Wang and Zhou *Fixed Point Theory and Applications* (2015) 2015:154
DOI 10.1186/s13663-015-0400-8

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Fixed point theorems and the Krein-Šmulian property in locally convex spaces

Fuli Wang* and Hua Zhou

*Correspondence:
win-fully@163.com
School of Mathematics and Physics, Changzhou University, Changzhou, 213164, China

Abstract

The aim of this paper is to establish some new fixed point theorems for the superposition operators in locally convex spaces which satisfy the Krein-Šmulian property. We employ a family of measures of noncompactness in conjunction with the Schauder-Tychonoff fixed point theorem. As an application, the existence of solutions to a quite general nonlinear Volterra type integral equation is considered in locally integrable spaces.

MSC: Primary 47H10; 47H08; 47H30

Keywords: fixed point theorem; locally convex space; Krein-Šmulian property; family of measures of weak noncompactness

1 Introduction

As an example of algebraic settings, the captivating Krasnosel'skii fixed point theorem leads to the consideration of fixed points for the sum of two operators. It asserts that, if M is a nonempty, bounded, closed, and convex subset of a Banach space X and A, B are two maps from M into X such that $A(M) + B(M) \subseteq M$, A is compact and B is a contraction, then $A + B$ has at least one fixed point in M (see [1] or [2], p.31). Since then, there has been a vast literature dealing with the improvement of such a result. In the previous decade, several papers have given generalizations of this theorem involving the weak topology of Banach spaces by using the De Blasi measure of weak noncompactness (see [3–12]). The novelty of their results is that the involved operators need not to be weakly continuous. Some weak and strong compactness are addressed instead of the weak continuity since the condition of weak continuity is usually not easy to verify.

On the other hand, the Krasnosel'skii fixed point theorem has been generalized to locally convex spaces or Fréchet spaces by some authors (see [13–18]), and a family of measures of noncompactness has been introduced to the concrete spaces by Olszowy [19, 20]. All of these results naturally cause us to consider the Krasnosel'skii type fixed point theorems in locally convex spaces by means of a family of measures of weak noncompactness. This problem will be followed with interest in some special situations in the present paper.

As a more general consideration, there is another setting which leads to the investigation of the fixed points for so-called nonautonomous type superposition operators, that is, to study the existence of solutions to the following operator equation:

$$x = F(x, Ax) \tag{1.1}$$

for given $A : X \to Y$ and $F : X \times Y \to X$, where X is a locally convex space and Y a topological vector space. In [12, 21], one of the authors of this paper has established some fixed point results for (1.1) in the framework of Banach spaces. In this paper, by means of a family of measures of weak noncompactness we will establish some new fixed point theorems for (1.1) in the framework of locally convex spaces. Our results extend several Krasnosel'skii type fixed point theorems from the previous literature.

As an application we will also study the solvability for the following quite general nonlinear Volterra integral equation:

$$x(t) = f\left(t, x(t), \int_0^t \kappa(t,s)\nu\bigl(s, x(s)\bigr)\, ds\right) \tag{1.2}$$

in $\mathcal{L}^1_{\mathrm{loc}} := \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_+)$, the space consisting of all locally integrable functions on $\mathbb{R}_+ := [0, \infty)$. Here $f : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ and $\nu : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are given functions, while κ is a given real function defined on $\Delta := \{(t,s) \in \mathbb{R}^2 : 0 \leq s \leq t\}$. We say the nonlinear integral equation (1.2) is of Volterra type, since within its form an operator of Volterra type appears.

Our goals in this paper are to establish new fixed point theorems for the solvability of (1.1) in locally convex spaces, and to study under what conditions (1.2) is solvable in $\mathcal{L}^1_{\mathrm{loc}}$ by applying our new theorems.

This paper is organized as follows. In Section 2, we present the relevant definitions and results needed in our work. In Section 3, we establish some new fixed point theorems for (1.1) in the framework of locally convex spaces. In Section 4, we prove the existence of locally integrable solutions for (1.2) by virtue of our fixed point theorems and a family of measures of weak noncompactness.

2 Definitions and preliminaries

Throughout this paper, X will denote a Hausdorff locally convex topological vector space, and $\{|\cdot|_\rho\}_{\rho\in\Lambda}$ a family of seminorms which generates the topology of X.

Recall that the weak topology on X is the weakest topology (the topology with the fewest open sets) such that all elements of X' (the topological dual of X) remain continuous. Explicitly, the neighborhood system of the origin for the weak topology is the collection of sets of the form $\phi^{-1}(U)$ where $\phi \in X'$ and U is a neighborhood of the origin.

Definition 2.1 (see [5], Remark 2.1) We say that X satisfies the Krein-Šmulian property if the closed convex hull of each weakly compact set is weakly compact.

Remark 2.2 Each Fréchet space satisfies the Krein-Šmulian property (see [22], p.233), particularly for each Banach space (see [23], p.434).

In our considerations, a family of measures of weakly noncompactness in locally convex spaces will play an important role. A single measure of weak noncompactness in Banach spaces may refer to the definition of Banaś and Rivero [24]. Next, we will use $\mathfrak{B}(X)$ denoting the collection of all nonempty bounded subsets of X, and $\mathfrak{W}(X)$ a subset of $\mathfrak{B}(X)$

consisting of all weakly compact subsets of X. For $\rho \in \Lambda$ and $r > 0$, the set $\{x : |x - x_0|_\rho < r\}$ is denoted by $V_\rho(x_0, r)$. The closure of this set is denoted by $\mathrm{B}_\rho(x_0, r)$. We shall also sometimes use $V(\rho)$ to stand for $V_\rho(0,1)$.

