use the assumption: there exists a positive number M_3 such that $\|\nabla f_\ell(x)\| \leq M_3$ for each $x \in \mathbb{R}^p$ and each $\ell = 1, 2, \ldots, L$. Here, we need a similar assumption for our algorithm and convergence theorem in this paper.

Theorem 3.1 *Let H_1 and H_2 be infinite dimensional Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator, and A^* be the adjoint of A. Let $B_1 : H_1 \multimap H_1$ and $B_2 : H_2 \multimap H_2$ be set-valued maximal monotone mappings. Let $\{a_n\}$, $\{\eta_n\}$, $\{\gamma_n\}$ be sequences in $[0,1]$. Choose $\delta \in (0, 1/2)$, and let $\{\rho_n\}$ be a sequence in $(0, \min\{\frac{\delta}{\|A\|^2}, \frac{2}{\|A\|^2+2}\})$. Let Ω be the solution set of problem (SFVIP) and assume that $\Omega \neq \emptyset$. For the sequence $\{x_n\}$ in Algorithm 1.2, and we further assume that:*

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \eta_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$, $\liminf_{n\to\infty} \rho_n > 0$, *and* $\liminf_{n\to\infty} \beta_n > 0$;

(ii) $\lim_{n\to\infty} \frac{\gamma_n}{a_n} = t$ *for some* $t \geq 0$, *and* $\{x_n\}$ *is a bounded sequence.*

Then $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} := P_\Omega 0$.

Proof Clearly, Ω is a closed and convex subset of H_1. Let $\bar{x} = P_\Omega 0$. Since $\liminf_{n\to\infty} \rho_n > 0$, we may assume that $\rho_n \geq \rho$ for some $\rho > 0$. Without loss of generality, we may assume that $x_n \neq y_n$ for each $n \in \mathbb{N}$. Take any $w \in \Omega$ and let w be fixed. Take any $n \in \mathbb{N}$, and let n be fixed. Let $\bar{x} = P_\Omega 0$. Since $w \in \Omega$, we know that $Aw \in B_2^{-1}(0)$. By Lemma 2.2(ii), we know that

$$A^*\big(I - J_{\beta_n}^{B_2}\big)Aw = A^*\big(Aw - J_{\beta_n}^{B_2}Aw\big) = A^*(Aw - Aw) = 0. \tag{3.1}$$

By Lemma 2.2(v) and (1.5),

$$\|x_{n+1} - w\|^2 + \big\|x_{n+1} - x_n + \alpha_n D(x_n, \rho_n)\big\|^2 \leq \big\|x_n - \alpha_n D(x_n, \rho_n) - w\big\|^2. \tag{3.2}$$

By (3.2),

$$\begin{aligned}
&\|x_n - w\|^2 - \|x_{n+1} - w\|^2 \\
&\quad \geq \|x_n - w\|^2 - \big\|x_n - \alpha_n D(x_n, \rho_n) - w\big\|^2 + \big\|x_{n+1} - x_n + \alpha_n D(x_n, \rho_n)\big\|^2 \\
&\quad \geq \|x_n - w\|^2 - \big\|x_n - \alpha_n D(x_n, \rho_n) - w\big\|^2
\end{aligned}$$

$$= \|x_n - w\|^2 - \|x_n - w\|^2 - \big\|\alpha_n D(x_n, \rho_n)\big\|^2 + 2\big\langle x_n - w, \alpha_n D(x_n, \rho_n)\big\rangle$$
$$= 2\big\langle x_n - w, \alpha_n D(x_n, \rho_n)\big\rangle - \alpha_n^2 \big\|D(x_n, \rho_n)\big\|^2. \tag{3.3}$$

Besides, by Lemma 2.3,

$$\big\langle A^*\big(I - J_{\beta_n}^{B_2}\big)Ay_n - A^*\big(I - J_{\beta_n}^{B_2}\big)Aw, y_n - w\big\rangle \geq 0. \tag{3.4}$$

By (3.1) and (3.4),

$$\big\langle A^*\big(I - J_{\beta_n}^{B_2}\big)Ay_n, y_n - w\big\rangle \geq 0. \tag{3.5}$$

By Lemma 2.2(vi) and (1.3),

$$\big\langle x_n - y_n - \rho_n A^*\big(I - J_{\beta_n}^{B_2}\big)Ax_n, y_n - w\big\rangle \geq \langle a_n \rho_n x_n - \gamma_n d_n, y_n - w\rangle. \tag{3.6}$$

By (3.5) and (3.6), we have

$$\langle a_n \rho_n x_n - \gamma_n d_n, y_n - w\rangle$$
$$\leq \big\langle x_n - y_n - \rho_n A^*\big(I - J_{\beta_n}^{B_2}\big)Ax_n + \rho_n A^*\big(I - J_{\beta_n}^{B_2}\big)Ay_n, y_n - w\big\rangle$$
$$= \big\langle D(x_n, \rho_n), y_n - w\big\rangle. \tag{3.7}$$

By (3.7), we know that

$$\big\langle D(x_n, \rho_n), x_n - y_n\big\rangle + \langle a_n \rho_n x_n - \gamma_n d_n, y_n - w\rangle \leq \big\langle D(x_n, \rho_n), x_n - w\big\rangle. \tag{3.8}$$

Here, we set

$$\varepsilon_n := \rho_n\big[A^*\big(I - J_{\beta_n}^{B_2}\big)Ay_n - A^*\big(I - J_{\beta_n}^{B_2}\big)Ax_n\big]. \tag{3.9}$$

Then it follows from (1.9) and (3.9) that

$$\big\langle D(x_n, \rho_n), x_n - y_n\big\rangle = \langle x_n - y_n, x_n - y_n + \varepsilon_n\rangle$$
$$= \|x_n - y_n\|^2 + \langle x_n - y_n, \varepsilon_n\rangle$$
$$\geq \|x_n - y_n\|^2 - \big|\langle x_n - y_n, \varepsilon_n\rangle\big|$$
$$\geq (1 - \delta)\|x_n - y_n\|^2 \tag{3.10}$$

and

$$\big\langle D(x_n, \rho_n), x_n - y_n\big\rangle = \langle x_n - y_n, x_n - y_n + \varepsilon_n\rangle$$
$$= \|x_n - y_n\|^2 + \langle x_n - y_n, \varepsilon_n\rangle$$
$$= \frac{1}{2}\|x_n - y_n\|^2 + \langle x_n - y_n, \varepsilon_n\rangle + \frac{1}{2}\|x_n - y_n\|^2$$
$$\geq \frac{1}{2}\|x_n - y_n\|^2 + \langle x_n - y_n, \varepsilon_n\rangle + \frac{1}{2}\|\varepsilon_n\|^2$$

$$= \frac{1}{2}\|x_n - y_n + \varepsilon_n\|^2$$
$$= \frac{1}{2}\big\|D(x_n, \rho_n)\big\|^2. \tag{3.11}$$

