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A comparative study on the convergence rate of some iteration methods involving contractive mappings

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Abstract

We compare the rate of convergence for some iteration methods for contractions. We conclude that the coefficients involved in these methods have an important role to play in determining the speed of the convergence. By using Matlab software, we provide numerical examples to illustrate the results. Also, we compare mathematical and computer-calculating insights in the examples to explain the reason of the existence of the old difference between the points of view.

MSC: 47H09; 47H10

Keywords: contractive map; fixed point; iteration method; rate of convergence

1 Introduction

Iteration schemes for numerical reckoning fixed points of various classes of nonlinear operators are available in the literature. The class of contractive mappings via iteration methods is extensively studied in this regard. In 1952, Plunkett published a paper on the rate of convergence for relaxation methods [1]. In 1953, Bowden presented a talk in a symposium on digital computing machines entitled 'Faster than thought' [2]. Later, this basic idea has been used in engineering, statistics, numerical analysis, approximation theory, and physics for many years (see, for example, [3–9] and [10]). In 1991, Argyros published a paper about iterations converging faster than Newton's method to the solutions of nonlinear equations in Banach spaces [11, 12]. In 1997, Lucet presented a method faster than the fast Legendre transform [13]. In 2004, Berinde used the notion of rate of convergence for iterations method and showed that the Picard iteration converges faster than the Mann iteration for a class of quasi-contractive operators [14]. Later, he provided some results in this area [15, 16]. In 2006, Babu and Vara Prasad showed that the Mann iteration converges faster than the Ishikawa iteration for the class of Zamfirescu operators [17]. In 2007, Popescu showed that the Picard iteration converges faster than the Mann iteration for the class of quasi-contractive operators [18]. Recently, there have been published some papers about introducing some new iterations and comparing of the rates of convergence for some iteration methods (see, for example, [19–22] and [23]).

In this paper, we compare the rates of convergence of some iteration methods for contractions and show that the involved coefficients in such methods have an important role to play in determining the rate of convergence. During the preparation of this work, we found that the efficiency of coefficients had been considered in [24] and [25]. But we obtained our results independently, before reading these works, and one can see it by comparing our results and those ones.

2 Preliminaries

As we know, the Picard iteration has been extensively used in many works from different points of view. Let (X,d) be a metric space, $x_0 \in X$, and $T: X \to X$ a selfmap. The Picard iteration is defined by

$$x_{n+1} = Tx_n$$

for all $n \geq 0$. Let $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$, and $\{\gamma_n\}_{n\geq 0}$ be sequences in $[0,1]$. Then the Mann iteration method is defined by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n \tag{2.1}$$

for all $n \geq 0$ (for more information, see [26]). Also, the Ishikawa iteration method is defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n Ty_n, \\ y_n &= (1-\beta_n)x_n + \beta_n Tx_n \end{aligned} \tag{2.2}$$

for all $n \geq 0$ (for more information, see [27]). The Noor iteration method is defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n Ty_n, \\ y_n &= (1-\beta_n)x_n + \beta_n Tz_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n Tx_n \end{aligned} \tag{2.3}$$

for all $n \geq 0$ (for more information, see [28]). In 2007, Agarwal *et al.* defined their new iteration methods by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n &= (1-\beta_n)x_n + \beta_n Tx_n \end{aligned} \tag{2.4}$$

for all $n \geq 0$ (for more information, see [29]). In 2014, Abbas *et al.* defined their new iteration methods by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)Ty_n + \alpha_n Tz_n, \\ y_n &= (1-\beta_n)Tx_n + \beta_n Tz_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n Tx_n \end{aligned} \tag{2.5}$$

for all $n \geq 0$ (for more information, see [30]). In 2014, Thakur *et al.* defined their new iteration methods by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n &= (1-\beta_n)z_n + \beta_n Tz_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n Tx_n \end{aligned} \tag{2.6}$$

for all $n \geq 0$ (for more information, see [23]). Also, the Picard S-iteration was defined by

$$\begin{aligned} x_{n+1} &= Ty_n, \\ y_n &= (1-\beta_n)Tx_n + \beta_n Tz_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n Tx_n \end{aligned} \tag{2.7}$$

for all $n \geq 0$ (for more information, see [20] and [22]).

3 Self-comparing of iteration methods

Now, we are ready to provide our main results for contractive maps. In this respect, we assume that $(X, \|\cdot\|)$ is a normed space, $x_0 \in X$, $T: X \to X$ is a selfmap and $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$ are sequences in $(0,1)$.

The Mann iteration is given by $x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n$ for all $n \geq 0$.

Note that we can rewrite it as $x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n$ for all $n \geq 0$.

We call these cases the first and second forms of the Mann iteration method.

In the next result we show that choosing a type of sequence $\{\alpha_n\}_{n\geq 0}$ in the Mann iteration has a notable role to play in the rate of convergence of the sequence $\{x_n\}_{n\geq 0}$.

Let $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ be two fixed point iteration procedures that converge to the same fixed point p and $\|u_n - p\| \leq a_n$ and $\|v_n - p\| \leq b_n$ for all $n \geq 0$. If the sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ converge to a and b, respectively, and $\lim_{n\to\infty} \frac{\|a_n - a\|}{\|b_n - b\|} = 0$, then we say that $\{u_n\}_{n\geq 0}$ converges faster than $\{v_n\}_{n\geq 0}$ to p (see [14] and [23]).

Proposition 3.1 *Let C be a nonempty, closed, and convex subset of a Banach space X, $x_1 \in C$, $T: C \to C$ a contraction with constant $k \in (0,1)$ and p a fixed point of T. Consider the first case for Mann iteration. If the coefficients of Tx_n are greater than the coefficients of x_n, that is, $1-\alpha_n < \alpha_n$ for all $n \geq 0$ or equivalently $\{\alpha_n\}_{n\geq 0}$ is a sequence in $(\frac{1}{2}, 1)$, then the Mann iteration converges faster than the Mann iteration which the coefficients of x_n are greater than the coefficients of Tx_n.*

Proof Let $\{x_n\}$ be the sequence in the Mann iteration which the coefficients of Tx_n are greater than the coefficients of x_n, that is,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n \tag{3.1}$$

for all n. In this case, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \big\|(1-\alpha_n)x_n + \alpha_n Tx_n - p\big\| \leq (1-\alpha_n)\|x_n - p\| + \alpha_n \|Tx_n - p\| \\ &\leq \big(1 - \alpha_n(1-k)\big)\|x_n - p\| \end{aligned}$$

for all n. Since $\alpha_n \in (\frac{1}{2}, 1)$, $1 - \alpha_n(1-k) < 1 - \frac{1}{2}(1-k)$. Put $a_n = (1 - \frac{1}{2}(1-k))^n \|x_1 - p\|$ for all n. Now, let $\{x_n\}$ be the sequence in the Mann iteration of which the coefficients of x_n are greater than the coefficients of Tx_n. In this case, we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n x_n + (1-\alpha_n)Tx_n - p\| \leq \alpha_n \|x_n - p\| + (1-\alpha_n)\|Tx_n - p\| \\ &\leq \bigl(1 - (1-\alpha_n)(1-k)\bigr)\|x_n - p\|\end{aligned}$$

for all n. Since $1 - \alpha_n < \alpha_n$ for all $n \geq 0$, we get $1 - (1-\alpha_n)(1-k) < 1$ for all $n \geq 0$. Put $b_n = \|x_1 - p\|$ for all n. Note that $\lim \frac{a_n}{b_n} = \lim \frac{(1-\frac{1}{2}(1-k))^n \|x_1 - p\|}{\|x_1 - p\|} = 0$. This completes the proof. □

Note that we can use $1 - \alpha_n < \alpha_n$, for n large enough, instead of the condition $1 - \alpha_n < \alpha_n$, for all $n \geq 0$. One can use similar conditions instead of the conditions which we will use in our results.

As we know, we can consider four cases for writing the Ishikawa iteration method. In the next result, we indicate each case by different enumeration. Similar to the last result, we want to compare the Ishikawa iteration method with itself in the four possible cases. Again, we show that the coefficient sequences $\{\alpha_n\}_{n\geq 0}$ and $\{\beta_n\}_{n\geq 0}$ have effective roles to play in the rate of convergence of the sequence $\{x_n\}_{n\geq 0}$ in the Ishikawa iteration method.