Definition 2.3 Let X satisfy the Krein-Šmulian property. A family of functions $\omega_\rho : \mathfrak{B}(X) \to \mathbb{R}_+$ ($\rho \in \Lambda$) is said to be a family of the measures of weak noncompactness of X if this family satisfies the following conditions:

(1) The family $\ker(\omega_\rho) := \{M \in \mathfrak{B}(X) : \omega_\rho(M) = 0 \text{ for all } \rho \in \Lambda\}$ is nonempty and $\ker(\omega_\rho)$ is contained in the subfamily consisting of all relatively weakly compact sets of X;
(2) $N \subseteq M \Rightarrow \omega_\rho(N) \leq \omega_\rho(M)$ for each $\rho \in \Lambda$, where $M, N \in \mathfrak{B}(X)$;
(3) $\omega_\rho(\overline{\mathrm{co}}(M)) = \omega_\rho(M)$ for each $\rho \in \Lambda$, where $\overline{\mathrm{co}}(M)$ is the closed convex hull of $M \in \mathfrak{B}(X)$;
(4) $\omega_\rho(\lambda M + (1-\lambda)N) \leq \lambda\omega_\rho(M) + (1-\lambda)\omega_\rho(N)$ for each $\rho \in \Lambda$, $\lambda \in [0,1]$ and $M, N \in \mathfrak{B}(X)$;
(5) if $(M_n)_{n=1}^\infty$ is a decreasing sequence of nonempty, bounded, and weakly closed subsets of X with $\lim_{n\to\infty} \omega_\rho(M_n) = 0$ for each $\rho \in \Lambda$, then $M_\infty := \bigcap_{n=1}^\infty M_n$ is nonempty.

The family $\ker(\omega_\rho)$ described in (1) is called the kernel of the measure of weak noncompactness ω_ρ. Note that the intersection M_∞ from (5) belongs to $\ker(\omega_\rho)$ since we have $\omega_\rho(M_\infty) \leq \omega_\rho(M_n)$ for each $\rho \in \Lambda$ and all $n \in \mathbb{N}$, and $\lim_{n\to\infty} \omega_\rho(M_n) = 0$.

Following the notion of a single measure of weak noncompactness in Banach spaces which is introduced by De Blasi [25], a family of measures of weak noncompactness in locally convex spaces for $\rho \in \Lambda$ may be defined by

$$\omega_\rho(M) = \inf\bigl\{r > 0 : \exists W \in \mathfrak{W}(X) \text{ such that } M \subseteq W + \mathrm{B}_\rho(0,r)\bigr\}. \tag{2.1}$$

Nevertheless, it is rather difficult to express this family of weak noncompactness with the help of a convenient formula in a concrete locally convex space. Such a formula is known in the case of the space $\mathcal{L}^1_{\mathrm{loc}}$ which is endowed with the family of seminorms $|\varphi|_I := \int_I |\varphi(t)|\,dt$ for $\varphi \in \mathcal{L}^1_{\mathrm{loc}}$, where I is a bounded subinterval of $\mathbb{R}_+$. In [26], Appell and De Pascale give to each ω_I the following simple form:

$$\omega_I(M) = \lim_{\varepsilon\to 0} \sup_{\varphi\in M} \biggl\{\sup\biggl\{\int_D \bigl|\varphi(t)\bigr|\,dt : D \subseteq I, \mathrm{meas}(D) \leq \varepsilon\biggr\}\biggr\}, \tag{2.2}$$

for all bounded subsets M of $\mathcal{L}^1_{\mathrm{loc}}$, where $\mathrm{meas}(\cdot)$ denotes the Lebesgue measure. It is easily verified that the family $\{\omega_I\}_{I\in\Lambda}$ satisfies Definition 2.3.

Following the definitions given by [27, 28] in Banach spaces, we introduce the concepts of ws-compactness and ww-compactness in locally convex spaces as follows.

Definition 2.4 Let Y be a topological vector space. A mapping $T : \mathcal{D} \subseteq X \to Y$ is said to be:

(i) ws-compact if it is continuous and maps relatively weakly compact sets of $\mathcal{D}$ into relatively strongly compact ones of Y;
(ii) ww-compact if it is continuous and maps relatively weakly compact sets of $\mathcal{D}$ into relatively weakly compact ones of Y.

Next, we collect a few auxiliary facts concerning the superposition operator required in the sequel.

Consider a function $\psi(t,x) = \psi : I \times \mathbb{R}^d \to \mathbb{R}$, where I is an interval of $\mathbb{R}$, and $\mathbb{R}^d$ a real Euclidean space of d-dimensions. We say that ψ is a Carathéodory function if it is measurable in t for each x in $\mathbb{R}^d$ and continuous in x for almost everywhere (a.e., for short) given $t \in I$.

Let $m(I)$ be the set of all measurable functions $x : I \to \mathbb{R}^d$. If ψ is a Carathéodory function, then ψ defines a mapping $\mathcal{N}_\psi : m(I) \to m(I)$ by $(\mathcal{N}_\psi x)(t) = \psi(t, x(t))$. This mapping is called the superposition operator (or Nemytskii operator) associated to ψ. For a given measurable function $\varphi : I \to \mathbb{R}^d$, the composite operator $\mathcal{N}_\psi \varphi(\cdot) := \psi(\cdot, \varphi(\cdot))$ which maps I into $\mathbb{R}$ is said to be a nonautonomous type superposition operator.

By generalizing the above concept, the solvability of (1.1) may be thought of as the existence of fixed points for the nonautonomous type superposition operator $\mathcal{N}_F A$, where $\mathcal{N}_F A(\cdot) := F(\cdot, A(\cdot))$ for simplicity.

The following theorem was proved by Krasnosel'skii [29] (see also [30]) in the case when I is a bounded interval and has been extended to an unbounded interval by Appell and Zabrejko [31].

Theorem 2.5 ([31], Theorem 3.1, p.93) *Let I be an interval in $\mathbb{R}$. The superposition operator $\mathcal{N}_\psi$ maps $L^1(I)$ into $L^1(I)$ if and only if there exist a function $a \in L^1_+(I)$ and a constant $b > 0$ such that*

$$\left|\psi(t,x)\right| \le a(t) + b|x|,$$

where $L^1_+(I)$ denotes a positive cone of the space $L^1(I)$.

In this case, the operator $\mathcal{N}_\psi$ is continuous and bounded in the sense that it maps bounded sets into bounded ones.

3 Fixed point theorems

Let $\mathcal{U}$ be the neighborhood system of the origin obtained from Λ. Thus if $U \in \mathcal{U}$, there is a finite number of seminorms $\rho_1, \rho_2, \ldots, \rho_n$ in Λ and real numbers $r_1, r_2, \ldots, r_n$ such that $U = \bigcap_{i=1}^n r_i V(\rho_i)$.

A mapping $T : X \to X$ is said to be a $|\cdot|_\rho$-contraction for $\rho \in \Lambda$ if there exists $\alpha_\rho \in [0,1)$ such that $|Tx_1 - Tx_2|_\rho \le \alpha_\rho |x_1 - x_2|_\rho$ for all $x_1, x_2 \in X$; if $\alpha_\rho = 1$ then the mapping T is said to be a $|\cdot|_\rho$-nonexpansion for $\rho \in \Lambda$.