By (3.3) and (3.8),

$$\begin{aligned}
&\|x_n - w\|^2 - \|x_{n+1} - w\|^2 \\
&\quad \geq 2\alpha_n\big\langle x_n - w, D(x_n, \rho_n)\big\rangle - \alpha_n^2\big\|D(x_n, \rho_n)\big\|^2 \\
&\quad \geq 2\alpha_n\big\langle D(x_n, \rho_n), x_n - y_n\big\rangle + 2\alpha_n a_n \rho_n\langle x_n, y_n - w\rangle - 2\alpha_n\gamma_n\langle d_n, y_n - w\rangle \\
&\qquad - \alpha_n^2\big\|D(x_n, \rho_n)\big\|^2 \\
&\quad = \alpha_n\big\langle D(x_n, \rho_n), x_n - y_n\big\rangle + 2\alpha_n a_n \rho_n\langle x_n, y_n - w\rangle - 2\alpha_n\gamma_n\langle d_n, y_n - w\rangle.
\end{aligned} \tag{3.12}$$

By (1.7) and (3.11), $\alpha_n \geq \frac{1}{2}$ for each $n \in \mathbb{N}$. It follows from (1.9) and $1 > 2\delta$ that

$$\begin{aligned}
\|x_n - y_n + \varepsilon_n\|^2 &= \|x_n - y_n\|^2 + \|\varepsilon_n\|^2 + 2\langle x_n - y_n, \varepsilon_n\rangle \\
&\geq \|x_n - y_n\|^2 + \|\varepsilon_n\|^2 - 2\big|\langle x_n - y_n, \varepsilon_n\rangle\big| \\
&\geq \|x_n - y_n\|^2 + \|\varepsilon_n\|^2 - 2\delta\|x_n - y_n\|^2 \\
&\geq (1 - 2\delta)\|x_n - y_n\|^2 > 0.
\end{aligned} \tag{3.13}$$

By (1.6), (1.7), (3.9), and (3.13),

$$\alpha_n^2 \leq \left(\frac{\|x_n - y_n\| \cdot \|x_n - y_n + \varepsilon_n\|}{\|x_n - y_n + \varepsilon_n\|^2}\right)^2 \leq \frac{\|x_n - y_n\|^2}{(1 - 2\delta)\|x_n - y_n\|^2} = \frac{1}{1 - 2\delta}. \tag{3.14}$$

So, $\{\alpha_n\}$ is a bounded sequence. By (3.12) and (3.10),

$$\begin{aligned}
&\|x_{n+1} - w\|^2 \\
&\quad \leq \|x_n - w\|^2 - \alpha_n\big\langle D(x_n, \rho_n), x_n - y_n\big\rangle + 2\alpha_n a_n \rho_n\langle x_n, w - y_n\rangle + 2\alpha_n\gamma_n\langle d_n, y_n - w\rangle \\
&\quad \leq \|x_n - w\|^2 - \alpha_n(1 - \delta)\|x_n - y_n\|^2 + 2\alpha_n a_n \rho_n\langle x_n, w - y_n\rangle + 2\alpha_n\gamma_n\langle d_n, y_n - w\rangle \\
&\quad \leq \|x_n - w\|^2 - \frac{1 - \delta}{2}\|x_n - y_n\|^2 + 2\alpha_n a_n \rho_n\langle x_n, w - y_n\rangle + 2\alpha_n\gamma_n\langle d_n, y_n - w\rangle.
\end{aligned} \tag{3.15}$$

It follows from (2.3) and (3.15) that

$$\begin{aligned}
&\|x_{n+1} - w\|^2 \\
&\quad \leq \|x_n - w\|^2 - \alpha_n(1 - \delta)\|x_n - y_n\|^2 + 2\alpha_n\gamma_n\langle d_n, y_n - w\rangle \\
&\qquad + \alpha_n a_n \rho_n\big(\|x_n - y_n\|^2 + \|w\|^2 - \|x_n - w\|^2 - \|y_n\|^2\big) \\
&\quad \leq (1 - \alpha_n a_n \rho_n)\|x_n - w\|^2 - \alpha_n[1 - \delta - a_n \rho_n]\|x_n - y_n\|^2 \\
&\qquad + \alpha_n a_n \rho_n\big(\|w\|^2 - \|y_n\|^2\big) + 2\alpha_n\gamma_n\langle d_n, y_n - w\rangle.
\end{aligned} \tag{3.16}$$

Since $\lim_{n\to\infty} a_n = 0$, and two sequences $\{\rho_n\}$ and $\{\alpha_n\}$ are bounded, we may assume that $a_n\rho_n < 1 - \delta$ and $0 < \alpha_n a_n \rho_n < 1$ for each $n \in \mathbb{N}$. Since $\{x_n\}$ is a bounded sequence, it is easy to see that $\{A^*(I-J_{\beta_n}^{B_2})Ax_n\}$ is a bounded sequence. Then there exists $M>0$ such that $\|A^*(I-J_{\beta_n}^{B_2})Ax_n\|\leq M$ for each $n\in\mathbb{N}$.

Since $\lim_{n\to\infty}\eta_n=0$, there exists $k\in\mathbb{N}$ such that $\eta_n<1/2$ for each $n>k$. Let $M^*=\max\{M,\|d_k\|\}$. Then $\|d_k\|<2M^*$. Suppose that $\|d_n\|\leq 2M^*$ for some $n>k$. Then we have

$$\|d_{n+1}\|\leq \bigl\|A^*\bigl(I-J_{\beta_{n+1}}^{B_2}\bigr)Ax_{n+1}\bigr\|+\eta_{n+1}\|d_n\|\leq M+\frac{1}{2}\|d_n\|\leq 2M^*.$$

By the induction method, we know that $\|d_n\|\leq 2M^*$ for each $n\geq k$. So, $\{d_n\}$ is a bounded sequence.

Next, we know that

$$\begin{aligned}
&\|y_n-w\|\\
&\quad\leq \bigl\|J_{\beta_n}^{B_1}\bigl[(1-a_n\rho_n)x_n-\rho_nA^*\bigl(I-J_{\beta_n}^{B_2}\bigr)Ax_n+\gamma_nd_n\bigr]\\
&\qquad-J_{\beta_n}^{B_1}\bigl[(1-a_n\rho_n)w-\rho_nA^*\bigl(I-J_{\beta_n}^{B_2}\bigr)Aw\bigr]\bigr\|\\
&\qquad+\bigl\|J_{\beta_n}^{B_1}\bigl[(1-a_n\rho_n)w-\rho_nA^*\bigl(I-J_{\beta_n}^{B_2}\bigr)Aw\bigr]-J_{\beta_n}^{B_1}\bigl[w-\rho_nA^*\bigl(I-J_{\beta_n}^{B_2}\bigr)Aw\bigr]\bigr\|\\
&\quad\leq (1-a_n\rho_n)\|x_n-w\|+\gamma_n\|d_n\|+\bigl\|J_{\beta_n}^{B_1}\bigl[(1-a_n\rho_n)w\bigr]-J_{\beta_n}^{B_1}[w]\bigr\|\\
&\quad\leq (1-a_n\rho_n)\|x_n-w\|+a_n\rho_n\|w\|+\gamma_n\|d_n\|.
\end{aligned}\tag{3.17}$$

Hence, if follows from (3.17) and the two sequences $\{x_n\}$ and $\{d_n\}$ being bounded that sequence $\{y_n\}$ is bounded.