Proposition 3.2 *Let C be a nonempty, closed, and convex subset of a Banach space X, $x_1 \in C$, $T: C \to C$ a contraction with constant $k \in (0,1)$, and p a fixed point of T. Consider the following cases of the Ishikawa iteration method*:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n, \end{cases} \tag{3.2}$$

$$\begin{cases} x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \end{cases} \tag{3.3}$$

$$\begin{cases} x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n, \end{cases} \tag{3.4}$$

and

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n \end{cases} \tag{3.5}$$

for all $n \geq 0$. If $1 - \alpha_n < \alpha_n$ and $1 - \beta_n < \beta_n$ for all $n \geq 0$, then the case (3.2) *converges faster than the others. In fact, the Ishikawa iteration method is faster whenever the coefficients of Ty_n and Tx_n simultaneously are greater than the related coefficients of x_n for all $n \geq 0$.*

Proof Let $\{x_n\}_{n\geq 0}$ be the sequence in the case (3.2). Then we have

$$\begin{aligned}\|y_n - p\| &= \|(1-\beta_n)x_n + \beta_n Tx_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n \|Tx_n - p\| \\ &\leq \bigl((1-\beta_n) + \beta_n k\bigr)\|x_n - p\|\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1}-p\| &= \big\|(1-\alpha_n)x_n+\alpha_n Ty_n-p\big\| \\ &\leq (1-\alpha_n)\|x_n-p\|+\alpha_n\|Ty_n-p\| \\ &\leq (1-\alpha_n)\|x_n-p\|+k\alpha_n\|y_n-p\| \\ &\leq \big(1-\alpha_n+k\alpha_n\big[(1-\beta_n)+\beta_n k\big]\big)\|x_n-p\| \\ &\leq \big(1-\alpha_n+\alpha_n k-\alpha_n\beta_n k+\alpha_n\beta_n k^2\big)\|x_n-p\| \\ &\leq \big(1-\alpha_n(1-k)-\alpha_n\beta_n k(1-k)\big)\|x_n-p\|\end{aligned}$$

for all $n\geq 0$. Since $\alpha_n,\beta_n\in(\frac{1}{2},1)$, $1-\alpha_n(1-k)-\alpha_n\beta_n k(1-k)<1-\frac{1}{2}(1-k)-\frac{1}{4}k(1-k)$ for all $n\geq 0$. Put $a_n=(1-\frac{1}{2}(1-k)-\frac{1}{4}k(1-k))^n\|x_1-p\|$ for all $n\geq 0$. If $\{x_n\}_{n\geq 0}$ is the sequence in the case (3.3), then we get

$$\begin{aligned}\|y_n-p\| &= \big\|\beta_n x_n+(1-\beta_n)Tx_n-p\big\| \\ &\leq \beta_n\|x_n-p\|+(1-\beta_n)\|Tx_n-p\| \\ &\leq \big(1-(1-\beta_n)(1-k)\big)\|x_n-p\|\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1}-p\| &= \big\|\alpha_n x_n+(1-\alpha_n)Ty_n-p\big\| \\ &\leq \alpha_n\|x_n-p\|+(1-\alpha_n)\|Ty_n-p\| \\ &\leq \alpha_n\|x_n-p\|+k(1-\alpha_n)\|y_n-p\| \\ &\leq \big(\alpha_n+k(1-\alpha_n)\big(1-(1-\beta_n)(1-k)\big)\big)\|x_n-p\| \\ &= \big(\alpha_n+(1-\alpha_n)k-k(1-\alpha_n)(1-\beta_n)(1-k)\big)\|x_n-p\| \\ &= \big(1-(1-\alpha_n)(1-k)-(1-\alpha_n)(1-\beta_n)k(1-k)\big)\|x_n-p\|\end{aligned}$$

for all $n\geq 0$. Since $\alpha_n,\beta_n\in(\frac{1}{2},1)$, $1-(1-\alpha_n)(1-k)-(1-\alpha_n)(1-\beta_n)(1-k)<1$ for all $n\geq 0$. Put $b_n=\|x_1-p\|$ for all $n\geq 0$. Since

$$1-\frac{1}{2}(1-k)-\frac{1}{4}k(k-1)<1+\frac{1}{2}k(1-k),$$

we get $\lim\frac{a_n}{b_n}=\lim\frac{(1-\frac{1}{2}(1-k)-\frac{1}{4}k(1-k))^n\|x_1-p\|}{\|x_1-p\|}=0$ and so the iteration (3.2) converges faster than the case (3.3). Now, let $\{x_n\}_{n\geq 0}$ be the sequence in the case (3.4). Then

$$\begin{aligned}\|y_n-p\| &= \big\|\beta x_n+(1-\beta_n)Tx_n-p\big\| \\ &\leq \beta_n\|x_n-p\|+(1-\beta_n)\|Tx_n-p\| \\ &\leq \big(\beta_n+k(1-\beta_n)\big)\|x_n-p\| \\ &= \big(1-(1-\beta_n)(1-k)\big)\|x_n-p\|\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1}-p\| &= \big\|(1-\alpha_n)x_n+\alpha_n Ty_n-p\big\| \\ &\leq (1-\alpha_n)\|x_n-p\|+\alpha_n\|Ty_n-p\| \\ &\leq \big(1-\alpha_n+k\alpha_n\big[\big(1-(1-\beta_n)(1-k)\big)\big]\big)\|x_n-p\| \\ &= \big(1-\alpha_n+k\alpha_n-\alpha_n(1-\beta_n)k(1-k)\big)\|x_n-p\| \\ &= \big(1-\alpha_n(1-k)-\alpha_n(1-\beta_n)k(1-k)\big)\|x_n-p\|\end{aligned}$$

for all $n\geq 0$. Since $\alpha_n,\beta_n\in(\frac{1}{2},1)$ for all $n\geq 0$, $-(1-k)<-\alpha_n(1-k)<-\frac{1}{2}(1-k)$ and $\frac{-1}{2}k(1-k)<-\alpha_n(1-\beta_n)k(1-k)<0$ for all n. Hence,

$$1-\alpha_n(1-k)-\alpha_n(1-\beta_n)k(1-k)<1-\frac{1}{2}(1-k)$$

for all $n\geq 0$. Put $c_n=(1-\frac{1}{2}(1-k))^n\|x_1-p\|$ for all $n\geq 0$. Thus, we obtain

$$\lim\frac{a_n}{c_n}=\lim\frac{(1-\frac{1}{2}(1-k)-\frac{1}{4}k(1-k))^n\|x_1-p\|}{(1-\frac{1}{2}(1-k))^n\|x_1-p\|}=0$$

and so the iteration (3.2) converges faster than the case (3.4). Now, let $\{x_n\}_{n\geq 0}$ be the sequence in the case (3.5). Then we have

$$\begin{aligned}\|y_n-p\| &= \big\|(1-\beta)x_n+\beta_n Tx_n-p\big\| \\ &\leq (1-\beta_n)\|x_n-p\|+\beta_n\|Tx_n-p\| \\ &\leq \big(1-\beta_n(1-k)\big)\|x_n-p\|\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1}-p\| &= \big\|\alpha_n x_n+(1-\alpha_n)Ty_n\big\| \\ &\leq \alpha_n\|x_n-p\|+(1-\alpha_n)\|Ty_n-p\| \\ &\leq \alpha_n\|x_n-p\|+k(1-\alpha_n)\|y_n-p\| \\ &\leq \big(\alpha_n+k(1-\alpha_n)\big[1-\beta_n(1-k)\big]\big)\|x_n-p\| \\ &\leq \big(\alpha_n+k(1-\alpha_n)-(1-\alpha_n)\beta_n k(1-k)\big)\|x_n-p\| \\ &\leq \big(1-(1-\alpha_n)+k(1-\alpha_n)-(1-\alpha_n)\beta_n k(1-k)\big)\|x_n-p\| \\ &\leq \big(1-(1-\alpha_n)(1-k)-(1-\alpha_n)\beta_n k(1-k)\big)\|x_n-p\|\end{aligned}$$

for all $n\geq 0$. Since $\alpha_n,\beta_n\in(\frac{1}{2},1)$ for all n, $-(1-k^2)<-\alpha_n(1-k^2)<-\frac{1}{2}(1-k^2)$, and $-\frac{1}{2}k(1-k)<-(1-\alpha_n)\beta_n k(1-k)<0$ and so

$$1-\alpha_n(1-k)-(1-\alpha_n)\beta_n k(1-k)<1-\frac{1}{2}(1-k)$$

for all $n \geq 0$. Put $d_n = (1 - \frac{1}{2}(1-k))^n \|x_1 - p\|$ for all $n \geq 0$. Then we have

$$\lim \frac{a_n}{d_n} = \lim \frac{(1 - \frac{1}{2}(1-k) - \frac{1}{4}k(1-k))^n \|x_1 - p\|}{(1 - \frac{1}{2}(1-k))^n \|x_1 - p\|} = 0$$

and so the iteration (3.2) converges faster than the case (3.5). □

By using a similar condition, one can show that the iteration (3.5) is faster than the case (3.3).

Now consider eight cases for writing the Noor iteration method. By using a condition, we show that the coefficient sequences $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$, and $\{\gamma_n\}_{n\geq 0}$ have effective roles to play in the rate of convergence of the sequence $\{x_n\}_{n\geq 0}$ in the Noor iteration method. We enumerate the cases of the Noor iteration method during the proof of our next result.

