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A viscous thin-film equation with a singular diffusion

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Abstract

The paper is devoted to studying a viscous thin-film equation with a singular diffusion term and the periodic boundary conditions in multidimensional space, which has a lot of applications in fluids theory such as draining of foams and the movement of contact lenses. In order to obtain the necessary uniform estimates and overcome the difficulty of a singular diffusion term, the entropy functional method is used. Finally, the existence of nonnegative weak solutions is obtained by some compactness arguments.

Keywords: fourth-order parabolic; thin-film equation; entropy functional; singular diffusion

1 Introduction

The research of the Cahn-Hilliard equation and the thin-film equation has become a hot topic recently. The Cahn-Hilliard equation (see [1]) can describe the evolution of a conserved concentration field during phase separation, which has the form $u_t + \nabla \cdot (m\nabla(\varepsilon^2 \Delta u + f'(u))) = 0$ where m, f, ε^2 denote the atomic mobility, the free energy, the parameter proportional to the interface energy, respectively. $-(\varepsilon^2 \Delta u + f'(u))$ can be taken as the chemical potential. For the linear or degenerate mobility, Elliott, Zheng, and Garcke [2, 3] have studied its existence and obtained some properties of solutions. Besides, Liang and Zheng [4] obtained the existence and stability results for this model with a gradient mobility by studying the corresponding semi-discrete problems.

The thin-film equation is usually used to describe the motion of a very thin layer of viscous incompressible fluids along an inclined plane such as the draining of foams and the movement of contact lenses. It can be taken as a class of fourth-order degenerate parabolic equations [5]:

$$u_t + \big(m(u)u_{xxx} + f(u, u_x, u_{xx})\big)_x = 0,$$

where the mobility $m(u)$ degenerates at $u = 0$. For example, thin-film flows driven by the surface tension can be modeled by the following fourth-order degenerate parabolic equations:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^3}{3}(Ch_{xxx} - \delta Bh_x \cos\alpha + B\sin\alpha) + A\frac{u_x}{u} + \frac{M}{2}\sigma_x u^2\right) = 0.$$

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For the simplified thin-film equation $u_t + (u^n u_{xxx})_x = 0$, Bernis and Friedman [6] gave the first result to the existence and nonnegativity of weak solutions. Bertozzi and Pugh [7] have studied the existence in the distributional sense and the long time decay for the model of the thin-film equation with a second-order diffusion term. Boutat *et al.* [8] studied a generalized thin-film equation with period boundary in multidimensional space. Furthermore, Liang [9] has investigated the existence of the weak solutions and strong solutions with the initial function near a steady state solution. For other results, the reader may refer to [10–14] and [15].

In this paper, we study the following viscous thin-film equation with a singular diffusion:

$$\begin{cases} u_t - \nabla \cdot (u^n \nabla w) + A \nabla \cdot (\frac{\nabla u}{u^\alpha}) = 0 & \text{in } Q_T, \\ w = -\Delta u + \nu u_t & \text{in } Q_T, \\ u \text{ is } \Omega\text{-periodic}, \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases} \tag{1}$$

where $\Omega = (-1,1)^N$, $Q_T = \Omega \times (0,T)$. n, A, α, and ν are all constants with $n, \alpha, \nu > 0$.

For convenience, we introduce some notations:

- C is denoted as a positive constant and may change from line to line.
- $\Omega = (-1,1)^N$, $\Gamma_j = \partial\Omega \cap \{x_j = -1\}$, $\Gamma_{j+N} = \partial\Omega \cap \{x_j = 1\}$.
- $H^m_{\text{per}}(\Omega)$ is the periodic Sobolev space *i.e.*

$$H^m_{\text{per}}(\Omega) = \left\{ u \in H^m(\Omega) \big| D^\xi u \big|_{\Gamma_j} = D^\xi u|_{\Gamma_{j+N}}, j = 1, \ldots, N, |\xi| \le m-1 \right\}.$$

- The following norms on $H^m_{\text{per}}(\Omega)$ ($m \ge 1$) are equivalent:

$$\|u\|_{H^m(\Omega)}, \qquad \|u\|_{L^2(\Omega)} + \|D^m u\|_{L^2(\Omega)} \quad \text{and} \quad |\overline{u}| + \|D^m u\|_{L^2(\Omega)},$$

  where $\overline{u} = \frac{1}{2^N} \int_\Omega u(x)\,dx$ (see [8]).
- $C^m_{\text{per}}(\overline{\Omega}) = \{u \in C^m(\overline{\Omega}) | D^\xi u|_{\Gamma_j} = D^\xi u|_{\Gamma_{j+N}}, j = 1, \ldots, N, |\xi| \le m-1\}$.
- $a_+ = \max\{a, 0\}$, $a_- = \min\{a, 0\}$ for $a \in R$.