Besides, we have

$$\begin{aligned}
&\|y_n-w\|^2\\
&\quad\leq \bigl\|\bigl[(1-a_n\rho_n)x_n-\rho_nA^*\bigl(I-J_{\beta_n}^{B_2}\bigr)Ax_n\bigr]-\bigl[w-\rho_nA^*\bigl(I-J_{\beta_n}^{B_2}\bigr)Aw\bigr]+\gamma_nd_n\bigr\|^2\\
&\quad=\bigl\|\bigl[x_n-\rho_nA^*\bigl(I-J_{\beta_n}^{B_2}\bigr)Ax_n\bigr]-\bigl[w-\rho_nA^*\bigl(I-J_{\beta_n}^{B_2}\bigr)Aw\bigr]\bigr\|^2\\
&\qquad+2\bigl\langle\bigl[x_n-\rho_nA^*\bigl(I-J_{\beta_n}^{B_2}\bigr)Ax_n\bigr]-\bigl[w-\rho_nA^*\bigl(I-J_{\beta_n}^{B_2}\bigr)Aw\bigr],\gamma_nd_n-a_n\rho_nx_n\bigr\rangle\\
&\qquad+\|\gamma_nd_n-a_n\rho_nx_n\|^2\\
&\quad\leq \bigl\|\bigl[x_n-\rho_nA^*\bigl(I-J_{\beta_n}^{B_2}\bigr)Ax_n\bigr]-\bigl[w-\rho_nA^*\bigl(I-J_{\beta_n}^{B_2}\bigr)Aw\bigr]\bigr\|^2\\
&\qquad+2\bigl\langle\bigl[x_n-\rho_nA^*\bigl(I-J_{\beta_n}^{B_2}\bigr)Ax_n\bigr]-\bigl[w-\rho_nA^*\bigl(I-J_{\beta_n}^{B_2}\bigr)Aw\bigr],\gamma_nd_n-a_n\rho_nx_n\bigr\rangle\\
&\qquad+\gamma_n^2\|d_n\|^2+(a_n\rho_n)^2\|x_n\|^2+2\gamma_na_n\rho_n\|d_n\|\cdot\|x_n\|.
\end{aligned}\tag{3.18}$$

By (3.18) and Lemma 2.4,

$$\begin{aligned}
&\|y_n-w\|^2\\
&\quad\leq \|x_n-w\|^2-\bigl(2\rho_n-\rho_n^2\|A\|^2\bigr)\bigl\|\bigl(I-J_{\beta_n}^{B_2}\bigr)Ax_n-\bigl(I-J_{\beta_n}^{B_2}\bigr)Aw\bigr\|^2\\
&\qquad+2\bigl\langle\bigl[x_n-\rho_nA^*\bigl(I-J_{\beta_n}^{B_2}\bigr)Ax_n\bigr]-\bigl[w-\rho_nA^*\bigl(I-J_{\beta_n}^{B_2}\bigr)Aw\bigr],\gamma_nd_n-a_n\rho_nx_n\bigr\rangle\\
&\qquad+\gamma_n^2\|d_n\|^2+(a_n\rho_n)^2\|x_n\|^2+2\gamma_na_n\rho_n\|d_n\|\cdot\|x_n\|\\
&\quad\leq \|x_n-w\|^2-\bigl(2\rho_n-\rho_n^2\|A\|^2\bigr)\bigl\|\bigl(I-J_{\beta_n}^{B_2}\bigr)Ax_n\bigr\|^2+\gamma_n^2\|d_n\|^2\\
&\qquad+2\|x_n-w\|\cdot\bigl(\gamma_n\|d_n\|+a_n\rho_n\|x_n\|\bigr)+(a_n\rho_n)^2\|x_n\|^2+2\gamma_na_n\rho_n\|d_n\|\cdot\|x_n\|.
\end{aligned}\tag{3.19}$$

Next, we know that

$$\begin{aligned}
&\left\|y_n - J_{\beta_n}^{B_1}x_n\right\| \\
&= \left\|J_{\beta_n}^{B_1}\left[(1-a_n\rho_n)x_n - \rho_n A^*\left(I - J_{\beta_n}^{B_2}\right)Ax_n + \gamma_n d_n\right] - J_{\beta_n}^{B_1}x_n\right\| \\
&\le \left\|\left[(1-a_n\rho_n)x_n - \rho_n A^*\left(I - J_{\beta_n}^{B_2}\right)Ax_n\right] - x_n\right\| + \gamma_n\|d_n\| \\
&\le a_n\rho_n\|x_n\| + \rho_n\left\|A^*\left(I - J_{\beta_n}^{B_2}\right)Ax_n\right\| + \gamma_n\|d_n\| \\
&\le a_n\rho_n\|x_n\| + \rho_n\|A\| \cdot \left\|Ax_n - J_{\beta_n}^{B_2}Ax_n\right\| + \gamma_n\|d_n\|. \qquad (3.20)
\end{aligned}$$

Further, by Lemma 2.2, we have

$$\begin{aligned}
&\|y_n - \bar{x}\|^2 \\
&= \left\|J_{\beta_n}^{B_1}\left[(1-a_n\rho_n)x_n - \rho_n A^*\left(I - J_{\beta_n}^{B_2}\right)Ax_n + \gamma_n d_n\right] - J_{\beta_n}^{B_1}\left[\bar{x} - \rho_n A^*\left(I - J_{\beta_n}^{B_2}\right)A\bar{x}\right]\right\|^2 \\
&\le \left\langle(1-a_n\rho_n)x_n - \rho_n A^*\left(I - J_{\beta_n}^{B_2}\right)Ax_n + \gamma_n d_n - \bar{x} + \rho_n A^*\left(I - J_{\beta_n}^{B_2}\right)A\bar{x}, y_n - \bar{x}\right\rangle \\
&= \left\langle(1-a_n\rho_n)x_n - \rho_n A^*\left(I - J_{\beta_n}^{B_2}\right)Ax_n - (1-a_n\rho_n)\bar{x} + \rho_n A^*\left(I - J_{\beta_n}^{B_2}\right)A\bar{x}, y_n - \bar{x}\right\rangle \\
&\quad - a_n\rho_n\langle\bar{x}, y_n - \bar{x}\rangle + \gamma_n\langle d_n, y_n - \bar{x}\rangle \\
&\le \left\|(1-a_n\rho_n)x_n - \rho_n A^*\left(I - J_{\beta_n}^{B_2}\right)Ax_n - (1-a_n\rho_n)\bar{x} + \rho_n A^*\left(I - J_{\beta_n}^{B_2}\right)A\bar{x}\right\| \\
&\quad \cdot \|y_n - \bar{x}\| + a_n\rho_n\langle -\bar{x}, y_n - \bar{x}\rangle + \gamma_n\langle d_n, y_n - \bar{x}\rangle \\
&\le (1-a_n\rho_n)\|x_n - \bar{x}\| \cdot \|y_n - \bar{x}\| + a_n\rho_n\langle -\bar{x}, y_n - \bar{x}\rangle + \gamma_n\langle d_n, y_n - \bar{x}\rangle \\
&\le \frac{(1-a_n\rho_n)^2}{2}\|x_n - \bar{x}\|^2 + \frac{1}{2}\|y_n - \bar{x}\|^2 + a_n\rho_n\langle -\bar{x}, y_n - \bar{x}\rangle + \gamma_n\langle d_n, y_n - \bar{x}\rangle \\
&\le \left(\frac{1-a_n\rho_n}{2}\right)\|x_n - \bar{x}\|^2 + \frac{1}{2}\|y_n - \bar{x}\|^2 + a_n\rho_n\langle -\bar{x}, y_n - \bar{x}\rangle + \gamma_n\langle d_n, y_n - \bar{x}\rangle. \qquad (3.21)
\end{aligned}$$