Theorem 3.1 *Let C be a nonempty, closed, and convex subset of a Banach space X, $x_1 \in C$, $T: C \to C$ a contraction with constant $k \in (0,1)$ and p a fixed point of T. Consider the case* (2.3) *of the Noor iteration method*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tz_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n Tx_n \end{cases}$$

for all $n \geq 0$. If $1-\alpha_n < \alpha_n$, $1-\beta_n < \beta_n$, and $1-\gamma_n < \gamma_n$ for all $n \geq 0$, then the iteration (2.3) *is faster than the other possible cases.*

Proof First, we compare the case (2.3) with the following Noor iteration case:

$$\begin{cases} u_{n+1} = (1-\alpha_n)u_n + \alpha_n Tv_n, \\ v_n = (1-\beta_n)u_n + \beta_n Tw_n, \\ w_n = \gamma_n u_n + (1-\gamma_n)Tu_n \end{cases} \tag{3.6}$$

for all $n \geq 0$. Note that

$$\begin{aligned} \|z_n - p\| &= \big\|(1-\gamma_n)x_n + \gamma_n Tx_n - p\big\| \\ &\leq (1-\gamma_n)\|x_n - p\| + k\gamma_n\|x_n - p\| \\ &= \big(1 - (1-k)\gamma_n\big)\|x_n - p\| \end{aligned}$$

and

$$\begin{aligned} \|y_n - p\| &= \big\|(1-\beta_n)x_n + \beta_n Tz_n - p\big\| \\ &\leq (1-\beta_n)\|x_n - p\| + k\beta_n\|z_n - p\| \\ &\leq (1-\beta_n) + k\beta_n\big(\big(1-(1-k)\gamma_n\big)\big)\|x_n - p\| \\ &\leq \big[1 - \beta_n(1-k) - \beta_n\gamma_n k(1-k)\big]\|x_n - p\| \end{aligned}$$

for all $n \geq 0$. Also, we have

$$\begin{aligned}\|x_{n+1}-p\| &= \big\|(1-\alpha_n)x_n+\alpha_n Ty_n-p\big\| \\ &\leq (1-\alpha_n)\|x_n-p\|+k\alpha_n\|y_n-p\| \\ &\leq (1-\alpha_n)\|x_n-p\|+k\alpha_n\big[1-\beta_n(1-k)-\beta_n\gamma_n k(1-k)\big]\|x_n-p\| \\ &\leq \big(1-\alpha_n+k\alpha_n\big(1-\beta_n(1-k)-\beta_n\gamma_n k(1-k)\big)\big)\|x_n-p\| \\ &\leq \big(1-\alpha_n+k\alpha_n-k(1-k)\beta_n\alpha_n-\alpha_n\beta_n\gamma_n k^2(1-k)\big)\|x_n-p\| \\ &\leq \big(1-(1-k)\alpha_n-k(1-k)\beta_n\alpha_n-\alpha_n\beta_n\gamma_n k^2(1-k)\big)\|x_n-p\|\end{aligned}$$

for all $n \geq 0$. Since $\alpha_n, \beta_n, \gamma_n \in (\frac{1}{2}, 1)$ for all n, $-(1-k^2) < -\alpha_n(1-k^2) < -\frac{1}{2}(1-k^2)$, $-k(1-k) < -\alpha_n\beta_n k(1-k) < -\frac{1}{4}k(1-k)$, and

$$-k^2(1-k) < -\alpha_n\beta_n\gamma_n k^2(1-k) < -\frac{1}{8}k^2(1-k)$$

for all n. This implies that

$$1-(1-k)\alpha_n-k(1-k)\beta_n\alpha_n-\alpha_n\beta_n\gamma_n k^2(1-k) < 1-\frac{1}{2}(1-k)-\frac{1}{4}k(1-k)-\frac{1}{8}k^2(1-k)$$

for all n. Put $a_n = (1-\frac{1}{2}(1-k)-\frac{1}{8}k^2(1-k))^n\|x_1-p\|$ for all $n \geq 0$. Now for the sequences $\{u_n\}_{n\geq 0}$ with $u_1 = x_1$ and $\{v_n\}_{n\geq 0}$ in (3.6), we have

$$\begin{aligned}\|w_n-p\| &= \big\|\gamma_n u_n+(1-\gamma_n)Tu_n-p\big\| \\ &\leq \gamma_n\|u_n-p\|+k(1-\gamma_n)\|u_n-p\| \\ &= \big(1-(1-\gamma_n)(1-k)\big)\|u_n-p\|\end{aligned}$$

and

$$\begin{aligned}\|v_n-p\| &= \big\|(1-\beta_n)u_n+\beta_n Tw_n-p\big\| \\ &\leq (1-\beta_n)\|u_n-p\|+k\beta_n\|w_n-p\| \\ &\leq (1-\beta_n)+k\beta_n\big(1-(1-\gamma_n)(1-k)\big)\|u_n-p\| \\ &\leq \big(1-\beta_n+k\beta_n-\beta_n(1-\gamma_n)k(1-k)\big)\|u_n-p\| \\ &\leq \big(1-\beta_n(1-k)-\beta_n(1-\gamma_n)k(1-k)\big)\|u_n-p\|\end{aligned}$$

for all $n \geq 0$. Hence,

$$\begin{aligned}\|u_{n+1}-p\| &= \big\|(1-\alpha_n)u_n+\alpha_n Tv_n-p\big\| \\ &\leq (1-\alpha_n)\|u_n-p\|+k\alpha_n\|v_n-p\| \\ &\leq (1-\alpha_n)\|u_n-p\|+k\alpha_n\big(1-\beta_n(1-k)-\beta_n(1-\gamma_n)k(1-k)\big)\|u_n-p\| \\ &\leq \big((1-\alpha_n)+k\alpha_n-\alpha_n\beta_n k(1-k)-\alpha\beta_n(1-\gamma_n)K^2(1-k)\big)\|u_n-p\| \\ &\leq \big(1-\alpha_n(1-k)-\alpha_n\beta_n k(1-k)-\alpha\beta_n(1-\gamma_n)K^2(1-k)\big)\|u_n-p\|\end{aligned}$$

for all n. Since $\alpha_n, \beta_n, \gamma_n \in (\frac{1}{2}, 1)$ for all n, $-k(1-k) < -\alpha_n\beta_n k(1-k) < -\frac{1}{4}k(1-k)$ and $\frac{1}{2}k^2(1-k) < -\alpha_n\beta_n(1-\gamma_n)k^2(1-k) < 0$ for all n. Hence,

$$1-\alpha_n(1-k)-\alpha_n\beta_n k(1-k)-\alpha\beta_n(1-\gamma_n)k^2(1-k) < 1-\frac{1}{2}(1-k)-\frac{1}{4}k(1-k)$$

for all n. Put $b_n = (1-\frac{1}{2}(1-k)-\frac{1}{4}k(1-k))^n\|u_1-p\|$ for all $n \geq 0$. Then we have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{(1-\frac{1}{2}(1-k)-\frac{1}{4}k(1-k)-\frac{1}{8}k^2(1-k))^n\|x_1-p\|}{(1-\frac{1}{2}(1-k)-\frac{1}{4}k(1-k))^n\|u_1-p\|} = 0.$$

Thus, $\{x_n\}_{n\geq 0}$ converges faster than the sequence $\{u_n\}_{n\geq 0}$. Now, we compare the case (2.3) with the following Noor iteration case:

$$\begin{cases} u_{n+1} = (1-\alpha_n)u_n + \alpha_n Tv_n, \\ v_n = \beta_n u_n + (1-\beta_n)Tw_n, \\ w_n = (1-\gamma_n)u_n + \gamma_n Tu_n \end{cases} \tag{3.7}$$

for all $n \geq 0$. Note that

$$\begin{aligned} \|w_n-p\| &= \big\|(1-\gamma_n)u_n+\gamma_n Tu_n-p\big\| \\ &\leq (1-\gamma_n)\|u_n-p\| + k\gamma_n\|u_n-p\| \\ &= \big(1-(1-k)\gamma_n\big)\|u_n-p\| \end{aligned}$$

and

$$\begin{aligned} \|v_n-p\| &= \big\|\beta_n u_n+(1-\beta_n)Tw_n-p\big\| \\ &\leq \beta_n\|u_n-p\| + k(1-\beta_n)\|w_n-p\| \\ &\leq \big(\beta_n+k(1-\beta_n)-\beta_n\gamma_n k(1-k)\big)\|u_n-p\| \\ &\leq \big(1-(1-k)(1-\beta_n)-\beta_n\gamma_n k(1-k)\big)\|u_n-p\| \end{aligned}$$

for all $n \geq 0$. Hence,

$$\begin{aligned} \|u_{n+1}-p\| &= \big\|(1-\alpha_n)u_n+\alpha_n Tv_n-p\big\| \\ &\leq (1-\alpha_n)\|u_n-p\| + k\alpha_n\|w_n-p\| \\ &\leq (1-\alpha_n)\|u_n-p\| + k\alpha_n\big(1-(1-k)(1-\beta_n)-\beta_n\gamma_n k(1-k)\big)\|u_n-p\| \\ &\leq \big((1-\alpha_n)+k\alpha_n-k(1-k)\alpha_n(1-\beta_n)-\alpha_n\beta_n\gamma_n k^2(1-k)\big)\|u_n-p\| \\ &\leq \big(1-(1-k)\alpha_n-\alpha_n(1-\beta_n)k(1-k)-\alpha_n\beta_n\gamma_n k^2(1-k)\big)\|u_n-p\| \end{aligned}$$

for all $n \geq 0$. Since $\alpha_n, \beta_n, \gamma_n \in (\frac{1}{2}, 1)$ for all n, $-\frac{1}{2}k(1-k) < -\alpha_n(1-\beta_n)k(1-k) < 0$, and $-k^2(1-k) < -\alpha_n\beta_n(1-\gamma_n)k^2(1-k) < -\frac{1}{8}k^2(1-k)$ and so

$$1-(1-k)\alpha_n-\alpha_n(1-\beta_n)k(1-k)-\alpha_n\beta_n\gamma_n k^2(1-k) < 1-\frac{1}{2}(1-k)-\frac{1}{8}k^2(1-k)$$

for all n. Put $c_n = (1 - \frac{1}{2}(1-k) - \frac{1}{8}k^2(1-k))^n \|u_1 - p\|$ for all $n \geq 0$. Then we have

$$\lim_{n\to\infty} \frac{a_n}{c_n} = \frac{(1 - \frac{1}{2}(1-k) - \frac{1}{4}k(1-k) - \frac{1}{8}k^2(1-k))^n \|x_1 - p\|}{(1 - \frac{1}{2}(1-k) - \frac{1}{8}k^2(1-k))^n \|u_1 - p\|} = 0.$$