Theorem 3.1 *Let M be a nonempty, closed, and convex subset of X, and let the Krein-Šmulian property be satisfied. Suppose that $T : M \to M$ is ws-compact such that $T(M)$ is relatively weakly compact, then T has at least one fixed point.*

Proof Let $\mathcal{N} := \overline{\mathrm{co}}(T(M))$. Since M is closed and convex satisfying $T(M) \subseteq M$, then $\mathcal{N} \subseteq M$ and therefore $T(\mathcal{N}) \subseteq T(M) \subseteq \mathcal{N}$.

It is clear that $\mathcal{N}$ is weakly compact according to the relatively weak compactness of $T(M)$ and the Krein-Šmulian property. Moreover, $T(\mathcal{N})$ is relatively compact since T is ws-compact. Now applying the Schauder-Tychonoff fixed point theorem (see [13], Theorem 2.1(b) or [2], p.32), we conclude that T has at least one fixed point $x \in \mathcal{N} \subseteq M$ such that $Tx = x$. □

Theorem 3.2 *Let X be sequentially complete and the Krein-Šmulian property be satisfied. Let Y be a topological vector space. Suppose that M is a nonempty, bounded, closed, and convex subset of X and the operators $A : M \to Y$, $F : X \times Y \to X$ satisfy the following conditions*:

(i) *$A(M)$ is relatively weakly compact and A is ws-compact*;

(ii) *for each $\rho \in \Lambda$ there exists $\alpha_\rho \in [0,1)$ such that*

$$\bigl|F(x_1,y) - F(x_2,y)\bigr|_\rho \le \alpha_\rho |x_1 - x_2|_\rho, \quad \forall x_1, x_2 \in X \text{ and } y \in Y,$$

and F is ww-compact;

(iii) $[x = F(x, Az), z \in M] \Rightarrow x \in M$.

Then there is a point x in M such that $x = F(x, Ax)$.

Proof For a given $y \in A(M)$ the mapping $F(\cdot, y)$ is a $|\cdot|_\rho$-contraction for each $\rho \in \Lambda$, so by Cain-Nashed theorem ([13], Theorem 2.2) it has a unique fixed point in X. Let us denote by $J : A(M) \to X$ the mapping which assigns each $y \in A(M)$ to the unique point in X such that $Jy = F(Jy, y)$. Thus, J is well defined.

For arbitrarily given $y, y_0 \in A(M)$ and $\rho \in \Lambda$, from the inequality

$$\begin{aligned}
|Jy - Jy_0|_\rho &= \bigl|F(Jy,y) - F(Jy_0,y_0)\bigr|_\rho \\
&\le \bigl|F(Jy,y) - F(Jy_0,y)\bigr|_\rho + \bigl|F(Jy_0,y) - F(Jy_0,y_0)\bigr|_\rho \\
&\le \alpha_\rho |Jy - Jy_0|_\rho + \bigl|F(Jy_0,y) - F(Jy_0,y_0)\bigr|_\rho,
\end{aligned}$$

we obtain

$$|Jy - Jy_0|_\rho \le (1-\alpha_\rho)^{-1} \bigl|F(Jy_0,y) - F(Jy_0,y_0)\bigr|_\rho.$$

Let $U = \bigcap_{i=1}^n r_i V(\rho_i)$ be an arbitrarily given neighborhood of the origin in X. By the continuity of F, there exists a neighborhood W_{y_0} of $y_0 \in Y$ such that, for all $y \in W_{y_0} \cap A(M)$,

$$\bigl|F(Jy_0,y) - F(Jy_0,y_0)\bigr|_{\rho_i} < 1 - \alpha_{\rho_i} \quad (i = 1, \ldots, n),$$

which implies that

$$|Jy - Jy_0|_{\rho_i} \le (1-\alpha_{\rho_i})^{-1} \bigl|F(Jy_0,y) - F(Jy_0,y_0)\bigr|_{\rho_i} < 1.$$

It follows that

$$W_{y_0} \cap A(M) \subset J^{-1}(Jy_0 + U),$$

and therefore J is continuous.

For any $z \in M$, by assumption (iii) we infer that there is $x = (JA)z \in M$ such that $x = F(x, Az)$. This shows that $JA(M) \subseteq M$. Let $M_1 := M$ and $M_{n+1} := \overline{\mathrm{co}}(JA(M_n))$. By induction, we infer that $(M_n)_{n\in\mathbb{N}}$ is a decreasing sequence of nonempty, bounded, weakly closed, and convex subsets of X. Moreover, we obtain

$$JA(M_n) \subseteq F\bigl(JA(M_n), A(M)\bigr) \subseteq F\bigl(M_{n+1}, A(M)\bigr) \subseteq F\bigl(M_n, A(M)\bigr).$$

Next, according to (2.1) for an arbitrarily given $\rho \in \Lambda$ we may take $r > 0$ and $W \in \mathfrak{W}(X)$ such that $M_n \subseteq W + \mathrm{B}_\rho(0,r)$. By assumption (ii), there exists a constant $\alpha_\rho \in [0,1)$ such that

$$\big|F(x,y) - F(w,y)\big|_\rho \le \alpha_\rho |x-w|_\rho \le \alpha_\rho r,$$

for all $x \in M_n$, $w \in W$, and $y \in A(M)$. Accordingly, for all $y \in A(M)$ we infer that

$$F(M_n,y) \subseteq F(W,y) + \mathrm{B}_\rho(0,\alpha_\rho r) \subseteq F\big(W,A(M)\big) + \mathrm{B}_\rho(0,\alpha_\rho r),$$

it follows that

$$F\big(M_n,A(M)\big) \subseteq F\big(W,A(M)\big) + \mathrm{B}_\rho(0,\alpha_\rho r) \subseteq \overline{F\big(W,A(M)\big)}^w + \mathrm{B}_\rho(0,\alpha_\rho r).$$