This implies that

$$\begin{aligned}
&\|y_n - \bar{x}\|^2 \\
&\le (1-a_n\rho_n)\|x_n - \bar{x}\|^2 + 2a_n\rho_n\langle -\bar{x}, y_n - \bar{x}\rangle + 2\gamma_n\langle d_n, y_n - \bar{x}\rangle \\
&= (1-a_n\rho_n)\|x_n - \bar{x}\|^2 + 2a_n\rho_n\langle -\bar{x}, y_n - x_n\rangle + 2a_n\rho_n\langle -\bar{x}, x_n - \bar{x}\rangle \\
&\quad + 2\gamma_n\langle d_n, y_n - \bar{x}\rangle \\
&\le (1-a_n\rho_n)\|x_n - \bar{x}\|^2 + 2a_n\rho_n\|\bar{x}\| \cdot \|y_n - x_n\| + 2a_n\rho_n\langle -\bar{x}, x_n - \bar{x}\rangle \\
&\quad + 2\gamma_n\|d_n\| \cdot \|y_n - \bar{x}\|. \qquad (3.22)
\end{aligned}$$

By (3.22), we also have

$$\begin{aligned}
&\|x_{n+1} - \bar{x}\|^2 - \|x_{n+1} - y_n\|^2 - 2\langle x_{n+1} - x_n, y_n - \bar{x}\rangle \\
&= \|y_n - \bar{x}\|^2 \\
&\le (1-a_n\rho_n)\|x_n - \bar{x}\|^2 + 2a_n\rho_n\|\bar{x}\| \cdot \|y_n - x_n\| + 2a_n\rho_n\langle -\bar{x}, x_n - \bar{x}\rangle \\
&\quad + 2\gamma_n\|d_n\| \cdot \|y_n - \bar{x}\|. \qquad (3.23)
\end{aligned}$$

That is,

$$\begin{aligned}
&\|x_{n+1}-\bar{x}\|^2 \\
&\quad\le (1-a_n\rho_n)\|x_n-\bar{x}\|^2 + 2a_n\rho_n\|\bar{x}\|\cdot\|y_n-x_n\| + 2a_n\rho_n\langle -\bar{x}, x_n-\bar{x}\rangle \\
&\qquad + \|x_{n+1}-y_n\|^2 + 2\langle x_{n+1}-x_n, y_n-\bar{x}\rangle + 2\gamma_n\|d_n\|\cdot\|y_n-\bar{x}\|.
\end{aligned} \tag{3.24}$$

By (3.2) again, we have

$$\begin{aligned}
&\|x_{n+1}-\bar{x}\|^2 + \|x_{n+1}-x_n\|^2 + \big\|\alpha_n D(x_n,\rho_n)\big\|^2 + 2\big\langle x_{n+1}-x_n, \alpha_n D(x_n,\rho_n)\big\rangle \\
&\quad\le \|x_n-\bar{x}\|^2 + \big\|\alpha_n D(x_n,\rho_n)\big\|^2 - 2\big\langle x_n-\bar{x}, \alpha_n D(x_n,\rho_n)\big\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|x_{n+1}-x_n\|^2 \le{}& \|x_n-\bar{x}\|^2 - \|x_{n+1}-\bar{x}\|^2 - 2\big\langle x_{n+1}-x_n, \alpha_n D(x_n,\rho_n)\big\rangle \\
&- 2\big\langle x_n-\bar{x}, \alpha_n D(x_n,\rho_n)\big\rangle.
\end{aligned} \tag{3.25}$$

By (2.1) and (3.16), we have

$$\begin{aligned}
&\|x_{n+1}-\bar{x}\|^2 - 2\alpha_n\gamma_n\langle d_n, y_n-\bar{x}\rangle \\
&\quad\le (1-\alpha_n a_n\rho_n)\|x_n-\bar{x}\|^2 + \alpha_n a_n\rho_n\big(\|\bar{x}\|^2 - \|y_n\|^2\big) \\
&\quad= (1-\alpha_n a_n\rho_n)\|x_n-\bar{x}\|^2 + \alpha_n a_n\rho_n\big(\|\bar{x}-y_n+y_n\|^2 - \|y_n\|^2\big) \\
&\quad\le (1-\alpha_n a_n\rho_n)\|x_n-\bar{x}\|^2 + \alpha_n a_n\rho_n\big(2\langle\bar{x}-y_n,\bar{x}\rangle + \|y_n\|^2 - \|y_n\|^2\big) \\
&\quad\le (1-\alpha_n a_n\rho_n)\|x_n-\bar{x}\|^2 + 2\alpha_n a_n\rho_n\langle\bar{x}-y_n,\bar{x}\rangle \\
&\quad= (1-\alpha_n a_n\rho_n)\|x_n-\bar{x}\|^2 + 2\alpha_n a_n\rho_n\langle -\bar{x}, y_n-\bar{x}\rangle \\
&\quad= (1-\alpha_n a_n\rho_n)\|x_n-\bar{x}\|^2 + 2\alpha_n a_n\rho_n\big(\langle -\bar{x}, y_n-x_n\rangle + \langle -\bar{x}, x_n-\bar{x}\rangle\big).
\end{aligned} \tag{3.26}$$

This implies that

$$\begin{aligned}
&\|x_{n+1}-\bar{x}\|^2 \\
&\quad\le \left(1-\frac{1}{2}a_n\rho\right)\|x_n-\bar{x}\|^2 + \frac{a_n\rho}{2}\cdot\frac{4\alpha_n\rho_n}{\rho}\big(\langle -\bar{x}, y_n-x_n\rangle + \langle -\bar{x}, x_n-\bar{x}\rangle\big) \\
&\qquad + \frac{a_n\rho}{2}\cdot\frac{4\alpha_n}{\rho}\cdot\frac{\gamma_n}{a_n}\langle d_n, y_n-\bar{x}\rangle.
\end{aligned} \tag{3.27}$$

Case 1: there exists a natural number N such that $\|x_{n+1}-\bar{x}\| \le \|x_n-\bar{x}\|$ for each $n \ge N$. Clearly, $\lim_{n\to\infty}\|x_n-\bar{x}\|$ exists. By (3.15) and $\lim_{n\to\infty}\gamma_n = 0$, we know that

$$\lim_{n\to\infty}\|x_n-y_n\| = \lim_{n\to\infty}\big\|D(x_n,\rho_n)\big\| = 0. \tag{3.28}$$

By (3.25) and (3.28),

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}\|x_{n+1}-y_n\| = 0. \tag{3.29}$$