Thus, $\{x_n\}_{n\geq 0}$ converges faster than the sequence $\{u_n\}_{n\geq 0}$. Now, we compare the case (2.3) with the following Noor iteration case:

$$\begin{cases} u_{n+1} = (1-\alpha_n)u_n + \alpha_n T v_n, \\ v_n = \beta_n u_n + (1-\beta_n) T w_n, \\ w_n = \gamma_n u_n + (1-\gamma_n) T u_n \end{cases} \tag{3.8}$$

for all $n \geq 0$. Note that

$$\begin{aligned} \|w_n - p\| &= \big\| \gamma_n u_n + (1-\gamma_n) T u_n - p \big\| \\ &\leq \gamma_n \|u_n - p\| + k(1-\gamma_n)\|u_n - p\| \\ &= \big(1 - (1-\gamma_n)(1-k)\big)\|u_n - p\| \end{aligned}$$

and

$$\begin{aligned} \|v_n - p\| &= \big\| (1-\beta_n)u_n + \beta_n T w_n - p \big\| \\ &\leq (1-\beta_n)\|u_n - p\| + k\beta_n \|w_n - p\| \\ &\leq \big(1 - \beta_n + k\beta_n\big(1 - (1-\gamma_n)(1-k)\big)\big)\|u_n - p\| \\ &\leq \big(1 - \beta_n + k\beta_n - \beta_n(1-\gamma_n)k(1-k)\big)\|u_n - p\| \\ &\leq \big(1 - \beta_n(1-k) - \beta_n(1-\gamma_n)k(1-k)\big)\|u_n - p\| \end{aligned}$$

and so

$$\begin{aligned} \|u_{n+1} - p\| &= \big\| (1-\alpha_n)u_n + \alpha_n T v_n - p \big\| \\ &\leq (1-\alpha_n)\|u_n - p\| + k\alpha_n \|w_n - p\| \\ &\leq (1-\alpha_n)\|u_n - p\| + k\alpha_n\big(1 - \beta_n(1-k) - \beta_n(1-\gamma_n)k(1-k)\big)\|u_n - p\| \\ &\leq \big(1 - \alpha_n + k\alpha_n - \alpha_n\beta_n k(1-k) - \alpha_n\beta_n(1-\gamma_n)k^2(1-k)\big)\|u_n - p\| \\ &\leq \big(1 - (1-k)\alpha_n - \alpha_n\beta_n k(1-k) - \alpha_n\beta_n(1-\gamma_n)k^2(1-k)\big)\|u_n - p\| \end{aligned}$$

for all n. Since $\alpha_n, \beta_n, \gamma_n \in (\frac{1}{2}, 1)$ for all n, $-k(1-k) < -\alpha_n\beta_n k(1-k) < -\frac{1}{4}k(1-k)$, and $-\frac{1}{2}k^2(1-k) < -\alpha_n\beta_n(1-\gamma_n)k^2(1-k) < 0$ for all n. This implies that

$$1 - (1-k)\alpha_n - \alpha_n\beta_n k(1-k) - \alpha_n\beta_n(1-\gamma_n)k^2(1-k) < 1 - \frac{1}{2}(1-k) - \frac{1}{4}k(1-k)$$

for all n. Put $d_n = (1 - \frac{1}{2}(1-k) - \frac{1}{4}k(1-k))^n \|u_1 - p\|$ for all $n \geq 0$. Then we get

$$\lim_{n\to\infty} \frac{a_n}{d_n} = \frac{(1 - \frac{1}{2}(1-k) - \frac{1}{4}k(1-k) - \frac{1}{8}k^2(1-k))^n \|x_1 - p\|}{(1 - \frac{1}{2}(1-k) - \frac{1}{4}k(1-k))^n \|u_1 - p\|} = 0$$

and so the sequence $\{x_n\}_{n\geq 0}$ converges faster than the sequence $\{u_n\}_{n\geq 0}$. By using similar proofs, one can show that the case (2.3) is faster than the following cases of the Noor iteration method:

$$\begin{cases} u_{n+1} = \alpha_n u_n + (1-\alpha_n)Tv_n, \\ v_n = (1-\beta_n)u_n + \beta_n Tw_n, \\ w_n = (1-\gamma_n)u_n + \gamma_n Tu_n, \end{cases} \tag{3.9}$$

$$\begin{cases} u_{n+1} = \alpha_n u_n + (1-\alpha_n)Tv_n, \\ v_n = (1-\beta_n)u_n + \beta_n Tw_n, \\ w_n = \gamma_n u_n + (1-\gamma_n)Tu_n, \end{cases} \tag{3.10}$$

$$\begin{cases} u_{n+1} = \alpha_n u_n + (1-\alpha_n)Tv_n, \\ v_n = \beta_n u_n + (1-\beta_n)Tw_n, \\ w_n = (1-\gamma_n)u_n + \gamma_n Tu_n, \end{cases} \tag{3.11}$$

and

$$\begin{cases} u_{n+1} = \alpha_n u_n + (1-\alpha_n)Tv_n, \\ v_n = \beta_n u_n + (1-\beta_n)Tw_n, \\ w_n = \gamma_n u_n + (1-\gamma_n)Tu_n \end{cases} \tag{3.12}$$

for all $n \geq 0$. This completes the proof. □

By using similar conditions, one can show that the case (3.7) converges faster than (3.8), (3.9) converges faster than (3.11), (3.11) converges faster than (3.10) and (3.10) converges faster than (3.12).

As we know, the Agarwal iteration method could be written in the following four cases:

$$\begin{cases} x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n, \end{cases} \tag{3.13}$$

$$\begin{cases} x_{n+1} = \alpha_n Tx_n + (1-\alpha_n)Ty_n, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \end{cases} \tag{3.14}$$

$$\begin{cases} x_{n+1} = \alpha_n Tx_n + (1-\alpha_n)Ty_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n, \end{cases} \tag{3.15}$$

and

$$\begin{cases} x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n \end{cases} \tag{3.16}$$

for all $n \geq 0$. One can easily show that the case (3.13) converges faster than the other ones for contractive maps. We record it as the next lemma.

Lemma 3.1 *Let C be a nonempty, closed, and convex subset of a Banach space X, $x_1 \in C$, $T: C \to C$ a contraction with constant $k \in (0,1)$ and p a fixed point of T. If $1-\alpha_n < \alpha_n$ and $1-\beta_n < \beta_n$ for all $n \geq 0$, then the case* (3.13) *converges faster than* (3.14), (3.15), *and* (3.16).

Also by using a similar condition, one can show that the case (3.16) converges faster than (3.14). Similar to Theorem 3.1, we can prove that for contractive maps one case in the Abbas iteration method converges faster than the other possible cases whenever the elements of the sequences $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$, and $\{\gamma_n\}_{n\geq 0}$ are in $(\frac{1}{2}, 1)$ for sufficiently large n. Also, one can show that for contractive maps the case (2.6) of the Thakur-Thakur-Postolache iteration method converges faster than the other possible cases whenever elements of the sequences $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$, and $\{\gamma_n\}_{n\geq 0}$ are in $(\frac{1}{2}, 1)$ for sufficiently large n. We record these results as follows.

Lemma 3.2 *Let C be a nonempty, closed, and convex subset of a Banach space X, $u_1 \in C$, $T: C \to C$ a contraction with constant $k \in (0,1)$, and p a fixed point of T. Consider the following case in the Abbas iteration method*:

$$\begin{cases} u_{n+1} = \alpha_n Tv_n + (1-\alpha_n)Tw_n, \\ v_n = (1-\beta_n)Tu_n + \beta_n Tw_n, \\ w_n = (1-\gamma_n)u_n + \gamma_n Tu_n \end{cases} \tag{3.17}$$

for all n. If $1-\alpha_n < \alpha_n$, $1-\beta_n < \beta_n$, and $1-\gamma_n < \gamma_n$ for sufficiently large n, then the case (3.17) *converges faster than the other possible cases.*

Also by using similar conditions in the Abbas iteration method, one can show that the cases

$$\begin{cases} u_{n+1} = \alpha_n Tv_n + (1-\alpha_n)Tw_n, \\ v_n = \beta_n Tu_n + (1-\beta_n)Tw_n, \\ w_n = (1-\gamma_n)u_n + \gamma_n Tu_n \end{cases} \tag{3.18}$$

and

$$\begin{cases} u_{n+1} = \alpha_n Tv_n + (1-\alpha_n)Tw_n, \\ v_n = (1-\beta_n)Tu_n + \beta_n Tw_n, \\ w_n = \gamma_n u_n + (1-\gamma_n)Tu_n \end{cases} \tag{3.19}$$

converge faster than the case

$$\begin{cases} u_{n+1} = \alpha_n Tv_n + (1-\alpha_n)Tw_n, \\ v_n = \beta_n Tu_n + (1-\beta_n)Tw_n, \\ w_n = \gamma_n u_n + (1-\gamma_n)Tu_n. \end{cases} \tag{3.20}$$