Our main result is the following theorem.

Theorem 1 *Let $\alpha \in (0, \frac{1}{2}]$, $u_0 \in H^1(\Omega)$,*

$$n \in \begin{cases} (\frac{6}{7}, 2), & N = 1; \\ (\frac{8}{9}, 2), & N = 2; \\ (\frac{16}{17}, 2), & N = 3. \end{cases}$$

Suppose $A \le 0$ or $\alpha \le 1 - n$. Then there exist at least one pair of solutions (u, w) satisfying

1. *$u \in L^2(0, T; H^2_{\text{per}}(\Omega)) \cap C([0,T]; H^1(\Omega))$, $u^{-\frac{1}{2}}|\nabla u| \in L^4(Q_T)$, $w, u_t \in L^2(Q_T)$;*
2. *for any test function $\phi \in C^1([0,T]; C^2_{\text{per}}(\Omega))$, one has*

$$\iint_{Q_T} u_t \phi\,dx\,dt + \iint_{Q_T} u^n w \Delta\phi\,dx\,dt$$

$$+ n \iint_{Q_T} u^{n-1} \nabla u w \nabla\phi\,dx\,dt - A \iint_{Q_T} \frac{\nabla u \nabla\phi}{u^\alpha}\,dx\,dt = 0,$$

$$\iint_{Q_T} w\phi\,dx\,dt = -\iint_{Q_T} \Delta u \phi\,dx\,dt + \nu \iint_{Q_T} u_t \phi\,dx\,dt.$$

The following lemmas are needed in the paper.

Lemma 1 (Bernis, see [8]) *Let $u \in H^2_{\text{per}}(\Omega)$ be a nonnegative function. There exists a constant $\mu > 0$ such that the following inequality holds*:

$$\int_\Omega \frac{|\nabla u|^4}{u^2} \, \mathrm{d}x \leq \mu \|u\|^2_{H^2(\Omega)}.$$

Lemma 2 (Aubin-Lions, see [16]) *Let X, B, and Y be Banach spaces and assume $X \hookrightarrow B \hookrightarrow Y$ with compact imbedding $X \hookrightarrow B$.*

(1) *Let $\mathfrak{F}$ be bounded in $L^p(0, T; X)$ where $1 \leq p < \infty$, and $\frac{\partial \mathfrak{F}}{\partial t} = \{\frac{\partial f}{\partial t} : f \in \mathfrak{F}\}$ be bounded in $L^1(0, T; Y)$. Then $\mathfrak{F}$ is relatively compact in $L^p(0, T; B)$.*

(2) *Let $\mathfrak{F}$ be bounded in $L^\infty(0, T; X)$, and $\frac{\partial \mathfrak{F}}{\partial t} = \{\frac{\partial f}{\partial t} : f \in \mathfrak{F}\}$ be bounded in $L^r(0, T; Y)$ where $r > 1$. Then $\mathfrak{F}$ is relatively compact in $C([0, T]; B)$.*

Lemma 3 (see [17] or [18]) *Let V be a real, separable, reflexive Banach space and H is a real, separable, Hilbert space. $V \hookrightarrow H$ is continuous and V is dense in H. Then $\{u \in L^2(0, T; V) | u_t \in L^2(0, T; V')\}$ is continuously imbedded in $C([0, T]; H)$.*

The paper is arranged as follows. The existence of solutions to the approximate problem will be proved in Section 2. In Sections 3 and 4, we will take the limit for small parameters $\delta \to 0$ and $\varepsilon \to 0$, respectively.

2 Approximate problem

This section is devoted to studying the following approximate problem:

$$\begin{cases} u_t - \nabla \cdot ((u_+ + \delta)^n \nabla w) + A \nabla \cdot \left(\frac{(u_+ + \delta)^n \nabla u}{(u_+ + \varepsilon)^{n+\alpha}(1+\varepsilon|\nabla u|^2)}\right) = 0 & \text{in } Q_T, \\ w = -\Delta u + \nu u_t & \text{in } Q_T, \\ u \text{ is } \Omega\text{-periodic}, \\ u(x, 0) = u_{0\delta\varepsilon}(x) & \text{on } \Omega \end{cases} \qquad (2)$$

for $0 < \delta < \varepsilon < 1$ and $u_+ = \max\{u, 0\}$.