By (3.23), (3.28), (3.29), and $\lim_{n\to\infty} \gamma_n = 0$,

$$\lim_{n\to\infty} \|y_n - \bar{x}\| = \lim_{n\to\infty} \|x_n - \bar{x}\|. \tag{3.30}$$

By (3.19), (3.30), and $\lim_{n\to\infty} \gamma_n = 0$,

$$\lim_{n\to\infty} \left(2\rho_n - \rho_n^2\|A\|^2\right)\left\|Ax_n - J_{\beta_n}^{B_2}Ax_n\right\|^2 = 0. \tag{3.31}$$

By (3.31),

$$\lim_{n\to\infty} \left\|Ax_n - J_{\beta_n}^{B_2}Ax_n\right\| = 0. \tag{3.32}$$

By (3.20), (3.32), and $\lim_{n\to\infty} a_n = 0$,

$$\lim_{n\to\infty} \left\|y_n - J_{\beta_n}^{B_1}x_n\right\| = 0. \tag{3.33}$$

By (3.28) and (3.33),

$$\lim_{n\to\infty} \left\|x_n - J_{\beta_n}^{B_1}x_n\right\| = 0. \tag{3.34}$$

Since $\liminf_{n\to\infty} \beta_n > 0$, we may assume that $\beta_n \geq \beta$ for some $\beta > 0$. By (3.32), (3.34) and Lemma 2.2(iii),

$$\lim_{n\to\infty} \left\|x_n - J_{\beta}^{B_1}x_n\right\| = \lim_{n\to\infty} \left\|Ax_n - J_{\beta}^{B_2}Ax_n\right\| = 0. \tag{3.35}$$

Since $\{x_n\}$ is a bounded sequence, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $z \in H$ such that $x_{n_k} \rightharpoonup z$ and

$$\limsup_{n\to\infty} \langle -\bar{x}, x_n - \bar{x}\rangle = \lim_{k\to\infty} \langle -\bar{x}, x_{n_k} - \bar{x}\rangle = \langle -\bar{x}, z - \bar{x}\rangle. \tag{3.36}$$

It follows from $x_{n_k} \rightharpoonup z$ and (3.35) that $z \in \operatorname{Fix}(J_\beta^{B_1}) = B_1^{-1}(0)$. Besides, since $x_{n_k} \rightharpoonup z$, we have

$$\lim_{k\to\infty} \langle Ax_{n_k} - Az, y\rangle = \lim_{k\to\infty} \left\langle x_{n_k} - z, A^*y\right\rangle = 0. \tag{3.37}$$

Then $Ax_{n_k} \rightharpoonup Az$. Similarly, we know that $Az \in \operatorname{Fix}(J_\beta^{B_2}) = B_2^{-1}(0)$. So, $z \in \Omega$. By (3.36) and Lemma 2.1, we know that

$$\limsup_{n\to\infty} \langle -\bar{x}, x_n - \bar{x}\rangle = \lim_{k\to\infty} \langle -\bar{x}, x_{n_k} - \bar{x}\rangle = \langle -\bar{x}, z - \bar{x}\rangle \leq 0. \tag{3.38}$$

We also have

$$\begin{aligned} &\limsup_{n\to\infty} \langle d_n, x_n - \bar{x}\rangle \\ &\quad = \limsup_{n\to\infty} \left\langle -A^*\left(I - J_{\beta_n}^{B_2}\right)Ax_n + \eta_n d_{n-1}, x_n - \bar{x}\right\rangle \end{aligned}$$

$$
\begin{aligned}
&= \limsup_{n\to\infty}\bigl(\bigl\langle -A^*\bigl(I-J_{\beta_n}^{B_2}\bigr)Ax_n + A^*\bigl(I-J_{\beta_n}^{B_2}\bigr)A\bar{x}, x_n-\bar{x}\bigr\rangle + \langle \eta_n d_{n-1}, x_n-\bar{x}\rangle\bigr) \\
&\le \limsup_{n\to\infty} \eta_n \langle d_{n-1}, x_n-\bar{x}\rangle = 0. \qquad (3.39)
\end{aligned}
$$

Hence, it follows from (3.28) and (3.39) that

$$
\begin{aligned}
&\limsup_{n\to\infty}\langle d_n, y_n-\bar{x}\rangle \\
&\quad= \limsup_{n\to\infty}\bigl(\langle d_n, y_n-x_n\rangle + \langle d_n, x_n-\bar{x}\rangle\bigr) \\
&\quad\le \limsup_{n\to\infty}\langle d_n, y_n-x_n\rangle + \limsup_{n\to\infty}\langle d_n, x_n-\bar{x}\rangle \le 0. \qquad (3.40)
\end{aligned}
$$

By (3.27), (3.38), (3.40), and Lemma 2.7, we know that $\lim_{n\to\infty} x_n = \bar{x}$.

Case 2: suppose that there exists a subset $\{n_i\}$ of $\{n\}$ such that $\|x_{n_i}-\bar{x}\| \le \|x_{n_i+1}-\bar{x}\|$ for all $i\in\mathbb{N}$. By Lemma 2.6, there exists a nondecreasing sequence $\{m_k\}$ in $\mathbb{N}$ such that $m_k\to\infty$,

$$
\|x_{m_k}-\bar{x}\| \le \|x_{m_k+1}-\bar{x}\| \quad \text{and} \quad \|x_k-\bar{x}\| \le \|x_{m_k+1}-\bar{x}\| \qquad (3.41)
$$

for all $k\in\mathbb{N}$. By (3.26),

$$
\begin{aligned}
&\|x_{m_k+1}-\bar{x}\|^2 \\
&\quad\le (1-\alpha_{m_k}a_{m_k}\rho_{m_k})\|x_{m_k}-\bar{x}\|^2 + 2\alpha_{m_k}a_{m_k}\rho_{m_k}\langle -\bar{x}, y_{m_k}-x_{m_k}\rangle \\
&\qquad + 2\alpha_{m_k}a_{m_k}\rho_{m_k}\langle -\bar{x}, x_{m_k}-\bar{x}\rangle + 2\alpha_{m_k}\gamma_{m_k}\langle d_{m_k}, y_{m_k}-\bar{x}\rangle \qquad (3.42)
\end{aligned}
$$

for all $k\in\mathbb{N}$. By (3.41) and (3.42),

$$
\begin{aligned}
&\alpha_{m_k}a_{m_k}\rho_{m_k}\|x_{m_k}-\bar{x}\|^2 \\
&\quad\le \|x_{m_k}-\bar{x}\|^2 - \|x_{m_k+1}-\bar{x}\|^2 + 2\alpha_{m_k}a_{m_k}\rho_{m_k}\langle -\bar{x}, y_{m_k}-x_{m_k}\rangle \\
&\qquad + 2\alpha_{m_k}a_{m_k}\rho_{m_k}\langle -\bar{x}, x_{m_k}-\bar{x}\rangle + 2\alpha_{m_k}\gamma_{m_k}\langle d_{m_k}, y_{m_k}-\bar{x}\rangle \\
&\quad\le 2\alpha_{m_k}a_{m_k}\rho_{m_k}\langle -\bar{x}, y_{m_k}-x_{m_k}\rangle + 2\alpha_{m_k}a_{m_k}\rho_{m_k}\langle -\bar{x}, x_{m_k}-\bar{x}\rangle \\
&\qquad + 2\alpha_{m_k}\gamma_{m_k}\langle d_{m_k}, y_{m_k}-\bar{x}\rangle \qquad (3.43)
\end{aligned}
$$