Also the case

$$\begin{cases} u_{n+1} = (1-\alpha_n)Tv_n + \alpha_n Tw_n, \\ v_n = (1-\beta_n)Tu_n + \beta_n Tw_n, \\ w_n = (1-\gamma_n)u_n + \gamma_n Tu_n \end{cases} \tag{3.21}$$

converges faster than the cases

$$\begin{cases} u_{n+1} = (1-\alpha_n)Tv_n + \alpha_n Tw_n, \\ v_n = \beta_n Tu_n + (1-\beta_n)Tw_n, \\ w_n = (1-\gamma_n)u_n + \gamma_n Tu_n \end{cases} \tag{3.22}$$

and

$$\begin{cases} u_{n+1} = (1-\alpha_n)Tv_n + \alpha_n Tw_n, \\ v_n = (1-\beta_n)Tu_n + \beta_n Tw_n, \\ w_n = \gamma_n u_n + (1-\gamma_n)Tu_n, \end{cases} \tag{3.23}$$

and

$$\begin{cases} u_{n+1} = (1-\alpha_n)Tv_n + \alpha_n Tw_n, \\ v_n = \beta_n Tu_n + (1-\beta_n)Tw_n, \\ w_n = \gamma_n u_n + (1-\gamma_n)Tu_n. \end{cases} \tag{3.24}$$

Lemma 3.3 *Let C be a nonempty, closed, and convex subset of a Banach space X, $u_1 \in C$, $T: C \to C$ a contraction with constant $k \in (0,1)$ and p a fixed point of T. If $1-\alpha_n < \alpha_n$, $1-\beta_n < \beta_n$, and $1-\gamma_n < \gamma_n$ for sufficiently large n, then the case* (2.6) *in the Thakur-Thakur-Postolache iteration method converges faster than the other possible cases.*

Also by using similar conditions, one can show that the cases

$$\begin{cases} u_{n+1} = (1-\alpha_n)Tu_n + \alpha_n Tv_n, \\ v_n = \beta_n w_n + (1-\beta_n)Tw_n, \\ w_n = (1-\gamma_n)u_n + \gamma_n Tu_n \end{cases} \tag{3.25}$$

and

$$\begin{cases} u_{n+1} = (1-\alpha_n)Tu_n + \alpha_n Tv_n, \\ v_n = (1-\beta_n)w_n + \beta_n Tw_n, \\ w_n = \gamma_n u_n + (1-\gamma_n)Tu_n \end{cases} \tag{3.26}$$

converge faster than the case

$$\begin{cases} u_{n+1} = (1-\alpha_n)Tu_n + \alpha_n Tv_n, \\ v_n = \beta_n w_n + (1-\beta_n)Tw_n, \\ w_n = \gamma_n u_n + (1-\gamma_n)Tu_n. \end{cases} \tag{3.27}$$

Also the case

$$\begin{cases} u_{n+1} = \alpha_n Tu_n + (1-\alpha_n)Tv_n, \\ v_n = (1-\beta_n)w_n + \beta_n Tw_n, \\ w_n = (1-\gamma_n)u_n + \gamma_n Tu_n \end{cases} \tag{3.28}$$

converges faster than the cases

$$\begin{cases} u_{n+1} = \alpha_n Tu_n + (1-\alpha_n)Tv_n, \\ v_n = \beta_n w_n + (1-\beta_n)Tw_n, \\ w_n = (1-\gamma_n)u_n + \gamma_n Tu_n \end{cases} \tag{3.29}$$

and

$$\begin{cases} u_{n+1} = \alpha_n Tu_n + (1-\alpha_n)Tv_n, \\ v_n = (1-\beta_n)w_n + \beta_n Tw_n, \\ w_n = \gamma_n u_n + (1-\gamma_n)Tu_n, \end{cases} \tag{3.30}$$

and

$$\begin{cases} u_{n+1} = \alpha_n Tu_n + (1-\alpha_n)Tv_n, \\ v_n = \beta_n w_n + (1-\beta_n)Tw_n, \\ w_n = \gamma_n u_n + (1-\gamma_n)Tu_n. \end{cases} \tag{3.31}$$

Finally, we have a similar situation for the Picard S-iteration which we record here.

Lemma 3.4 *Let C be a nonempty, closed, and convex subset of a Banach space X, $x_1 \in C$, $T: C \to C$ a contraction with constant $k \in (0,1)$ and p a fixed point of T. If $1-\alpha_n < \alpha_n$ and $1-\beta_n < \beta_n$ for sufficiently large n, then the case* (2.7) *in the Picard S-iteration method converges faster than the other possible cases.*

4 Comparing different iterations methods

In this section, we compare the rate of convergence of some different iteration methods for contractive maps. Our goal is to show that the rate of convergence relates to the coefficients.

Theorem 4.1 *Let C be a nonempty, closed, and convex subset of a Banach space X, $u_1 \in C$, $T: C \to C$ a contraction with constant $k \in (0,1)$ and p a fixed point of T. Consider the case* (2.5) *in the Abbas iteration method*

$$\begin{cases} u_{n+1} = (1-\alpha_n)Tv_n + \alpha_n Tw_n, \\ v_n = (1-\beta_n)Tu_n + \beta_n Tw_n, \\ w_n = (1-\gamma_n)u_n + \gamma_n Tu_n, \end{cases}$$

the case (3.17) *in the Abbas iteration method*

$$\begin{cases} u_{n+1} = \alpha_n Tv_n + (1-\alpha_n)Tw_n, \\ v_n = (1-\beta_n)Tu_n + \beta_n Tw_n, \\ w_n = (1-\gamma_n)u_n + \gamma_n Tu_n, \end{cases}$$

and the case (2.6) *in the Thakur-Thakur-Postolache iteration method*

$$\begin{cases} u_{n+1} = (1-\alpha_n)Tu_n + \alpha_n Tv_n, \\ v_n = (1-\beta_n)w_n + \beta_n Tw_n, \\ w_n = (1-\gamma_n)u_n + \gamma_n Tu_n \end{cases}$$

for all $n \geq 0$. If $1-\alpha_n < \alpha_n$, $1-\beta_n < \beta_n$, and $1-\gamma_n < \gamma_n$ for sufficiently large n, then the case (3.17) *in the Abbas iteration method converges faster than the case* (2.6) *in the Thakur-Thakur-Postolache iteration method. Also, the case* (2.6) *in the Thakur-Thakur-Postolache iteration method is faster than the case* (2.5) *in the Abbas iteration method.*

Proof Let $\{u_n\}_{n\geq 0}$ be the sequence in the case (3.17). Then we have

$$\begin{aligned}\|w_n - p\| &= \big\|(1-\gamma_n)u_n + \gamma_n Tu_n - p\big\| \\ &\leq (1-\gamma_n)\|u_n - p\| + k\gamma_n\|u_n - p\| \\ &= \big(1-(1-k)\gamma_n\big)\|u_n - p\|, \\ \|v_n - p\| &= \big\|(1-\beta_n)Tu_n + \beta_n Tw_n - p\big\| \\ &\leq k(1-\beta_n)\|u_n - p\| + k\beta_n\|w_n - p\| \\ &\leq k\big[(1-\beta_n) + \beta_n\big(1-(1-k)\gamma_n\big)\big]\|u_n - p\| \\ &\leq k\big[1-\beta_n\gamma_n(1-k)\big]\|u_n - p\|,\end{aligned}$$

and

$$\begin{aligned}\|u_{n+1} - p\| &= \big\|\alpha_n Tv_n + (1-\alpha_n)Tw_n - p\big\| \\ &\leq \alpha_n k\|v_n - p\| + k\alpha_n\|w_n - p\| \\ &\leq \alpha_n k^2\big(1-\beta_n\gamma_n(1-k)\big)\|u_n - p\| + k(1-\alpha_n)\big(1-(1-k)\gamma_n\big)\|u_n - p\| \\ &\leq k\big[k\alpha_n - \alpha_n\beta_n\gamma_n k(1-k) + (1-\alpha_n)\big(1-(1-k)\gamma_n\big)\big]\|u_n - p\| \\ &= k\big[k\alpha_n - \alpha_n\beta_n\gamma_n k(1-k) + 1 - \alpha_n - (1-\alpha_n)\gamma_n(1-k)\big]\|u_n - p\| \\ &= k\big[1-\alpha_n(1-k) - (1-\alpha_n)\gamma_n(1-k) - \alpha_n\beta_n\gamma_n k(1-k)\big]\|u_n - p\|\end{aligned}$$

for all n. Since $\alpha_n, \beta_n, \gamma_n \in (\frac{1}{2}, 1)$ for sufficiently large n, we have

$$-(1-k) < -\alpha_n(1-k) < -\frac{1}{2}(1-k),$$

$-\frac{1}{2}(1-k) < -\alpha_n\gamma_n(1-k) < 0$, and $-k(1-k) < -\alpha_n\beta_n\gamma_n k(1-k) < -\frac{1}{8}k(1-k)$ for sufficiently large n. Hence,

$$1-\alpha_n(1-k) - (1-\alpha_n)\gamma_n(1-k) - \alpha_n\beta_n\gamma_n k(1-k) < 1 - \frac{1}{2}(1-k) - \frac{1}{8}k(1-k)$$