Moreover, from the weak compactness of W and $\overline{A(M)}^w$ we see that $W \times A(M)$ is relatively weakly compact by Tychonoff's product theorem. Thus, $\overline{F(W,A(M))}^w$ is weakly compact since F is ww-compact. Accordingly, we have

$$\omega_\rho\big(JA(M_n)\big) \le \omega_\rho\big(F\big(M_n,A(M)\big)\big) \le \alpha_\rho \omega_\rho(M_n),$$

which implies that $\omega_\rho(M_{n+1}) \le \alpha_\rho \omega_\rho(M_n)$. It follows by induction that $\omega_\rho(M_n) \le \alpha_\rho^n \omega_\rho(M)$ and therefore $\lim_{n\to\infty} \omega_\rho(M_n) = 0$. The arbitrariness of $\rho \in \Lambda$ shows that $\mathcal{N} := \bigcap_{n=0}^\infty M_n$ is a nonempty, closed, convex, and weakly compact subset of M by Definition 2.3. It is easily seen that $JA(\mathcal{N}) \subseteq \mathcal{N}$. Consequently, JA is ws-compact since A is ws-compact and J is continuous. Now by the use of Theorem 3.1 we conclude that JA has at least one fixed point $x \in \mathcal{N} \subseteq M$ such that $(JA)x = x$, which implies that

$$F(x,Ax) = F\big((JA)x,Ax\big) = (JA)x = x.$$

This completes the proof. □

In the framework of locally convex spaces, the following result can be thought of as an extension of Latrach *et al.* [9], Theorem 2.3, and also a variant of Cain and Nashed [13], Theorem 3.1, under the weak topology. This implies that we are to establish a new version of Krasnosel'skii's fixed point theorem in locally convex spaces.

Corollary 3.3 *Let X be sequentially complete and the Krein-Šmulian property be satisfied. Suppose that M be a nonempty, bounded, closed, and convex subset of X, and the operators $A: M \to X$, $B: X \to X$ satisfy the following conditions:*

(i) *$A(M)$ is relatively weakly compact, and A is ws-compact;*
(ii) *B is a $|\cdot|_\rho$-contraction for each $\rho \in \Lambda$, and B is ww-compact;*
(iii) $[x = Bx + Az, z \in M] \Rightarrow x \in M$.

Then there is a point x in M such that $Ax + Bx = x$.

Proof Let us take $F(x,y) := Bx + y$ and $Y = X$. All assumptions of Theorem 3.2 are easily verified, and then the proof immediately is achieved. □

Since assumption (iii) of Theorem 3.2 is hard to verify in real applications, we next establish a Schaefer type fixed point theorem for (1.1).

Theorem 3.4 *Let X be sequentially complete and the Krein-Šmulian property be satisfied. Let Y be a topological vector space. Suppose that the operators $A : X \to Y$, and $F : X \times Y \to X$ satisfy the following conditions*:

(i) *A maps bounded sets of X into relatively weakly compact ones of Y, and A is ws-compact*;

(ii) *for each $\rho \in \Lambda$ there exists $\alpha_\rho \in [0,1)$ such that*

$$\left|F(x_1,y) - F(x_2,y)\right|_\rho \le \alpha_\rho |x_1 - x_2|_\rho, \quad \forall x_1, x_2 \in X \text{ and } y \in Y,$$

and F is ww-compact.

Then either

(a) *there is a point x in X such that $x = F(x, Ax)$, or*

(b) *the set $\{x \in X : x = \lambda F(x/\lambda, Ax)\}$ is unbounded for $\lambda \in (0,1)$.*

Proof As in the proof of Theorem 3.2, let us denote by $J : A(X) \to X$ the mapping which assigns each $y \in A(X)$ to the unique point in X such that $Jy = F(Jy, y)$. We know that J is well defined and continuous.

Let $\overline{U}$ be a convex, symmetric, and closed neighborhood of the origin in X, and let $\mathrm{B}_n := n\overline{U}$ for $n \in \mathbb{N}$. We define the radial retraction onto B_n as follows:

$$r(x) := \frac{x}{\max\{1, \mu(x)\}}, \quad \text{for } x \in X,$$

where μ is the Minkowski functional on B_n, *i.e.*, $\mu(x) = \inf\{\alpha > 0 : x \in \alpha \mathrm{B}_n\}$. Then r is a continuous retraction of X onto B_n; and if $x \in \mathrm{B}_n$ then $r(x) = x$, otherwise $r(x) \in \partial \mathrm{B}_n$ (the boundary of B_n).

Then rJA has a fixed point x in B_n by Theorem 3.2. Either $(JA)x \in \mathrm{B}_n$, in which case

$$x = (rJA)x = (JA)x = F\bigl((JA)x, Ax\bigr) = F(x, Ax),$$

or $(JA)x \notin \mathrm{B}_n$, in which case $x = (rJA)x \in \mathrm{B}_n$ such that

$$x = (rJA)x = \frac{1}{\mu((JA)x)}(JA)x = \lambda(JA)x,$$

that is, $(JA)x = x/\lambda$, and the property of $(JA)x$ yields

$$\frac{x}{\lambda} = F\left(\frac{x}{\lambda}, Ax\right).$$

Thus either (a) for some $n \in \mathbb{N}$ we obtain a solution of (1.1), or (b) for each $n \in \mathbb{N}$ we obtain a solution of $x = \lambda F(x/\lambda, Ax)$ for some $\lambda \in (0,1)$; in the second case the set of such solutions is unbounded. □

Corollary 3.5 *Let X be sequentially complete and the Krein-Šmulian property be satisfied. Suppose that the operators $A, B : X \to X$ satisfy the following conditions*:

(i) *A maps bounded sets into relatively weakly compact ones, and A is ws-compact*;
(ii) *B is a $|\cdot|_\rho$-contraction for all $\rho \in \Lambda$, and B is ww-compact.*
Then either
(a) *there is a point x in X such that $x = Ax + Bx$, or*
(b) *the set $\{x \in X : x = \lambda Ax + \lambda B(x/\lambda)\}$ is unbounded for $\lambda \in (0,1)$.*

Recalling that a mapping $T : M \to X$ is said to be demiclosed at 0 if for every net (x_δ) in M converges weakly to x and (Tx_δ) converges to 0, then we have $Tx = 0$.