Lemma 4 *Let $u_{0\delta\varepsilon} \in H^1_{\text{per}}(\Omega)$, $\alpha > 0$, and $0 < n < 2$. Then there exist at least a pair of solutions (u, w) to (2) satisfying*

1. *$u \in L^2(0, T; H^3_{\text{per}}(\Omega)) \cap C([0, T]; H^2_{\text{per}}(\Omega))$, $u_t \in L^2(Q_T)$, $w \in L^2(0, T; H^1_{\text{per}}(\Omega))$, and $u(x, 0) = u_0$;*

2. *for any test function $\phi \in L^2(0, T; H^1_{\text{per}}(\Omega))$, one has*

$$\iint_{Q_T} u_t \phi \, \mathrm{d}x \, \mathrm{d}t + \iint_{Q_T} (u_+ + \delta)^n \nabla w \nabla \phi \, \mathrm{d}x \, \mathrm{d}t$$

$$- A \iint_{Q_T} \left(\frac{(u_+ + \delta)^n \nabla u}{(u_+ + \varepsilon)^{n+\alpha}(1 + \varepsilon|\nabla u|^2)}\right) \nabla \phi \, \mathrm{d}x \, \mathrm{d}t = 0,$$

$$\iint_{Q_T} w \phi \, \mathrm{d}x \, \mathrm{d}t = - \iint_{Q_T} \Delta u \phi \, \mathrm{d}x \, \mathrm{d}t + \nu \iint_{Q_T} u_t \phi \, \mathrm{d}x \, \mathrm{d}t.$$

Proof We will apply the Galerkin method to obtain the existence of solutions. Let $\{\phi_i\}_{i=1,2,3,...}$ be the eigenfunctions of the Laplace operator $-\Delta\phi_i = \lambda_i\phi_i$ with periodic boundary value conditions. Moreover, those eigenfunctions are orthogonal in H^1 and L^2 spaces and we can normalize ϕ_i such that $(\phi_i, \phi_j) = \delta_{ij} = \left\{ \begin{smallmatrix} 1, & i=j, \\ 0, & i\neq j, \end{smallmatrix} \right.$ where we define $\lambda_1 = 0$, $\phi_1 = 1$, and $(\cdot, \cdot)$ denotes the scalar product of the L^2 space.

Let M denote a positive integer and define $w^M(x,t) = \sum_{i=1}^{M} d_i(t)\phi_i(x)$, $u^M(x,t) = \sum_{i=1}^{M} c_i(t)\phi_i(x)$, $u^M(x,0) = \sum_{i=1}^{M}(u_0, \phi_i)\phi_i$. For $j = 1, \ldots, M$, we consider the following system of ordinary differential equations:

$$\frac{d}{dt}(u^M, \phi_j) = -\left((u_+^M + \delta)^n \nabla w^M, \nabla\phi_j\right)$$
$$+ A\left(\left(\frac{(u_+^M + \delta)^n \nabla u^M}{(u_+ + \varepsilon)^{n+\alpha}(1 + \varepsilon|\nabla u^M|^2)}\right), \nabla\phi_j\right), \tag{3}$$

$$(w^M, \phi_j) = -(\Delta u^M, \phi_j) + v\frac{d}{dt}(u^M, \phi_j). \tag{4}$$

The ODE existence theorem yields the local unique existence of this initial value problem since the right hand side depends on c_i continuously. In order to show the global solvability, we take $-\Delta u^M$ as the test function and apply the Young inequality to get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u^M|^2\, dx + v\int_\Omega |u_t^M|^2\, dx + \int_\Omega (u_+^M + \delta)^n |\nabla w^M|^2\, dx$$
$$= A\int_\Omega \frac{(u_+^M + \delta)^n \nabla u^M \nabla w^M}{(u_+^M + \varepsilon)^{n+\alpha}(1 + \varepsilon|\nabla u^M|^2)}\, dx$$
$$\leq \frac{1}{2}\int_\Omega (u_+^M + \delta)^n |\nabla w^M|^2\, dx + \frac{C}{\varepsilon^\alpha}\int_\Omega |\nabla u^M|^2\, dx. \tag{5}$$

It gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u^M|^2\, dx + v\int_\Omega |u_t^M|^2\, dx + \int_\Omega (u_+^M + \delta)^n |\nabla\Delta u^M|^2\, dx$$
$$= \frac{C}{\varepsilon^\alpha}\int_\Omega |\nabla u^M|^2\, dx. \tag{6}$$

The mass conservation property $\int_\Omega u^M(x,t)\, dx = \int_\Omega u_0^M(x)\, dx$ (by letting $j = 1$) ensures that Poincaré's inequality can be applied. On the other hand, the Gronwall inequality yields

$$\sup_{t\in(0,T)}\int_\Omega (|u^M|^2 + |\nabla u^M|^2)(x,t)\, dx + \iint_{Q_T} |u_t^M|^2\, dx\, dt + \iint_{Q_T} |\nabla w^M|^2\, dx$$
$$\leq C. \tag{7}$$

Therefore, we have obtained

$$u^M \in L^\infty(0, T; H^1_{per}(\Omega)), \qquad w^M \in L^2(0, T; H^1_{per}(\Omega)), \qquad u_t^M \in L^2(Q_T). \tag{8}$$

The classic L^p-estimate of the second-order elliptic equations implies

$$u^M \in L^2(0, T; H^3_{per}(\Omega)). \tag{9}$$

By (8), (10), and Lemma 2, we conclude that there exist a pair of functions (u, w) and a subsequence of (u^M, w^M) such that as $M \to \infty$,

$$u^M \rightharpoonup u \quad \text{weakly* in } L^\infty\big(0, T; H^1_{\text{per}}(\Omega)\big); \tag{10}$$

$$w^M \rightharpoonup w \quad \text{weakly in } L^2\big(0, T; H^1_{\text{per}}(\Omega)\big); \tag{11}$$

$$u^M_t \rightharpoonup u_t \quad \text{weakly in } L^2(Q_T); \tag{12}$$

$$u^M \to u \quad \text{strongly in } L^2\big(0, T; H^2_{\text{per}}(\Omega)\big); \tag{13}$$

$$\nabla u^M \to \nabla u \quad \text{a.e. in } (Q_T)^N; \tag{14}$$

$$u^M \to u \quad \text{a.e. in } Q_T \text{ and strongly in } C\big([0, T]; L^2(\Omega)\big). \tag{15}$$

Moreover, by Lemma 3, (12), (13), and the embedding $H^2(\Omega) \hookrightarrow W^{1,4}(\Omega)$, we have

$$u^M, u \in C\big([0, T]; H^1_{\text{per}}(\Omega)\big), \tag{16}$$

$$\nabla u^M, \nabla u \in L^4(Q_T). \tag{17}$$

From Vitali's theorem, we get

$$\big(u^M_+ + \delta\big)^n \to \big(u^M_+ + \delta\big)^n \quad \text{strongly in } L^4(Q_T); \tag{18}$$

$$\nabla u^M \to \nabla u \quad \text{strongly in } L^4(Q_T); \tag{19}$$

$$\frac{(u^M_+ + \delta)^n \nabla u^M}{(u^M_+ + \varepsilon)^{n+\alpha}(1 + \varepsilon|\nabla u^M|^2)} \to \frac{(u_+ + \delta)^n \nabla u}{(u_+ + \varepsilon)^{n+\alpha}(1 + \varepsilon|\nabla u|^2)} \quad \text{strongly in } L^2(Q_T). \tag{20}$$

Let T_M denote the projection from the space $L^2(\Omega)$ to $Span\{\phi_1, \ldots, \phi_M\}$. By multiplying equation (3) by $T_M \phi$ for $\phi \in L^2(0, T; H^1_{\text{per}}(\Omega))$, one has

$$\iint_{Q_T} u^M_t T_M \phi \, dt + \iint_{Q_T} \big(u^M_+ + \delta\big)^n \nabla w^M \nabla T_M \phi \, dx \, dt$$

$$= A \iint_{Q_T} \left(\frac{(u^M_+ + \delta)^n \nabla u^M}{(u^M_+ + \varepsilon)^{n+\alpha}(1 + \varepsilon|\nabla u^M|^2)} \right) \nabla T_M \phi \, dx \, dt, \tag{21}$$

$$\iint_{Q_T} w^M T_M \phi \, dx \, dt = - \iint_{Q_T} \Delta u^M T_M \phi \, dx \, dt + \nu \iint_{Q_T} u^M_t T_M \phi \, dx \, dt. \tag{22}$$

By (10)-(20), we can perform the limit $M \to \infty$ in each term of (21)-(22). □

3 The limit $\delta \to 0$

We shall perform the limit $\delta \to 0$ in the section to the solutions obtained by Lemma 4 and we suppose that the initial function $u_{0\delta\varepsilon} \to u_{0\varepsilon} \in H^1(\Omega)$ as $\delta \to 0$ and $u_{0\varepsilon} \geq 0$.