for sufficiently large n. Put $a_n = k^n(1-\frac{1}{2}(1-k) - \frac{1}{8}k(1-k))^n\|u_1 - p\|$ for all n. Now, let $\{u_n\}_{n\geq 0}$ be the sequence in the case (2.6). Then we have

$$\begin{aligned}\|w_n - p\| &= \big\|(1-\gamma_n)u_n + \gamma_n Tu_n - p\big\| \\ &\leq (1-\gamma_n)\|u_n - p\| + k\gamma_n\|u_n - p\| \\ &= \big(1-(1-k)\gamma_n\big)\|u_n - p\|, \\ \|v_n - p\| &= \big\|(1-\beta_n)w_n + \beta_n Tw_n - p\big\| \\ &\leq (1-\beta_n)\|u_n - p\| + k\beta_n\|w_n - p\| \\ &\leq (1-\beta_n)\big(1-(1-k)\gamma_n\big) + k\beta_n\big(\big(1-(1-k)\gamma_n\big)\big)\|u_n - p\| \\ &\leq \big[1-\beta_n(1-k)\big]\big[1-\gamma_n(1-k)\big]\|u_n - p\|,\end{aligned}$$

and

$$\begin{aligned}\|u_{n+1}-p\| &= \big\|(1-\alpha_n)Tu_n+\alpha_n Tv_n-p\big\| \\ &\le (1-\alpha_n)k\|u_n-p\|+k\alpha_n\|v_n-p\| \\ &\le k(1-\alpha_n)\|u_n-p\|+k\alpha_n\big[1-\beta_n(1-k)\big]\big[1-\gamma_n(1-k)\big]\|u_n-p\| \\ &\le k\big[1-\alpha_n+\alpha_n\big(1-\beta_n(1-k)\big)\big(1-\gamma_n(1-k)\big)\big]\|u_n-p\| \\ &\le k\big[1-\alpha_n+\big(\alpha_n-(1-k)\beta_n\alpha_n\big)\big((1-\gamma_n)+k\gamma_n\big)\big]\|u_n-p\| \\ &\le k\big[1-\alpha_n+\alpha_n(1-\gamma_n)+\alpha_n\gamma_n k-\beta_n\alpha_n(1-\gamma_n)(1-k) \\ &\quad -\alpha_n\beta_n\gamma_n k(1-k)\big]\|u_n-p\| \\ &\le k\big[1-\alpha_n\gamma_n(1-k)-\alpha_n\beta_n(1-\gamma_n)(1-k)-\alpha_n\beta_n\gamma_n k(1-k)\big]\|u_n-p\|\end{aligned}$$

for all n. Since $\alpha_n, \beta_n, \gamma_n \in (\frac{1}{2}, 1)$ for sufficiently large n, we have

$$-(1-k) < -\alpha_n\gamma_n(1-k) < -\frac{1}{4}(1-k),$$

$-\frac{1}{2}(1-k) < -\alpha_n\beta_n(1-\gamma_n)(1-k) < 0$, and $-k(1-k) < -\alpha_n\beta_n\gamma_n k(1-k) < -\frac{1}{8}k(1-k)$ for sufficiently large n. Hence,

$$1-\alpha_n\gamma_n(1-k)-\alpha_n\beta_n(1-\gamma_n)(1-k)-\alpha_n\beta_n\gamma_n k(1-k) < 1-\frac{1}{4}(1-k)-\frac{1}{8}k(1-k)$$

for sufficiently large n. Put $b_n = k^n(1-\frac{1}{4}(1-k)-\frac{1}{8}k(1-k))^n\|u_1-p\|$ for all n. Then

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{k^n(1-\frac{1}{2}(1-k)-\frac{1}{8}k(1-k))^n\|u_1-p\|}{k^n(1-\frac{1}{4}(1-k)-\frac{1}{8}k(1-k))^n\|u_1-p\|} = 0.$$

Thus, the case (3.17) in the Abbas iteration method converges faster than the case (2.6) in the Thakur-Thakur-Postolache iteration method.

Now for the case (2.5), we have

$$\begin{aligned}\|w_n-p\| &= \|1-\gamma_n u_n+\gamma_n Tu_n-p\| \\ &\le (1-\gamma_n)\|u_n-p\|+k\gamma_n\|u_n-p\| \\ &= \big(1-(1-k)\gamma_n\big)\|u_n-p\|, \\ \|v_n-p\| &= \big\|(1-\beta_n)Tu_n+\beta_n Tw_n-p\big\| \\ &\le k(1-\beta_n)\|u_n-p\|+k\beta_n\|w_n-p\| \\ &\le k\big[(1-\beta_n)+\beta_n\big(1-(1-k)\gamma_n\big)\big]\|u_n-p\| \\ &\le k\big[1-\beta_n\gamma_n(1-k)\big]\|u_n-p\|,\end{aligned}$$

and

$$\begin{aligned}\|u_{n+1}-p\| &= \big\|(1-\alpha_n)Tv_n+\alpha_n Tw_n-p\big\| \\ &\le (1-\alpha_n)k\|v_n-p\|+k\alpha_n\|w_n-p\|\end{aligned}$$

$$\le (1-\alpha_n)k^2\big(1-\beta_n\gamma_n(1-k)\big)\|u_n-p\| + k\alpha_n\big(1-(1-k)\gamma_n\big)\|u_n-p\|$$
$$\le k\big[(1-\alpha_n)k-(1-\alpha_n)\beta_n\gamma_n k(1-k)+\alpha_n-\alpha_n\gamma_n(1-k)\big]\|u_n-p\|$$
$$\le k\big[1-(1-\alpha_n)(1-k)-\alpha_n\gamma_n(1-k)-(1-\alpha_n)\beta_n\gamma_n k(1-k)\big]\|u_n-p\|$$

for all n. Since $\alpha_n, \beta_n, \gamma_n \in (\frac{1}{2}, 1)$ for sufficiently large n, $-\frac{1}{2}(1-k) < -(1-\alpha_n)(1-k) < 0$, $-(1-k) < -\alpha_n\gamma_n(1-k) < -\frac{1}{4}(1-k)$, and $-\frac{1}{2}k(1-k) < -(1-\alpha_n)\beta_n\gamma_n k(1-k) < 0$ for sufficiently large n. Hence,

$$1-(1-\alpha_n)(1-k)-\alpha_n\gamma_n(1-k)-(1-\alpha_n)\beta_n\gamma_n k(1-k) < 1-\frac{1}{4}(1-k)$$

for sufficiently large n. Put $c_n = k^n(1-\frac{1}{4}(1-k))^n\|x_1-p\|$ for all n. Then we have

$$\lim_{n\to\infty}\frac{b_n}{c_n} = \frac{k^n(1-\frac{1}{4}(1-k)-\frac{1}{8}k(1-k))^n\|u_1-p\|}{k^n(1-\frac{1}{4}(1-k))^n\|u_1-p\|} = 0$$

and so the case (2.6) in the Thakur-Thakur-Postolache iteration method is faster than the case (2.5) in the Abbas iteration method. □

By using a similar proof, we can compare the Thakur-Thakur-Postolache and the Agarwal iteration methods as follows.

Theorem 4.2 *Let C be a nonempty, closed, and convex subset of a Banach space X, $x_1 \in C$, $T: C \to C$ a contraction with constant $k \in (0,1)$ and p a fixed point of T. If $1-\alpha_n < \alpha_n$, $1-\beta_n < \beta_n$, and $1-\gamma_n < \gamma_n$ for sufficiently large n, then the case* (2.6) *in the Thakur-Thakur-Postolache iteration method converges faster than the case* (2.4) *in the Agarwal iteration method and the case* (2.4) *in the Agarwal iteration method is faster than the cases* (3.29) *and* (3.30) *in the Thakur-Thakur-Postolache iteration method.*

Also by using similar proofs, we can compare some another iteration methods. We record those as follows.

Theorem 4.3 *Let C be a nonempty, closed, and convex subset of a Banach space X, $x_1 \in C$, $T: C \to C$ a contraction with constant $k \in (0,1)$, and p a fixed point of T. If $1-\alpha_n < \alpha_n$, $1-\beta_n < \beta_n$, and $1-\gamma_n < \gamma_n$ for sufficiently large n, then the case* (2.3) *in the Abbas iteration method converges faster than the case* (2.2) *in the Ishikawa iteration method and the case* (2.2) *in the Ishikawa iteration method is faster than the cases* (3.11) *and* (3.12) *in the Abbas iteration method.*

It is notable that there are some cases which the coefficients have no effective roles to play in the rate of convergence. By using similar proofs, one can check the next result. One can obtain some similar cases. This shows us that researchers should stress more the probability of the efficiency of coefficients in the rate of convergence for iteration methods.

Theorem 4.4 *Let C be a nonempty, closed, and convex subset of a Banach space X, $x_1 \in C$, $T: C \to C$ a contraction with constant $k \in (0,1)$, p a fixed point of T, and $\alpha_n, \beta_n, \gamma_n \in (0,1)$ for all $n \geq 0$. Then the case* (2.4) *in the Agarwal iteration method is faster than the case* (2.1) *in the Mann iteration method, the case* (2.5) *in the Abbas iteration method is faster than the case* (2.1) *in the Mann iteration method, the case* (2.6) *in the Thakur-Thakur-Postolache iteration method is faster than the case* (2.1) *in the Mann iteration method, the case* (2.4) *in the Agarwal iteration method is faster than the case* (2.2) *in the Ishikawa iteration method, the case* (2.5) *in the Abbas iteration method is faster than the case* (2.2) *in the Ishikawa iteration method and the case* (2.6) *in the Thakur-Thakur-Postolache iteration method is faster than the case* (2.2) *in the Ishikawa iteration method*.