Theorem 3.6 *Let X be sequentially complete and the Krein-Šmulian property be satisfied. Let Y be a topological vector space. Suppose that M is a nonempty, bounded, closed, and convex subset of X, and the operators $A : M \to Y$, $F : X \times Y \to X$ satisfy the following conditions*:
(i) *$A(M)$ is relatively weakly compact, and A is ws-compact*;
(ii) *for each $\rho \in \Lambda$ and all $x_1, x_2 \in X$, $y \in Y$ we have*

$$\left|F(x_1, y) - F(x_2, y)\right|_\rho \le |x_1 - x_2|_\rho,$$

and F is ww-compact;
(iii) *if $(x_n)_{n\in\mathbb{N}}$ is a sequence such that $(x_n - F(x_n, Ax_n))_{n\in\mathbb{N}}$ is convergent, then $(x_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence*;
(iv) *$I - \mathcal{N}_F A$ is demiclosed at 0*;
(v) *if $\lambda \in (0,1)$ and $x = \lambda F(x, Az)$ for some $z \in M$, then $x \in M$.*
Then there is a point x in M such that $x = F(x, Ax)$.

Proof For each $\lambda \in (0,1)$, the operators A and λF satisfy the assumptions of Theorem 3.2, then there is a point $x_\lambda \in M$ such that $x_\lambda = \lambda F(x_\lambda, Ax_\lambda)$. Now, choose a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $(0,1)$ such that $\lambda_n \to 1$ and consider the corresponding sequence $(x_n)_{n\in\mathbb{N}}$ of elements of M satisfying

$$x_n = \lambda_n F(x_n, Ax_n).$$

Since M is bounded and $\lambda_n F(x_n, Ax_n) = x_n \in M$, $F(x_n, Ax_n)$ is bounded. Thus, for each $\rho \in \Lambda$ we have

$$\left|x_n - F(x_n, Ax_n)\right|_\rho = (1 - \lambda_n)\left|F(x_n, Ax_n)\right|_\rho \to 0,$$

which implies that $x_n - F(x_n, Ax_n) \to 0$.

By condition (iii) the sequence $(x_n)_{n\in\mathbb{N}}$ has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which converges weakly to some $x \in M$ (M is weakly closed because of its closeness and convexity). The demiclosedness of $I - \mathcal{N}_F A$ at 0 yields $(I - \mathcal{N}_F A)x = 0$, that is, $x = \mathcal{N}_F Ax = F(x, Ax)$. □

The following corollary extends [4], Theorem 2.1, and [11], Theorem 3.8, to locally convex spaces.

Corollary 3.7 *Let X be sequentially complete and the Krein-Šmulian property be satisfied. Suppose that M be a nonempty, bounded, closed, and convex subset of X, and the operators $A : M \to X$, $B : X \to X$ satisfy the following conditions*:

(i) *$A(M)$ is relatively weakly compact, and A is ws-compact*;
(ii) *B is a $|\cdot|_\rho$-nonexpansion for all $\rho \in \Lambda$, and B is ww-compact*;
(iii) *if $(x_n)_{n\in\mathbb{N}}$ is a sequence such that $(x_n - Bx_n - Ax_n)_{n\in\mathbb{N}}$ is convergent, then $(x_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence*;
(iv) *$I - A - B$ is demiclosed at* 0;
(v) *if $\lambda \in (0,1)$ and $x = \lambda Bx + \lambda Ay$ for some $y \in M$, then $x \in M$.*

Then there is a point x in M such that $Ax + Bx = x$.

4 Application to the existence of locally integrable solutions for a general nonlinear integral equation

In this section we mainly consider (1.2). Solutions to it will be sought in $\mathcal{L}^1_{\text{loc}}$, the space consisting of all real functions defined and locally Lebesgue integrable on $\mathbb{R}_+$, equipped with the family of seminorms

$$|x|_T := \int_0^T \bigl|x(t)\bigr|\,dt, \quad T > 0.$$

It is well known that $\mathcal{L}^1_{\text{loc}}$ is a locally convex space and becomes a Fréchet space furnished with the distance

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x-y|_n}{1 + |x-y|_n}.$$

Let $L^1[0,T]$ denote the Banach space consisting of all real functions defined and Lebesgue integrable on $[0,T]$, and let $\Pi_T : \mathcal{L}^1_{\text{loc}} \to L^1[0,T]$ denote the restrictive mapping.

Remark 4.1 Assume that M is a nonempty subset of $\mathcal{L}^1_{\text{loc}}$. The following facts are important in our further considerations:
(1) M is bounded if there exists a $L_T > 0$ for each $T > 0$ such that $|x|_T \leq L_T$ for all $x \in M$;
(2) M is relatively (strongly) compact if and only if $\Pi_T(M)$ is relatively (strongly) compact in Banach space $L^1[0,T]$ for each $T > 0$;
(3) M is relatively weakly compact if and only if $\Pi_T(M)$ is relatively weakly compact in Banach space $L^1[0,T]$ for each $T > 0$;
(4) a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{L}^1_{\text{loc}}$ is convergent to $x \in \mathcal{L}^1_{\text{loc}}$ if and only if $|x_n - x|_T \to 0$ for each $T > 0$.

We will discuss the solvability of (1.2) under the following hypotheses:

($\mathcal{H}1$) $\kappa : \Delta \to \mathbb{R}$ is measurable, where $\Delta = \{(t,s) \in \mathbb{R}^2 : 0 \leq s \leq t\}$;
($\mathcal{H}2$) $\nu : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and there exist a function $a \in \mathcal{L}^1_+$ and a constant $b > 0$ such that $|\nu(t,x)| \leq a(t) + b|x|$;
($\mathcal{H}3$) $f : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function, and there exist two positive functions $\alpha, \beta \in \mathcal{L}^\infty_{\text{loc}}$ such that

$$\bigl|f(t,x_1,y_1) - f(t,x_2,y_2)\bigr| \leq \alpha(t)|x_1 - x_2| + \beta(t)|y_1 - y_2|;$$

($\mathcal{H}4$) $\alpha_T + b\beta_T|K|_T < 1$ for all $T > 0$, where

$$\alpha_T := \operatorname{ess}\sup_{t\in[0,T]} \alpha(t), \qquad \beta_T := \operatorname{ess}\sup_{t\in[0,T]} \beta(t), \qquad |K|_T := \operatorname{ess}\sup_{s\in[0,T]} \int_s^T \bigl|\kappa(t,s)\bigr|\,dt.$$

Note that (1.2) may be written in the abstract form $x = F(x, Ax)$, where F is the superposition operator associated to f (*i.e.*, $F = \mathcal{N}_f$):

$$F : \mathcal{L}^1_{\mathrm{loc}} \times \mathcal{L}^1_{\mathrm{loc}} \to \mathcal{L}^1_{\mathrm{loc}}, \qquad (x,y) \mapsto F(x,y),$$
$$F(x,y)(t) = f\bigl(t, x(t), y(t)\bigr);$$

and $A := K\mathcal{N}_v$ appears as the composition of the superposition operator associated to v with the linear operator K defined by

$$K : \mathcal{L}^1_{\mathrm{loc}} \to \mathcal{L}^1_{\mathrm{loc}}, \qquad \varphi \mapsto K\varphi,$$
$$(K\varphi)(t) = \int_\Omega k(t,s)\varphi(s)\,ds.$$

Our aim is now to prove that the nonautonomous type superposition operator $\mathcal{N}_F A$ has a fixed point in $\mathcal{L}^1_{\mathrm{loc}}$. Before starting to prove the solvability of (1.2), we make some remarks.