The main result of this section is the following.

Proposition 1 *Let*

$$n \in \begin{cases} (\frac{6}{7}, 2), & N = 1; \\ (\frac{8}{9}, 2), & N = 2; \\ (\frac{16}{17}, 2), & N = 3. \end{cases}$$

Then there exist at least a pair of functions $(\overline{u}, \overline{w})$ satisfying

1. $\overline{w} \in L^2(Q_T), \overline{u} \in L^2(0, T; H^2_{per}(\Omega)) \cap C([0, T]; H^1_{per}(\Omega)), \overline{u}_t \in L^2(Q_T)$, and $\overline{u}(x, 0) = u_{0\varepsilon}$;
2. *for any test function $\phi \in L^2(0, T; C^\infty_{per}(\overline{\Omega}))$, one has*

$$\iint_{Q_T} \overline{u}_t \phi \, dx \, dt - \iint_{Q_T} \overline{u}^n \overline{w} \Delta \phi \, dx \, dt - n \iint_{Q_T} \overline{u}^{n-1} \overline{w} \nabla \overline{u} \nabla \phi \, dx \, dt$$

$$- A \iint_{Q_T} \frac{\overline{u}^n \nabla \overline{u} \nabla \phi}{(\overline{u} + \varepsilon)^{n+\alpha}(1 + \varepsilon |\nabla \overline{u}|^2)} \, dx \, dt = 0,$$

$$\iint_{Q_T} \overline{w} \phi \, dx \, dt = - \iint_{Q_T} \Delta \overline{u} \phi \, dx \, dt + \nu \iint_{Q_T} \overline{u}_t \phi \, dx \, dt.$$

In order to prove this proposition, we have to establish some uniform energy estimates independent of δ and thus we introduce a nonnegative convex functional $\Phi_\delta(\cdot)$ (see[8]):

If $0 \le n < 2, n \ne 1$,

$$\Phi_\delta(\sigma) = \begin{cases} \frac{1}{(1-n)(2-n)}(\sigma + \delta)^{2-n} - \frac{1}{1-n}(\sigma + \delta) + \frac{1}{2-n}, & \sigma \ge 0; \\ \frac{(\sigma)^2}{2\delta^n} + \frac{1}{1-n}(\delta^{1-n} - 1)\sigma + \frac{1}{2-n}, & \sigma < 0. \end{cases}$$

If $n = 1$,

$$\Phi_\delta(\sigma) = \begin{cases} (\sigma + \delta)Ln(\sigma + \delta) - (\sigma + \delta) + 1, & \sigma \ge 0; \\ \frac{(\sigma)^2}{2\delta} + \sigma(Ln\delta) + \delta(Ln\delta) - \delta + 1, & \sigma < 0. \end{cases}$$

It is easy to check that $\Phi_\delta \in W^{2,+\infty}_{loc}(R)$, $\Phi''_\delta(\sigma) = \frac{1}{(\sigma_+ + \delta)^n}$.

By applying this functional, we can get the following estimates.

Lemma 5 *There exist some constants C independent of δ (may depend on ε) such that*

1. $\frac{d}{dt} \int_\Omega \Phi(u(x, t)) \, dx + \int_\Omega |w|^2 \, dx + \nu \int_\Omega |u_t|^2 \, dx \le C$;
2. $\|w\|_{L^2(Q_T)} \le C, \|u\|_{L^2(0,T;H^2_{per}(\Omega))} \le C$;
3. $\|u\|_{L^\infty(0,T;H^1_{per}(\Omega))} \le C$;
4. $\iint_{Q_T} (u_+ + \delta)^n |\nabla w|^2 \, dx \, dt \le C$;
5. $\|u_t\|_{L^2(Q_T)} \le C$.