5 Examples and figures

In this section, we provide some examples to illustrate our results.

Example 1 Let $X = \mathbb{R}$, $C = [1, 60]$, $x_0 = 20$, $\alpha_n = 0.7$, and $\beta_n = 0.85$ for all $n \geq 0$. Define the map $T: C \to C$ by the formula $T(x) = (3x+18)^{\frac{1}{3}}$ for all $x \in C$. It is easy to see that T is a contraction. In Tables 1-3, we first compare two cases of the Mann iteration method and also four cases of the Ishikawa and Agarwal iteration methods separately. From a mathematical point of view, one can see that the Mann iteration (3.1) is more than 2.82 times faster than the Mann iteration (2.1), the Ishikawa iteration (3.2) is more than 1.07 times faster than the Ishikawa iteration (3.4), the Ishikawa iteration (3.2) is more than 11.33 times faster than the Ishikawa iteration (3.3), the Ishikawa iteration (3.2) is more than 11 times faster

Table 1 Cases of Mann iteration

Step	Mann (2.1)	Mann (3.1)
1	15.2817976045	8.9908610772
2	11.8962912491	5.186577882
3	9.4591508761	3.8138707904
4	7.6992520365	3.305644632
5	6.4247631019	3.1152016077
6	5.4994648986	3.0434826465
7	4.8262347919	3.0164213456
8	4.3355308466	3.0062028434
9	3.977352589	3.0023431856
10	3.7156123245	3.0008851876
11	3.5241766763	3.000334402
12	3.3840675849	3.0001263293
13	3.2814716521	3.0000477244
14	3.2063163994	3.0000180292
15	3.1512468009	3.000006811
16	3.11088634	3.0000025731
17	3.0813015724	3.000000972
18	3.0596130334	3.0000003672
19	3.0437118532	3.0000001387
20	3.0320530065	3.0000000524
21	3.0235042722	3.0000000198
22	3.0172357852	3.0000000075
23	3.0126392095	3.0000000028
24	3.0092685565	
25	3.0067968355	
26	3.004984289	
⋮	⋮	
63	3.0000000517	
64	3.0000000379	
65	3.0000000278	
66	3.0000000204	
CPU time	0.0010	0.0007

Table 2 Cases of Ishikawa iteration

Step	Ishikawa (3.2)	Ishikawa (3.3)	Ishikawa (3.5)	Ishikawa (3.4)
1	6.022745179	17.599516463	6.397259957	17.53342562
2	3.55504988	15.542488073	3.725044385	15.426710149
3	3.102829451	13.778956254	3.157958555	13.626959863
4	3.019085154	12.266356345	3.034584416	12.089125019
5	3.003543432	10.968408676	3.007580568	10.774826445
6	3.000657931	9.854176549	3.001661995	9.651358665
7	3.000122164	8.89726621	3.000364402	8.690843013
8	3.000022683	8.07514758	3.000079898	7.86950815
9	3.000004212	7.368577613	3.000017518	7.167078769
10	3.000000782	6.76111087	3.000003841	6.566256169
11	3.000000145	6.23868412	3.000000842	6.052276815
12	3.000000027	5.789263769	3.000000185	5.612537089
13	3.000000005	5.402546543	3.00000004	5.23627424
14	3.000000001	5.069705312	3.000000009	4.91429501
15		4.783173147	3.000000002	4.638744748
16		4.536459758		4.402910896
17		4.323995342		4.201055645
18		4.14099766		4.028273397
19		3.983358785		3.880369278
20		3.847548529		3.753755571
21		3.730532022		3.645363373
22		3.629699305		3.55256722
23		3.542805134		3.473120743
24		3.467917475		3.405101727
25		3.403373393		3.346865184
26		3.347741258		3.297003256
27		3.299788327		3.254310946
28		3.258452935		3.217756821
29		3.222820611		3.18645797
30		3.192103569		3.159658578
31		3.165623078		3.136711605
32		3.142794307		3.117063114
33		3.123113286		3.100238856
34		3.1061457		3.0858328
35		3.09151723		3.07349731
CPU time	0.00086	0.0035	0.0016	0.0085

Table 3 Cases of Agarwal iteration

Step	Agarwal (3.13)	Agarwal (3.14)	Agarwal (3.16)	Agarwal (3.15)
1	3.663643981	4.231276342	4.038158759	4.165185499
2	3.034148064	3.125898552	3.08652991	3.112771857
3	3.001785887	3.013368608	3.007415671	3.011314821
4	3.000093479	3.001425297	3.000637055	3.001139398
5	3.000004893	3.000152024	3.000054738	3.000114779
6	3.000000256	3.000016216	3.000004703	3.000011563
7	3.000000013	3.00000173	3.000000404	3.000001165
8	3.000000001	3.000000184	3.000000035	3.000000117
9	3	3.00000002	3.000000003	3.000000012
10		3.000000002	3	3.000000001
11		3		3
CPU time	0.00095	0.0034	0.0011	0.0011

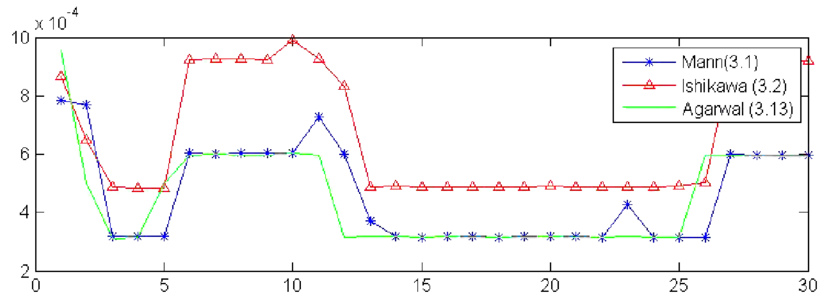

Figure 1 CPU time.

than the Ishikawa iteration (3.5), the Ishikawa iteration (3.4) is more than 8.75 times faster than the Ishikawa iteration (3.5), the Agarwal iteration (3.13) is 1.22 times faster than the Agarwal iteration (3.14), the Agarwal iteration (3.13) is 1.11 times faster than the Agarwal iteration (3.15), the Agarwal iteration (3.13) is 1.22 times faster than the Agarwal iteration (3.16) and so on. We first add our CPU time in Tables 1-3 for each iteration method. Also, we provide Figure 1 by using at least 30 times calculating of CPU times for our faster cases in the methods. From a computer-calculation point of view, we get a different answer. As one can see in the CPU time table, we found that the Agarwal iteration (3.13) and the Mann iteration (3.1) are faster than the Ishikawa iteration (3.2). This note emphasizes the difference of the mathematical results and computer-calculation results which have appeared many times in the literature.

The next example illustrates Lemma 3.2.

Example 2 Let $X = \mathbb{R}$, $C = [0, 2000]$, $x_0 = 1000$, $\alpha_n = 0.85$, $\beta_n = 0.65$, and $\gamma_n = 0.75$ for all $n \geq 0$. Define the map $T: C \to C$ by the formula $T(x) = \sqrt[3]{x^2}$ for all $x \in C$. Table 4 shows us that the Abbas iteration (3.17) converges faster than the other cases, the Abbas iteration (3.18) is 1.1 times faster than the Abbas iteration (3.20), the Abbas iteration (3.19) is 1.05 times faster than the Abbas iteration (3.20), the Abbas iteration (3.21) is 1.04 times faster than the Abbas iteration (3.22) and 1.3 times faster than the Abbas iteration (3.23) and the Abbas iteration (3.24). One can get similar results about difference of the mathematical and computer-calculating points of views for this example.

The next example illustrates Theorem 3.1.

Example 3 Let $X = \mathbb{R}$, $C = [1, 60]$, $x_0 = 40$, $\alpha_n = 0.9$, $\beta_n = 0.6$, and $\gamma_n = 0.8$ for all $n \geq 0$. Define the map $T: C \to C$ by $T(x) = \sqrt{x^2 - 8x + 40}$ for all $x \in C$ (see [23]). Table 5 shows the Abbas iteration (3.17) converges 1.09 times faster than the Thakur-Thakur-Postolache iteration (2.6) and the Thakur-Thakur-Postolache iteration (2.6) is 1.16 times faster than the Abbas iteration (2.5) from the mathematical point of view. Again, we get different results from the computer-calculating point of view by checking Table 5 and Figures 2 and 3.

The next example shows that choosing the coefficients is very important in the rate of convergence of an iteration method.