Remark 4.2 It should be noted that assumption ($\mathcal{H}_1$) leads to the estimate

$$\left|\int_0^t \kappa(t,s)\varphi(s)\,ds\right|_T \le \int_0^T \left(\int_s^T \bigl|\kappa(t,s)\bigr|\,dt\right)\bigl|\varphi(s)\bigr|\,ds \le |K|_T|\varphi|_T,$$

for all $\varphi \in \mathcal{L}^1_{\mathrm{loc}}$ and an arbitrarily given $T > 0$. This shows that the linear Volterra operator K is continuous on the Banach space $L^1[0,T]$, hence weakly continuous from $L^1[0,T]$ into itself.

Theorem 4.3 *Assume that the assumptions* ($\mathcal{H}1$)-($\mathcal{H}4$) *are satisfied, then* (1.2) *has at least one solution in* $\mathcal{L}^1_{\mathrm{loc}}$.

Proof We will apply Theorem 3.4 to prove the present theorem. Let us take the spaces X and Y of Theorem 3.4 to $\mathcal{L}^1_{\mathrm{loc}}$. Our proving is divided into several steps.

(1) Assumption ($\mathcal{H}_2$) shows that the superposition operator $\mathcal{N}_v$ is continuous and maps bounded sets of $\mathcal{L}^1_{\mathrm{loc}}$ into bounded sets of $\mathcal{L}^1_{\mathrm{loc}}$ by Theorem 2.5. It follows that the operator $\Pi_T K\mathcal{N}_v$ is continuous and maps $L^1[0,T]$ into itself since Remark 4.2 shows K is continuous. The arbitrariness of $T > 0$ implies that $A = K\mathcal{N}_v$ is continuous on $\mathcal{L}^1_{\mathrm{loc}}$.

Now we check that $A = K\mathcal{N}_v$ maps relatively weakly compact sets into relatively strongly compact ones. To this end, let $(x_n)_{n\in\mathbb{N}}$ be a weakly convergent sequence of $\mathcal{L}^1_{\mathrm{loc}}$. By Remark 4.1(3), $(\Pi_T x_n)_{n\in\mathbb{N}}$ is weakly convergent for an arbitrarily given $T > 0$. Since $\mathcal{N}_v$ is ww-compact on $L^1[0,T]$ by [10], Lemma 3.2, the sequence $(\Pi_T\mathcal{N}_v x_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence, say $(\Pi_T\mathcal{N}_v x_{n_k})_{k\in\mathbb{N}}$. Moreover, since the linear operator K is weakly continuous on $L^1[0,T]$ by Lemma 4.2, the sequence $(\Pi_T K\mathcal{N}_v x_{n_k})_{k\in\mathbb{N}}$, *i.e.*, $(\Pi_T A x_{n_k})_{k\in\mathbb{N}}$, converges pointwise almost everywhere on $[0,T]$. Using Vitali's convergence theorem [32], p.94, we conclude that $(\Pi_T A x_{n_k})_{k\in\mathbb{N}}$ is strongly convergent in $L^1[0,T]$. The arbitrariness of $T > 0$ implies that $(Ax_{n_k})_{k\in\mathbb{N}}$ is strongly convergent in $\mathcal{L}^1_{\mathrm{loc}}$. Accordingly, A is ws-compact and the condition (i) of Theorem 3.4 is fulfilled.

(2) For all x_1, x_2 and $y \in L^1[0,T]$ where $T > 0$ is given, by $(\mathcal{H}3)$ we obtain

$$\begin{aligned}\bigl|F(x_1,y) - F(x_2,y)\bigr|_T &= \int_0^T \bigl|f\bigl(t,x_1(t),y(t)\bigr) - f\bigl(t,x_2(t),y(t)\bigr)\bigr|\,dt \\ &\leq \alpha_T \int_0^T \bigl|x_1(t) - x_2(t)\bigr|\,dt = \alpha_T|x_1 - x_2|_T.\end{aligned}$$

(3) From assumption $(\mathcal{H}_3)$ it follows that, for a given $T > 0$,

$$\begin{aligned}\bigl|F(x_1,y_1) - F(x_2,y_2)\bigr|_T &\leq \int_0^T \bigl|f\bigl(t,x_1(t),y_1(t)\bigr) - f\bigl(t,x_2(t),y_2(t)\bigr)\bigr|\,dt \\ &\leq \int_0^T \alpha(t)\bigl|x_1(t) - x_2(t)\bigr|\,dt + \int_0^T \beta(t)\bigl|y_1(t) - y_2(t)\bigr|\,dt \\ &\leq \alpha_T|x_1 - x_2|_T + \beta_T|y_1 - y_2|_T \quad \bigl(x_i, y_i \in \mathcal{L}^1_{\mathrm{loc}}, i = 1,2\bigr),\end{aligned}$$

which implies the continuity of $\Pi_T F$. The arbitrariness of $T > 0$ shows that F is continuous on $\mathcal{L}^1_{\mathrm{loc}} \times \mathcal{L}^1_{\mathrm{loc}}$.