Proof By choosing $\Phi'(u)$ as the test function in (2), we get

$$\frac{d}{dt} \int_\Omega \Phi(u(x, t)) \, dx + \int_\Omega |w|^2 \, dx + \nu \int_\Omega |u_t|^2 \, dx$$

$$= A \int_\Omega \frac{|\nabla u|^2}{(u_+ + \varepsilon)^{n+\alpha}(1 + \varepsilon |\nabla u|^2)} \, dx \le \frac{|A|}{\varepsilon^{n+\alpha+1}}. \tag{23}$$

This implies

$$\|u\|_{L^2(0,T;H^2_{per}(\Omega))} \le C \|w\|_{L^2(Q_T)} \le C, \tag{24}$$

which yields the results 1-2. Similar to (5), we conclude that

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla u|^2 \, dx + \nu \int_\Omega |u_t|^2 \, dx + \frac{1}{2} \int_\Omega (u_+ + \delta)^n |\nabla w|^2 \, dx \le C(\varepsilon) \int_\Omega |\nabla u|^2 \, dx,$$

which gives 3 and 4. $\qquad\square$

Lemma 6 *There exist a pair of functions* $(\overline{u}, \overline{w})$ *such that, as* $\delta \to 0$,

1. $u \rightharpoonup \overline{u}$ *weakly in* $L^2(0, T; H^2_{\mathrm{per}}(\Omega))$;
2. $w \rightharpoonup \overline{w}$ *weakly in* $L^2(Q_T)$;
3. $u_t \to \overline{u}_t$ *weakly in* $L^2(Q_T)$;
4. $u \to \overline{u}$ *strongly in* $L^2(0, T; H^1_{\mathrm{per}}(\Omega))$ *and a.e. in* Q_T;
5. $u \to \overline{u}$ *strongly in* $C([0, T]; L^2(\Omega))$;
6. *if* $\sup_{\delta \in (0,1)} \int_\Omega \Phi(u_0) \, dx < \infty$, *then* $\overline{u} \geq 0$ *in* $\overline{Q}_T$ *and* $\sup_{t \leq T} \|u_-(t)\|_{L^2(\Omega)} \leq C \delta^{\frac{n}{2}}$ *when* $n \leq 1$;
7. $\overline{u} \in L^2(0, T; H^2_{\mathrm{per}}(\Omega)) \cap C([0, T]; L^2(\Omega))$, $\overline{u}_t \in L^2(Q_T)$, $\overline{w} \in L^2(Q_T)$.

Proof The results 1-3 can be obtained from Lemma 5, and Lemma 2 can give 4 and 5. In order to prove 6-7, we integrate (23) over $(0, T)$ to get

$$0 \leq \int_\Omega \Phi(u(x, t)) \, dx \leq \frac{n|A|T}{\varepsilon^{n+\alpha+1}} + \sup_{\delta \in (0,1)} \int_\Omega \Phi(u_0(x)) \, dx \leq C(\varepsilon).$$

If $n < 1$, we have

$$0 \leq \frac{1}{2} \int_\Omega u_-^2(x, t) \, dx \leq \frac{\delta^n}{n-1} (\delta^{1-n} - 1) \int_\Omega u_-(x, t) \, dx + C(\varepsilon)\delta^n.$$

If $n = 1$, we have

$$0 \leq \frac{1}{2} \int_\Omega u_-^2(x, t) \, dx \leq -\delta \ln \delta \int_\Omega u_-(x, t) \, dx + C(\varepsilon)\delta.$$

By performing the limit $\delta \to 0$, we get $\int_\Omega \overline{u}_-^2(x, t) \, dx = 0$, which implies 6. Besides, the result 7 can be obtained from 1-3 and Lemma 3. $\qquad\square$

Proof of Proposition 1 For any function $\phi \in L^2(0, T; C^\infty_{\mathrm{per}}(\Omega))$, Lemma 4 gives

$$\iint_{Q_T} u_t \phi \, dx \, dt - \iint_{Q_T} (u_+ + \delta)^n w \Delta\phi \, dx \, dt - n \iint_{Q_T} (u_+ + \delta)^{n-1} w \nabla u \nabla\phi \, dx \, dt$$

$$+ A \iint_{Q_T} \left(\frac{(u_+ + \delta)^n \nabla u \nabla\phi}{(u_+ + \varepsilon)^{n+\alpha}(1 + \varepsilon|\nabla u|^2)} \right) dx \, dt = 0, \tag{25}$$

$$\iint_{Q_T} w \phi \, dx \, dt = -\iint_{Q_T} \Delta u \phi \, dx \, dt + \nu \iint_{Q_T} u_t \phi \, dx \, dt. \tag{26}$$