Table 4 Cases of Abbas iteration

Step	Abbas (3.17)	Abbas (3.18)	Abbas (3.19)	Abbas (3.20)	Abbas (3.21)	Abbas (3.22)	Abbas (3.23)	Abbas (3.24)
1	20.933947	23.074444	29.706456	30.294581	42.622758	43.000492	74.725586	74.829373
2	3.501533	3.915728	4.912771	5.052334	6.872931	6.975246	14.057893	14.097781
3	1.650123	1.789347	2.07514	2.127569	2.605814	2.644699	4.919453	4.938021
4	1.218545	1.278689	1.392374	1.417334	1.596596	1.615195	2.581994	2.592232
5	1.080705	1.109014	1.161005	1.174049	1.254442	1.264388	1.750749	1.757015
6	1.030883	1.044439	1.069469	1.076461	1.115609	1.121158	1.389425	1.39348
7	1.011982	1.018426	1.030642	1.034379	1.054109	1.057231	1.212285	1.214975
8	1.004673	1.007695	1.013649	1.015622	1.025684	1.02743	1.119022	1.120821
9	1.001827	1.003223	1.006106	1.007132	1.012273	1.013239	1.067815	1.069015
10	1.000715	1.001351	1.002737	1.003264	1.005884	1.006411	1.038999	1.039794
11	1.00028	1.000567	1.001228	1.001495	1.002825	1.00311	1.022548	1.02307
12	1.000109	1.000238	1.000551	1.000685	1.001357	1.00151	1.013078	1.013417
13	1.000043	1.0001	1.000247	1.000314	1.000653	1.000733	1.007599	1.007818
14	1.000017	1.000042	1.000111	1.000144	1.000314	1.000356	1.00442	1.00456
15	1.000007	1.000018	1.00005	1.000066	1.000151	1.000173	1.002572	1.002661
16	1.000003	1.000007	1.000022	1.00003	1.000073	1.000084	1.001498	1.001554
17	1.000001	1.000003	1.00001	1.000014	1.000035	1.000041	1.000872	1.000907
18		1.000001	1.000005	1.000006	1.000017	1.00002	1.000508	1.00053
19		1.000001	1.000002	1.000003	1.000008	1.00001	1.000296	1.00031
20		1	1.000001	1.000001	1.000004	1.000005	1.000172	1.000181
21		1	1	1.000001	1.000002	1.000002	1.0001	1.000106
22		1	1	1	1.000001	1.000001	1.000058	1.000062
23		1	1	1	1	1.000001	1.000034	1.000036
24			1	1	1	1	1.00002	1.000021
25				1	1	1	1.000012	1.000012
26				1	1	1	1.000007	1.000007
27					1	1	1.000004	1.000004
28					1	1	1.000002	1.000002
29					1	1	1.000001	1.000001
30							1.000001	1.000001
31							1	1
32							1	1
33							1	1
34							1	1
35							1	1
36							1	1
37							1	1

Table 5 Comparison between Thakur iteration and Abbas iteration

Step	Thakur (2.6)	Abbas (2.5)	Abbas (3.17)
1	31.77453587	33.18158852	31.22317681
2	23.81196041	26.52340588	22.75386567
3	16.33019829	20.11920431	14.88031305
4	9.89958703	14.1634562	8.4317634
5	5.97706669	9.11456867	5.36305686
6	5.07407177	5.96019967	5.01260299
7	5.00409402	5.0925653	5.00037245
8	5.00022019	5.00645474	5.00001094
9	5.00001182	5.00043527	5.00000032
10	5.00000063	5.00002928	5.00000001
11	5.00000003	5.00000197	5
12	5	5.00000013	
13	5	5.00000001	
14		5	
CPU time	0.0012	0.0012	0.0009

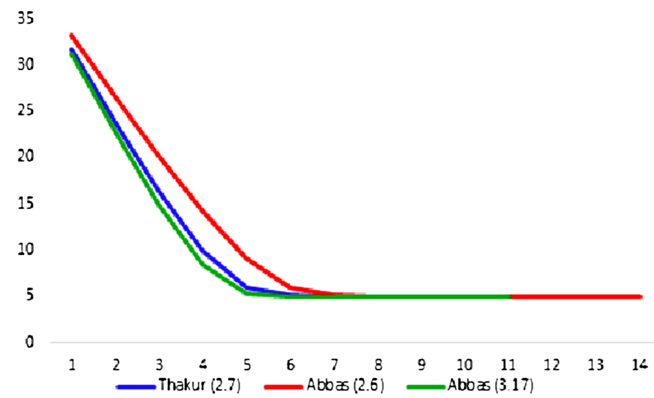

Figure 2 Convergence behavior of the iteration methods of Thakur equation (2.6), Abbas equation (2.5), and Abbas equation (3.17).

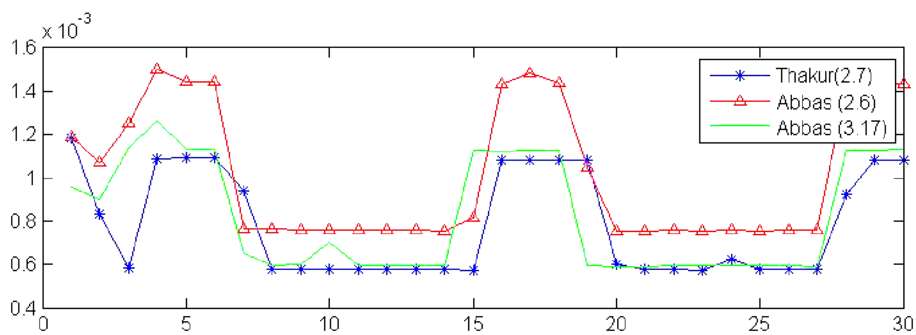

Figure 3 CPU time.

Example 4 Let $X = \mathbb{R}$, $C = [0, 30]$, and $x_0 = 20$. Define the map $T: \mathbb{R} \to \mathbb{R}$ by $T(x) = \frac{x}{2} + 1$ for all $x \in C$. Consider the following coefficients separately in the Thakur-Thakur-Postolache iteration (2.6):

(a) $\alpha_n = \beta_n = \gamma_n = 1 - \frac{1}{(n+1)^{10}}$,

(b) $\alpha_n = \beta_n = \gamma_n = 1 - \frac{1}{n+1}$,

(c) $\alpha_n = \beta_n = \gamma_n = 1 - \frac{1}{(n+1)^{\frac{1}{2}}}$,

(d) $\alpha_n = \beta_n = \gamma_n = 1 - \frac{1}{(n+1)^{\frac{1}{5}}}$

for all $n \geq 0$. Table 6 shows that the Thakur-Thakur-Postolache iteration (2.6) with coefficients (a) is 1.25 times faster than the Thakur-Thakur-Postolache iteration (2.6) with coefficients (b), the Thakur-Thakur-Postolache iteration (2.6) with coefficients (a) is 1.6 times faster than the Thakur-Thakur-Postolache iteration (2.6) with coefficients (c) and the Thakur-Thakur-Postolache iteration (2.6) with coefficients (a) is 2.16 times faster than the Thakur-Thakur-Postolache iteration (2.6) with coefficients (d). This note satisfies other iteration methods of course from the mathematical point of view. Here, we find a little different computer-calculating result for the CPU time table of this example, which one can check in Figure 4.

Table 6 Cases of Thakur iteration

Step	(a)	(b)	(c)	(d)
1	4.2609841803	9.03125	10.2844561595	10.8540663001
2	2.2826469537	4.2135416667	5.4804739263	6.2632682688
3	2.0353310377	2.6009419759	3.3595601275	4.0142167756
4	2.004416382	2.1466298421	2.5007642765	2.9360724936
5	2.0005520478	2.0330086855	2.1756764587	2.4287794141
6	2.000069006	2.0069770545	2.0591364356	2.1939030837
7	2.0000086257	2.0014018838	2.0192087915	2.0866824323
8	2.0000010782	2.0002701847	2.0060472121	2.0383477219
9	2.0000001348	2.0000502881	2.0018515929	2.0168034488
10	2.0000000168	2.0000090866	2.0005529869	2.0072985299
11	2.0000000021	2.0000016005	2.0001614712	2.0031443476
12	2.0000000003	2.0000002757	2.0000461907	2.0013443922
13		2.0000000466	2.0000129668	2.0005707329
14		2.0000000077	2.0000035774	2.0002406784
15		2.0000000013	2.0000009712	2.0001008564
16			2.0000002597	2.0000420126
17			2.0000000685	2.0000174019
18			2.0000000178	2.0000071693
19			2.0000000046	2.0000029385
20			2.0000000012	2.0000011985
21				2.0000004865
22				2.0000001966
23				2.0000000791
24				2.0000000317
25				2.0000000127
26				2.000000005
CPU time	0.0013	0.0014	0.0015	0.0017

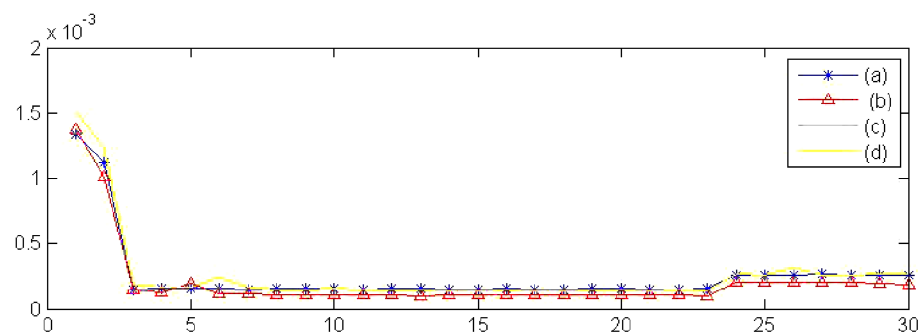

Figure 4 CPU time.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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