Now we check that F maps relatively weakly compact sets of $\mathcal{L}^1_{\mathrm{loc}} \times \mathcal{L}^1_{\mathrm{loc}}$ into relatively weakly compact ones of $\mathcal{L}^1_{\mathrm{loc}}$. Note that $(\mathcal{H}_3)$ leads to the estimate

$$\bigl|f(t,x,y)\bigr| \leq \bigl|f(t,0,0)\bigr| + \alpha(t)|x| + \beta(t)|y|.$$

Let $M, N \in \mathfrak{B}(\mathcal{L}^1_{\mathrm{loc}})$. For a given $T > 0$ and $D \subseteq [0,T]$ with $\mathrm{meas}(D) \leq \varepsilon$ we have

$$\int_D \bigl|f\bigl(t,x(t),y(t)\bigr)\bigr|\,dt \leq \int_D \bigl|f(t,0,0)\bigr|\,dt + \alpha_T \int_D \bigl|x(t)\bigr|\,dt + \beta_T \int_D \bigl|y(t)\bigr|\,dt,$$

where $x \in M$ and $y \in N$. Taking into account the fact that the set consisting of one element is weakly compact, by means of (2.2) we obtain

$$\omega_T\bigl(F(M \times N)\bigr) \leq \alpha_T\omega_T(M) + \beta_T\omega_T(N),$$

which implies that $\Pi_T F$ maps relatively weakly compact sets of $\mathcal{L}^1_{\mathrm{loc}} \times \mathcal{L}^1_{\mathrm{loc}}$ into relatively weakly compact ones of $\mathcal{L}^1_{\mathrm{loc}}$. The arbitrariness of $T > 0$ implies that the condition (ii) of Theorem 3.4 is fulfilled.

(4) We will prove that if there exists $x \in \mathcal{L}^1_{\mathrm{loc}}$ such that

$$x(t) = \lambda f\bigl(t, x(t)/\lambda, (Ax)(t)\bigr),$$

then x is bounded for any $\lambda \in (0,1)$. In fact, we have

$$\begin{aligned}\bigl|x(t)\bigr| &\leq \lambda\bigl|f(t,0,0)\bigr| + \lambda\bigl|f\bigl(t,x(t)/\lambda,(Ax)(t)\bigr) - f(t,0,0)\bigr| \\ &\leq \bigl|f(t,0,0)\bigr| + \alpha(t)\bigl|x(t)\bigr| + \beta(t)\bigl|(Ax)(t)\bigr|,\end{aligned}$$

by integrating on [0,T] for an arbitrarily given $T > 0$, it follows that

$$|x|_T \leq \bigl|f(t,0,0)\bigr|_T + \alpha_T|x|_T + \beta_T|K|_T\bigl(|a|_T + b|x|_T\bigr),$$

which implies that

$$|x|_T \leq \frac{|f(t,0,0)|_T + \beta_T|K|_T|a|_T}{1 - \alpha_T - b\beta_T|K|_T} := L_T.$$

The arbitrariness of $T > 0$ implies that the set $\{x \in \mathcal{L}^1_{\mathrm{loc}} : x = \lambda F(x/\lambda, Ax)\}$ is bounded for any $\lambda \in (0,1)$. According to Theorem 3.4, (1.2) has at least one solution x in $\mathcal{L}^1_{\mathrm{loc}}$, which completes the proof. □

Example 4.4 Consider the following nonlinear integral equation:

$$\begin{aligned}\varphi(t) &= \frac{\sin t}{2 + |\varphi(t)|}\sin\left(\frac{1}{2}\int_0^t e^{-\sqrt{t-s}}\sqrt{\frac{s^4 + \varphi^2(s)}{t-s}}\,ds\right) \\ &\quad + \frac{t}{8 + 2t^2}\int_0^t e^{-\sqrt{t-s}}\sqrt{\frac{s^4 + \varphi^2(s)}{t-s}}\,ds, \quad t \in \mathbb{R}^+.\end{aligned}$$

In order to show that such an equation admits a solution in $\mathcal{L}^1_{\mathrm{loc}}$, we are going to check that the conditions of Theorem 4.3 are satisfied. To this end, define the functions as follows:

$$\begin{aligned}&\kappa : \Delta \to \mathbb{R}, \qquad \kappa(t,s) = \frac{e^{-\sqrt{t-s}}}{2\sqrt{t-s}}, \qquad \Delta := \bigl\{(t,s) : 0 \leq s \leq t < \infty\bigr\}; \\ &\nu : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}, \qquad \nu(t,x) = \sqrt{t^4 + x^2}; \\ &f : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}, \qquad f(t,x,y) = \frac{\sin t \sin y}{2 + |x|} + \frac{ty}{4 + t^2}.\end{aligned}$$

Obviously, ν and f are Carathéodory functions, and κ is measurable. Furthermore, we have

$$\bigl|\nu(t,x)\bigr| = \sqrt{t^4 + x^2} \leq t^2 + |x| = a(t) + b|x|,$$

where $a(t) := t^2$ and $b := 1$; in addition, we have

$$\begin{aligned}&\bigl|f(t,x_1,y_1) - f(t,x_2,y_2)\bigr| \\ &\quad \leq |\sin t||\sin y_1|\left|\frac{1}{2 + |x_1|} - \frac{1}{2 + |x_2|}\right| + \frac{|\sin t|}{2 + |x_2|}|y_1 - y_2| + \frac{t}{4 + t^2}|y_1 - y_2| \\ &\quad \leq \frac{|\sin t|}{4}|x_1 - x_2| + \left(\frac{|\sin t|}{2} + \frac{t}{4 + t^2}\right)|y_1 - y_2| = \alpha(t)|x_1 - x_2| + \beta(t)|y_1 - y_2|,\end{aligned}$$

where $\alpha(t) := |\sin t|/4$ and $\beta(t) := |\sin t|/2 - t/(4+t^2)$. It follows that $(\mathcal{H}_1)$-$(\mathcal{H}_3)$ are satisfied. By a simple calculation, we obtain

$$|\alpha|_T = \operatorname{ess}\sup_{t\in[0,T]} \frac{|\sin t|}{4} \leq \frac{1}{4}, \qquad |\beta|_T = \operatorname{ess}\sup_{t\in[0,T]} \left(\frac{|\sin t|}{2} + \frac{t}{4+t^2}\right) \leq \frac{3}{4},$$

$$|K|_T = \operatorname{ess}\sup_{s\in[0,T]} \int_s^T \left|\kappa(t,s)\right| dt = \operatorname{ess}\sup_{s\in[0,T]} \int_s^T \frac{e^{-\sqrt{t-s}}}{2\sqrt{t-s}}\, dt = 1 - e^{-\sqrt{T}}.$$

It follows that

$$\alpha_T + b\beta_T|K|_T \leq \frac{1}{4} + \frac{3}{4}\left(1 - e^{-\sqrt{T}}\right) = 1 - \frac{3}{4e^{\sqrt{T}}} < 1,$$

which shows that $(\mathcal{H}_4)$ is satisfied.