for all $k\in\mathbb{N}$. This implies that

$$
\begin{aligned}
\|x_{m_k}-\bar{x}\|^2 &\le 2\langle -\bar{x}, y_{m_k}-x_{m_k}\rangle + 2\langle -\bar{x}, x_{m_k}-\bar{x}\rangle \\
&\quad + \frac{2\gamma_{m_k}}{a_{m_k}\rho_{m_k}}\langle d_{m_k}, y_{m_k}-\bar{x}\rangle \qquad (3.44)
\end{aligned}
$$

for all $k\in\mathbb{N}$. By (3.41), and following a similar argument to the above, we know that

$$
\begin{cases}
\lim_{k\to\infty}\|y_{m_k}-x_{m_k}\| = \lim_{k\to\infty}\|x_{m_k+1}-x_{m_k}\| = 0, \\
\limsup_{k\to\infty}\langle -\bar{x}, x_{m_k}-\bar{x}\rangle \le 0, \\
\limsup_{k\to\infty}\langle d_{m_k}, y_{m_k}-\bar{x}\rangle \le 0.
\end{cases} \qquad (3.45)
$$

By (3.41), (3.44), and (3.45), we can get the following and this is the conclusion of Theorem 3.1:

$$\lim_{k\to\infty}\|x_{m_k}-\bar{x}\| = \lim_{k\to\infty}\|x_k-\bar{x}\| = 0. \tag{3.46}$$

Now, for completeness, we show the proof of (3.45).

By (3.15),

$$\begin{aligned}&\|x_{m_k+1}-\bar{x}\|^2\\&\quad\le \|x_{m_k}-\bar{x}\|^2-\frac{1-\delta}{2}\|x_{m_k}-y_{m_k}\|^2+2\alpha_{m_k}a_{m_k}\rho_{m_k}\langle x_{m_k},\bar{x}-y_{m_k}\rangle\\&\qquad+2\alpha_{m_k}\gamma_{m_k}\langle d_{m_k},y_{m_k}-\bar{x}\rangle.\end{aligned} \tag{3.47}$$

By (3.41) and (3.47),

$$\begin{aligned}&\frac{1-\delta}{2}\|x_{m_k}-y_{m_k}\|^2\\&\quad\le \|x_{m_k}-\bar{x}\|^2-\|x_{m_k+1}-\bar{x}\|^2+2\alpha_{m_k}a_{m_k}\rho_{m_k}\langle x_{m_k},\bar{x}-y_{m_k}\rangle\\&\qquad+2\alpha_{m_k}\gamma_{m_k}\langle d_{m_k},y_{m_k}-\bar{x}\rangle\\&\quad\le 2\alpha_{m_k}a_{m_k}\rho_{m_k}\langle x_{m_k},\bar{x}-y_{m_k}\rangle+2\alpha_{m_k}\gamma_{m_k}\langle d_{m_k},y_{m_k}-\bar{x}\rangle.\end{aligned} \tag{3.48}$$

By (3.48), we know that

$$\lim_{k\to\infty}\|x_{m_k}-y_{m_k}\| = 0. \tag{3.49}$$

Further,

$$\lim_{k\to\infty}D(x_{m_k},\rho_{m_k}) = 0. \tag{3.50}$$

By (3.25) and (3.41),

$$\begin{aligned}&\|x_{m_k+1}-x_{m_k}\|^2\\&\quad\le -2\big\langle x_{m_k+1}-x_{m_k},\alpha_{m_k}D(x_{m_k},\rho_{m_k})\big\rangle-2\big\langle x_{m_k}-\bar{x},\alpha_{m_k}D(x_{m_k},\rho_{m_k})\big\rangle.\end{aligned} \tag{3.51}$$

By (3.50) and (3.51),

$$\lim_{k\to\infty}\|x_{m_k+1}-x_{m_k}\| = 0. \tag{3.52}$$

We also have

$$\begin{aligned}&\limsup_{n\to\infty}\langle d_n,x_n-\bar{x}\rangle\\&\quad=\limsup_{n\to\infty}\big\langle -A^*\big(I-J_{\beta_n}^{B_2}\big)Ax_n+\eta_n d_{n-1},x_n-\bar{x}\big\rangle\\&\quad=\limsup_{n\to\infty}\big(\big\langle -A^*\big(I-J_{\beta_n}^{B_2}\big)Ax_n+A^*\big(I-J_{\beta_n}^{B_2}\big)A\bar{x},x_n-\bar{x}\big\rangle+\langle\eta_n d_{n-1},x_n-\bar{x}\rangle\big)\\&\quad\le\limsup_{n\to\infty}\eta_n\langle d_{n-1},x_n-\bar{x}\rangle=0.\end{aligned} \tag{3.53}$$

Hence, it follows from (3.49) and (3.53) that

$$\begin{aligned}
&\limsup_{k\to\infty}\langle d_{m_k}, y_{m_k}-\bar{x}\rangle \\
&\quad=\limsup_{k\to\infty}\bigl(\langle d_{m_k}, y_{m_k}-x_{m_k}\rangle+\langle d_{m_k}, x_{m_k}-\bar{x}\rangle\bigr) \\
&\quad\le\limsup_{k\to\infty}\langle d_{m_k}, y_{m_k}-x_{m_k}\rangle+\limsup_{k\to\infty}\langle d_{m_k}, x_{m_k}-\bar{x}\rangle\le 0.
\end{aligned}\tag{3.54}$$

By (3.19),

$$\begin{aligned}
&\|y_{m_k}-\bar{x}\|^2 \\
&\quad\le\|x_{m_k}-\bar{x}\|^2-\bigl(2\rho_{m_k}-\rho_{m_k}^2\|A\|^2\bigr)\bigl\|\bigl(I-J_{\beta_{m_k}}^{B_2}\bigr)Ax_{m_k}\bigr\|^2 \\
&\qquad+M_1\cdot\bigl(\gamma_{m_k}\|d_{m_k}\|+a_{m_k}\rho_{m_k}\|x_{m_k}\|\bigr) \\
&\qquad+\gamma_{m_k}^2\|d_{m_k}\|^2+(a_{m_k}\rho_{m_k})^2\|x_{m_k}\|^2+2\gamma_{m_k}a_{m_k}\rho_{m_k}\|d_{m_k}\|\cdot\|x_{m_k}\|,
\end{aligned}\tag{3.55}$$

where

$$M_1:=\sup_{k\in\mathbb{N}}\bigl\{2\bigl\|\bigl[x_{m_k}-\rho_{m_k}A^*\bigl(I-J_{\beta_{m_k}}^{B_2}\bigr)Ax_{m_k}\bigr]-\bigl[\bar{x}-\rho_{m_k}A^*\bigl(I-J_{\beta_{m_k}}^{B_2}\bigr)A\bar{x}\bigr]\bigr\|\bigr\}.$$