Similar to the proof of (18)-(20) and applying Lemma 5, Lemma 6, and Vitali's theorem, we can get

$$(u_+ + \delta)^n \to \overline{u}^n \quad \text{strongly in } L^4(Q_T); \tag{27}$$

$$\nabla u \to \nabla\overline{u} \quad \text{strongly in } L^4(Q_T); \tag{28}$$

$$(u_+ + \delta)^{n-1}\nabla u \to \overline{u}^{n-1}\nabla\overline{u} \quad \text{strongly in } L^2(Q_T) \text{ if } n \geq 1; \tag{29}$$

$$\frac{(u_+ + \delta)^n \nabla u}{(u_+ + \varepsilon)^{n+\alpha}(1 + \varepsilon|\nabla u|^2)} \to \frac{(\overline{u}_+ + \delta)^n \nabla\overline{u}}{(\overline{u}_+ + \varepsilon)^{n+\alpha}(1 + \varepsilon|\nabla\overline{u}|^2)} \quad \text{strongly in } L^2(Q_T). \tag{30}$$

If (29) holds for $n < 1$, (25)-(30) ensure that the limit $\delta \to 0$ can be performed in (25)-(26) and then we can complete the proof of Proposition 1.

Therefore, we only need to prove

$$(u_+ + \delta)^{n-1} \nabla u \to \overline{u}^{n-\frac{1}{2}} \frac{\nabla \overline{u}}{\overline{u}^{\frac{1}{2}}} = \overline{u}^{n-1} \nabla \overline{u} \quad \text{strongly in } L^2(Q_T) \tag{31}$$

if $n < 1$.

From the following three steps, we can prove (29).

Step 1. Define $m(\delta) = \delta + \|u_-\|_{C(\overline{Q_T})}$ and we have $u + m(\delta) \geq \delta > 0$. By applying the Bernis inequality, we get

$$\iint_{Q_T} \frac{|\nabla u|^4}{(u + m(\delta))^2} \, dx \, dt \leq \iint_{Q_T} |\Delta u|^2 \, dx \, dt \leq C, \tag{32}$$

where C is independent of δ.

Step 2. In this step, we define $U_\delta = (u_+ + \delta)^{n-1}(u + m(\delta))^{\frac{1}{2}}$ and we want to prove that the limit $\lim_{\delta \to 0} \|U_\delta - \overline{u}^{n-\frac{1}{2}}\|_{L^4(Q_T)} = 0$ holds.

At first, it is obvious that we have

$$U_\delta \geq \left(u_+ + m(\delta)\right)^{n-\frac{1}{2}} \quad \text{in } Q_T. \tag{33}$$

Now we choose

$$r \begin{cases} = +\infty, & N = 1; \\ < +\infty, & N = 2; \\ < 6, & N = 3, \end{cases}$$

such that $H^s(\Omega) \hookrightarrow W^{1,r}(\Omega)$ with $\frac{7}{4} < s < 2$. By using the Gagliardo-Nirenberg interpolation inequality and Lemma 6, we get

$$\begin{aligned}
\|u_-(t)\|_{L^\infty(\Omega)} &\leq C\|u_-(t)\|_{W^{1,r}(\Omega)}^\gamma \|u_-(t)\|_{L^2(\Omega)}^{1-\gamma} \\
&\leq C\|u_-(t)\|_{H^s_{\text{per}}(\Omega)}^\gamma \delta^{\frac{n}{2}(1-\gamma)} \\
&\leq C(\varepsilon, s)\delta^{\frac{n}{2}(1-\gamma)}
\end{aligned} \tag{34}$$

with $\gamma = \frac{\frac{1}{2}}{\frac{N+2}{2N} - \frac{1}{r}}$. It implies

$$\begin{aligned}
U_\delta(x, t) &\leq (u_+ + \delta)^{n-1}\left(u_+ + \delta + 2\|u_-(t)\|_{L^\infty(\Omega)}\right)^{\frac{1}{2}} \\
&\leq (u_+ + \delta)^{n-\frac{1}{2}} + \delta^{n-1}\left(2\|u_-(t)\|_{L^\infty(\Omega)}\right)^{\frac{1}{2}} \\
&\leq (u_+ + \delta)^{n-\frac{1}{2}} + C(\varepsilon)\delta^{n-1+\frac{n}{4}(1-\gamma)}
\end{aligned} \tag{35}$$

with

$$n \in \begin{cases} (\frac{6}{7}, 2), & N = 1; \\ (\frac{8}{9}, 2), & N = 2; \\ (\frac{16}{17}, 2), & N = 3. \end{cases}$$