Since the assumptions $(\mathcal{H}_1)$-$(\mathcal{H}_4)$ are all satisfied, we apply Theorem 4.3 to derive the existence of solutions to the equation of this example.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing the article. All authors read and approved the final manuscript.

Acknowledgements
The authors would like to express their thanks to the referees for the careful reading and helpful remarks on the manuscript. This research was supported by Natural Science Foundation of the Jiangsu Higher Education Institutions of China (No. 15KJB110001).

Received: 16 June 2015 Accepted: 10 August 2015 Published online: 25 August 2015

References

1. Krasnosel'skii, MA: Two remarks on the method of successive approximations. Usp. Mat. Nauk **10**, 123-127 (1955) (in Russian)
2. Smart, DR: Fixed Points Theorems. Cambridge University Press, Cambridge (1980)
3. Agarwal, RP, O'Regan, D, Taoudi, MA: Fixed point theorems for ws-compact mappings in Banach spaces. Fixed Point Theory Appl. **2010**, Article ID 183596 (2010)
4. Agarwal, RP, O'Regan, D, Taoudi, MA: Browder-Krasnoselskii-type fixed point theorems in Banach spaces. Fixed Point Theory Appl. **2010**, Article ID 243716 (2010)
5. Agarwal, RP, O'Regan, D, Taoudi, MA: Fixed point theorems for general classes of maps acting on topological vector spaces. Asian-Eur. J. Math. **4**(3), 373-387 (2011)
6. Ben Amar, A, Garcia-Falset, J: Fixed point theorems for 1-set weakly contractive and pseudocontractive operators on an unbounded domain. Port. Math. **68**(2), 125-147 (2011)
7. Garcia-Falset, J: Existence of fixed points for the sum of two operators. Math. Nachr. **283**, 1736-1757 (2010)
8. Garcia-Falset, J, Latrach, K, Moreno-Gálvez, E, Taoudi, MA: Schaefer-Krasnoselskii fixed point theorems using a usual measure of weak noncompactness. J. Differ. Equ. **252**, 3436-3453 (2012)
9. Latrach, K, Taoudi, MA, Zeghal, A: Some fixed point theorems of the Schauder and Krasnosel'skii type and application to nonlinear transport equations. J. Differ. Equ. **221**, 256-271 (2006)
10. Latrach, K, Taoudi, MA: Existence results for a generalized nonlinear Hammerstein equation on L_1 spaces. Nonlinear Anal. **66**, 2325-2333 (2007)
11. Wang, F: Fixed-point theorems for the sum of two operators under ω-condensing. Fixed Point Theory Appl. **2013**, 102 (2013)
12. Wang, F: Solvability of a general nonlinear integral equation in L^1 spaces by means of a measure of weak noncompactness. J. Integral Equ. Appl. **27**(2), 1-15 (2015). doi:10.1216/JIE-2015-27-2-1
13. Cain, GL Jr., Nashed, MZ: Fixed points and stability for a sum of two operators in locally convex spaces. Pac. J. Math. **39**(3), 581-592 (1971)
14. Sehgal, VM, Singh, SP: On a fixed point theorem of Krasnoselskii for locally convex spaces. Pac. J. Math. **62**(2), 561-567 (1976)
15. Barroso, CS, Teixeira, EV: A topological and geometric approach to fixed points results for sum of operators and applications. Nonlinear Anal. **60**, 625-650 (2005)
16. Hoa, LH, Schmitt, K: Fixed point theorems of Krasnosel'skii type in locally convex spaces and applications to integral equations. Results Math. **25**, 290-314 (1994)
17. Ngoc, LTP, Long, NT: On a fixed point theorem of Krasnosel'skii type and application to integral equations. Fixed Point Theory Appl. **2006**, Article ID 30847 (2006)

18. Vladimirescu, C: Remark on Krasnoselskii's fixed point theorem. Nonlinear Anal. **71**, 876-880 (2009)
19. Olszowy, L: Fixed point theorems in the Fréchet space $C(\mathbb{R}_+)$ and functional integral equations on an unbounded interval. Appl. Math. Comput. **218**, 9066-9074 (2012)
20. Olszowy, L: A family of measures of noncompactness in the space $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and its application to some nonlinear Volterra integral equation. Mediterr. J. Math. **11**, 687-701 (2014)
21. Wang, F: A fixed point theorem for nonautonomous type superposition operators and integrable solutions of a general nonlinear functional integral equation. J. Inequal. Appl. **2014**, 487 (2014)
22. Wilansky, A: Modern Methods in Topological Vector Spaces. McGraw Hill, New York (1978)
23. Dunford, N, Schwartz, JT: Linear Operators Part I: General Theory. Interscience Publishers, New York (1958)
24. Banaś, J, Rivero, J: On measures of weak noncompactness. Ann. Math. Pures Appl. **151**, 213-224 (1988)
25. De Blasi, FS: On a property of the unit sphere in Banach spaces. Bull. Math. Soc. Sci. Math. Roum. **21**, 259-262 (1977)
26. Appell, J, De Pascale, E: Su alcuni parametri connessi con la misura di non compattezza di Hausdorff in spazi di funzioni misurabili. Boll. Unione Mat. Ital., B **3**(6), 497-515 (1984)
27. Isac, G, Gowda, MS: Operators of class $(S)^1_+$, Altman's condition and the complementarity problem. J. Fac. Sci., Univ. Tokyo, Sect. 1A, Math. **40**, 1-16 (1993)
28. Jachymski, J: On Isac's fixed point theorem for selfmaps of a Galerkin cone. Ann. Sci. Math. Qué. **18**(2), 169-171 (2004)
29. Krasnosel'skii, MA: On the continuity of the operator $Fu(x) = f(x, u(x))$. Dokl. Akad. Nauk SSSR **77**, 185-188 (1951) (in Russian)
30. Krasnosel'skii, MA: Topological Methods in the Theory of Nonlinear Integral Equations. Pergamon Press, New York (1964)
31. Appell, J, Zabrejko, PP: Nonlinear Superposition Operators. Cambridge Tracts in Math., vol. 95. Cambridge University Press, Cambridge (1990)
32. Royden, HL, Fitzpatrick, PM: Real Analysis, 4th edn. Pearson Education Ltd., Boston (2010)