By (3.55),

$$\begin{aligned}
&\bigl(2\rho_{m_k}-\rho_{m_k}^2\|A\|^2\bigr)\bigl\|\bigl(I-J_{\beta_{m_k}}^{B_2}\bigr)Ax_{m_k}\bigr\|^2 \\
&\quad\le\|x_{m_k}-y_{m_k}\|\cdot\bigl(\|x_{m_k}-\bar{x}\|+\|y_{m_k}-\bar{x}\|\bigr) \\
&\qquad+M_1\cdot\bigl(\gamma_{m_k}\|d_{m_k}\|+a_{m_k}\rho_{m_k}\|x_{m_k}\|\bigr) \\
&\qquad+\gamma_{m_k}^2\|d_{m_k}\|^2+(a_{m_k}\rho_{m_k})^2\|x_{m_k}\|^2+2\gamma_{m_k}a_{m_k}\rho_{m_k}\|d_{m_k}\|\cdot\|x_{m_k}\|.
\end{aligned}\tag{3.56}$$

By (3.49) and (3.56), we know that

$$\lim_{k\to\infty}\bigl(2\rho_{m_k}-\rho_{m_k}^2\|A\|^2\bigr)\bigl\|\bigl(I-J_{\beta_{m_k}}^{B_2}\bigr)Ax_{m_k}\bigr\|=0.\tag{3.57}$$

This implies that

$$\lim_{k\to\infty}\bigl\|Ax_{m_k}-J_{\beta_{m_k}}^{B_2}Ax_{m_k}\bigr\|=0.\tag{3.58}$$

By (3.20), (3.49), and (3.58), we have

$$\lim_{k\to\infty}\bigl\|y_{m_k}-J_{\beta_{m_k}}^{B_1}x_{m_k}\bigr\|=\lim_{k\to\infty}\bigl\|x_{m_k}-J_{\beta_{m_k}}^{B_1}x_{m_k}\bigr\|=0.\tag{3.59}$$

Since $\{x_{m_k}\}$ is bounded, there is a subsequence $\{z_k\}$ of $\{x_{m_k}\}$ such that $z_k\rightharpoonup u$ and

$$\limsup_{k\to\infty}\langle -\bar{x}, x_{m_k}-\bar{x}\rangle=\lim_{k\to\infty}\langle -\bar{x}, z_k-\bar{x}\rangle=\langle -\bar{x}, u-\bar{x}\rangle.\tag{3.60}$$

By (3.58), (3.59), Lemma 2.2, and Lemma 2.5, we know that $u \in \Omega$. So, by (3.60) and Lemma 2.1, we know that

$$\limsup_{k\to\infty}\langle -\bar{x}, x_{m_k} - \bar{x}\rangle \leq 0. \tag{3.61}$$

Therefore, the proof is completed. □

4 Application: split feasibility problems

Let C and Q be nonempty, closed, and convex subsets of infinite dimensional real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a linear and bounded operator. The split feasibility problem is the following problem:

$$\text{Find } \bar{x} \in H_1 \text{ such that } \bar{x} \in C \text{ and } A\bar{x} \in Q. \tag{SFP}$$

Let $\{a_n\}$, $\{\eta_n\}$, $\{\gamma_n\}$, and $\{\rho_n\}$ be real sequences. Let δ be a fixed real number. Let Ω_1 be the solution set of problem (SFP).

Algorithm 4.1

Step 0. Choose $x_1 \in H_1$ arbitrarily, set $r_1 \in (0,1)$ and $d_0 = 0$.
Step 1. $d_n := -A^*(I - P_Q)Ax_n + \eta_n d_{n-1}$.
Step 2. For $n \in \mathbb{N}$, set y_n as

$$y_n = P_C\bigl[(1 - a_n\rho_n)x_n - \rho_n A^*(I - P_Q)Ax_n + \gamma_n d_n\bigr], \tag{4.1}$$

where $\rho_n > 0$ satisfies

$$\rho_n\bigl\|A^*(I - P_Q)Ax_n - A^*(I - P_Q)Ay_n\bigr\| \leq \delta\|x_n - y_n\|, \quad 0 < \delta < 1. \tag{4.2}$$

Step 3. If $x_n = y_n$, then set $n := n + 1$ and go to Step 1. Otherwise, go to Step 3.
Step 4. The new iterative x_{n+1} is defined by

$$x_{n+1} = P_C\bigl[x_n - \alpha_n D(x_n, \rho_n)\bigr], \tag{4.3}$$

where

$$D(x_n, \rho_n) := x_n - y_n + \rho_n\bigl[A^*(I - P_Q)Ay_n - A^*(I - P_Q)Ax_n\bigr], \tag{4.4}$$

$$\alpha_n := \frac{\langle x_n - y_n, D(x_n, \rho_n)\rangle}{\|D(x_n, \rho_n)\|^2}. \tag{4.5}$$

Then update $n := n + 1$ and go to Step 1.

Following the same argument as in [5], we can get the following strong convergence theorem of the proposed algorithm for the split feasibility problem.

Theorem 4.1 *Let C and Q be nonempty, closed, and convex subsets of infinite dimensional real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a linear and bounded operator. Let $\{a_n\}$, $\{\eta_n\}$, $\{\gamma_n\}$ be sequences in $[0,1]$. Choose $\delta \in (0,1/2)$, and let $\{\rho_n\}$ be a sequence in $(0, \min\{\frac{\delta}{\|A\|^2}, \frac{2}{\|A\|^2+2}\})$. Let Ω_1 be the solution set of problem* (SFVIP) *and assume that*

$\Omega_1 \neq \emptyset$. *For the sequence* $\{x_n\}$ *in Algorithm* 4.1, *we further assume that*:

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \eta_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$, $\liminf_{n\to\infty} \rho_n > 0$;

(ii) $\lim_{n\to\infty} \frac{\gamma_n}{a_n} = t$ *for some* $t \geq 0$, *and* $\{x_n\}$ *is a bounded sequence*.

Then $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} := P_{\Omega_1} 0$.

5 Numerical results for (SFVIP)

All codes were written in R language (version 2.15.2 (2012-10-26)), and all numerical results run on ASUS All in one PC series with i3-2100 CPU.

Set $u = (1,1)$, $\beta_1 = 1$, $\beta_n = 1 + \frac{1}{n-1}$ for $n \geq 2$, $\eta_n = \frac{1}{n+1}$, $a_n = \frac{1}{n+1}$, $\gamma_1 = 1$, and $\gamma_n = \frac{1}{n-1}$ for $n \geq 2$, and $\beta = 1$. Let $\varepsilon > 0$ and the algorithm stop if $\|x_{n-1} - x_n\| < \varepsilon$.

Example 5.1 Let A and $B_1, B_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$A := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B_1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix},$$

$$B_2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \end{bmatrix}.$$

Find a point $\bar{x} = (\bar{x}_1, \bar{x}_2)^\top \in \mathbb{R}^2$ such that $B_1(\bar{x}) = (0,0)^\top$ and $B_2(A\bar{x}) = (0,0)^\top$. Indeed, $\bar{x}_1 = 1$ and $\bar{x}_2 = 0$.

Example 5.2 Let B_1 and B_2 be the same as in Example 5.1. Let

$$A := \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}.$$

Find a point $\bar{x} = (\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2$ such that $B_1(\bar{x}) = (0,0)^\top$ and $B_2(A\bar{x}) = (0,0)^\top$. Indeed, $\bar{x}_1 = 0.5$ and $\bar{x}_2 = -0.5$.

Example 5.3 Let $B_1 : \mathbb{R}^2 \to \mathbb{R}^2$, $B_2 : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$A := \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 2 & 2 \end{bmatrix},$$

$$B_1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \end{bmatrix},$$

$$B_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -2 & -2 \\ -2 & 2 & 2 \\ -2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Find a point $\bar{x} = (\bar{x}_1, \bar{x}_2)^\top \in \mathbb{R}^2$ such that $B_1(\bar{x}) = (0,0)^\top$ and $B_2(A\bar{x}) = (0,0,0)^\top$. Indeed, $\bar{x}_1 = 1.5$ and $\bar{x}_2 = -0.5$.

For the above examples, we give the numerical results (see Tables 1-3) for the proposed algorithm and related algorithms.

Table 1 Numerical results for Example 5.1 ($\rho = \rho_n = 0.01$)

$x_1 = (1,1)^\top$	$\varepsilon = 10^{-3}$			$\varepsilon = 10^{-4}$		
	Time	**Iteration**	**Approximate solution**	**Time**	**Iteration**	**Approximate solution**
Algorithm 1.2	0.02	19	(0.9853714, –0.01460836)	0.02	60	(0.9932032, –0.006789955)
Theorem 1.1	0.01	218	(1.087499, 0.08749895)	0.05	505	(1.008727, 0.008726755)
Theorem 1.2	0.04	213	(1.237467, 0.2433358)	0.08	939	(1.065859, 0.06719048)
Theorem 1.3	0.01	137	(1.376151, 0.3943719)	0.14	1,308	(1.094086, 0.09599743)
Theorem 1.4	0.02	206	(1.083916, 0.08392859)	0.06	484	(1.007433, 0.007437749)
$x_1 = (1,1)^\top$	$\varepsilon = 10^{-5}$			$\varepsilon = 10^{-6}$		
	Time	**Iteration**	**Approximate solution**	**Time**	**Iteration**	**Approximate solution**
Algorithm 1.2	0.06	287	(0.9977524, –0.002246182)	0.25	970	(0.9993371, –0.0006624296)
Theorem 1.1	0.06	792	(1.000870, 0.0008703675)	0.08	1,078	(1.000088, 8.750662e–05)
Theorem 1.2	0.25	2,974	(1.020805, 0.02122562)	0.99	9,403	(1.006580, 0.006713283)
Theorem 1.3	0.34	4,205	(1.029428, 0.03002226)	1.59	13,297	(1.009306, 0.009494477)
Theorem 1.4	0.07	749	(1.000037, 4.018706e–05)	0.07	953	(0.9994309, –5.664470e–04)
$x_1 = (1,1)^\top$	$\varepsilon = 10^{-7}$			$\varepsilon = 10^{-8}$		
	Time	**Iteration**	**Approximate solution**	**Time**	**Iteration**	**Approximate solution**
Algorithm 1.2	0.75	2,999	(0.9997898, –0.0002100474)	2.62	9,426	(0.9999335, –6.645199e–05)
Theorem 1.1	0.11	1,365	(1.000009, 8.727519e–06)	0.14	1,562	(1.000001, 8.704438e–07)
Theorem 1.2	5.03	29,732	(1.002081, 0.002123133)	19.63	94,018	(1.000658, 0.000671414)
Theorem 1.3	8.73	42,047	(1.002943, 0.003002578)	21.03	132,961	(1.000931, 0.0009495180)
Theorem 1.4	0.09	1,034	(0.9994033, –5.942738e–04)	0.09	1,047	(0.9994030, –5.945951e–04)

Table 2 Numerical results for Example 5.2 ($\rho = \rho_n = 0.001$)

$x_1 = (1,1)^\top$	$\varepsilon = 10^{-3}$			$\varepsilon = 10^{-4}$		
	Time	**Iteration**	**Approximate solution**	**Time**	**Iteration**	**Approximate solution**
Algorithm 1.2	≤	20	(0.4872068, –0.5128408)	0.02	61	(0.4953678 – 0.5046371)
Theorem 1.1	0.02	157	(1.382916, 0.3832697)	0.26	3,035	(0.5882673, –0.4116973328)
$x_1 = (1,1)^\top$	$\varepsilon = 10^{-5}$			$\varepsilon = 10^{-6}$		
	Time	**Iteration**	**Approximate solution**	**Time**	**Iteration**	**Approximate solution**
Algorithm 1.2	0.07	209	(0.4985109, –0.5014895)	0.19	716	(0.4996333, –0.5003667)
Theorem 1.1	0.56	5,912	(0.5088314, –0.4911650960)	0.90	8,790	(0.5008829, –0.4991167527)
$x_1 = (1,1)^\top$	$\varepsilon = 10^{-7}$			$\varepsilon = 10^{-8}$		
	Time	**Iteration**	**Approximate solution**	**Time**	**Iteration**	**Approximate solution**
Algorithm 1.2	0.49	1,984	(0.4999414, –0.5000586)	1.04	3,944	(0.4999930, –0.5000070)
Theorem 1.1	1.33	11,668	(0.5000883, –0.4999116996)	1.79	14,545	(0.5000088, –0.4999911653)

Table 3 Numerical results for Example 5.3 ($\rho = \rho_n = 0.001$)

$x_1 = (1,1)^\top$	$\varepsilon = 10^{-3}$			$\varepsilon = 10^{-4}$		
	Time	**Iteration**	**Approximate solution**	**Time**	**Iteration**	**Approximate solution**
Algorithm 1.2	0.02	40	(1.540193, –0.5400902)	0.04	162	(1.515364, –0.5153539)
Theorem 1.1	≤	7	(0.5038810, 0.4956130)	0.33	3,658	(1.3762567, –0.3763273899)
$x_1 = (1,1)^\top$	$\varepsilon = 10^{-5}$			$\varepsilon = 10^{-6}$		
	Time	**Iteration**	**Approximate solution**	**Time**	**Iteration**	**Approximate solution**
Algorithm 1.2	0.19	697	(1.504028, –0.5040270)	0.55	2,233	(1.500311, –0.5003114)
Theorem 1.1	0.76	7,689	(1.4876280, –0.4876350226)	1.38	11,719	(1.4987623, –0.4987630241)
$x_1 = (1,1)^\top$	$\varepsilon = 10^{-7}$			$\varepsilon = 10^{-8}$		
	Time	**Iteration**	**Approximate solution**	**Time**	**Iteration**	**Approximate solution**
Algorithm 1.2	1.06	4,175	(1.499761, –0.4997615)	1.32	5,188	(1.499727, –0.4997275)
Theorem 1.1	2.06	15,750	(1.4998763, –0.4998763253)	2.83	19,781	(1.4999876, –0.4999876348)

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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