Equations (33) and (35) yield

$$\lim_{\delta \to 0} U_\delta(x, t) = \overline{u}^{n - \frac{1}{2}} \quad \text{a.e. in } Q_T. \tag{36}$$

The Lebesgue-dominated theorem yields

$$\lim_{\delta \to 0} \iint_{Q_T} \left| U_\delta - \overline{u}^{n - \frac{1}{2}} \right|^4 \, dx \, dt = 0. \tag{37}$$

Step 3. This step is devoted to the proof of (31). For any positive constant η, one has

$$\iint_Q \left| (u_+ + \delta)^{n-1} \nabla u - \overline{u}^{n - \frac{1}{2}} \frac{\nabla \overline{u}}{\sqrt{\overline{u}}} \right|^2 \, dx \, dt$$

$$\leq \iint_Q \left| U_\delta - \overline{u}^{n - \frac{1}{2}} \right|^2 \left| \frac{\nabla u}{\sqrt{u + m(\delta)}} \right|^2 \, dx \, dt + \iint_Q \overline{u}^{2n-1} \left| \frac{\nabla u}{u + m(\delta)} - \frac{\nabla \overline{u}}{\sqrt{\overline{u}}} \right|^2 \, dx \, dt$$

$$\leq \left(\iint_Q \left| \frac{\nabla u}{\sqrt{u + m(\delta)}} \right|^4 \, dx \, dt \right)^{\frac{1}{2}} \left(\iint_Q \left| U_\delta - \overline{u}^{n - \frac{1}{2}} \right|^4 \, dx \, dt \right)^{\frac{1}{2}}$$

$$+ \iint_{\{\overline{u} \geq \eta\}} \overline{u}^{2n-1} \left| \frac{\nabla u}{u + m(\delta)} - \frac{\nabla \overline{u}}{\sqrt{\overline{u}}} \right|^2 \, dx \, dt$$

$$+ \iint_{\{\overline{u} < \eta\}} \overline{u}^{2n-1} \left| \frac{\nabla u}{u + m(\delta)} - \frac{\nabla \overline{u}}{\sqrt{\overline{u}}} \right|^2 \, dx \, dt$$

$$= I_1 + I_2 + I_3. \tag{38}$$

From Step 1 and Step 2, we know $I_1 \to 0$ as $\delta \to 0$ and by applying Lemma 6, we have $I_2 \to 0$ as $\delta \to 0$. For the last term, we have

$$I_3 = \iint_{\{\overline{u} < \eta\}} \overline{u}^{2n-1} \left| \frac{\nabla u}{u + m(\delta)} - \frac{\nabla \overline{u}}{\sqrt{\overline{u}}} \right|^2 \, dx \, dt$$

$$\leq \eta^{2n-1} \left[\iint_{\{\overline{u} < \eta\}} \frac{|\nabla u|^2}{|u + m(\delta)|} \, dx \, dt + \iint_{\{\overline{u} < \eta\}} \frac{|\nabla \overline{u}|^2}{|\overline{u}|} \, dx \, dt \right]$$

$$\leq C\eta^{2n-1} \left[\left(\iint_{\{\overline{u} < \eta\}} \frac{|\nabla u|^4}{|u + m(\delta)|^2} \, dx \, dt \right)^{\frac{1}{2}} + \left(\iint_{\{\overline{u} < \eta\}} \frac{|\nabla \overline{u}|^4}{|\overline{u}|^2} \, dx \, dt \right)^{\frac{1}{2}} \right]$$

$$\leq C\eta^{2n-1}. \tag{39}$$

Therefore, by performing the limit $\eta \to 0$, we get $I_3 \to 0$ and then the estimate (31) holds. □

4 The limit $\varepsilon \to 0$

We will perform the last limit $\varepsilon \to 0$ in this section and assume that the initial function $u_{0\varepsilon}$ converges to u_0 strongly in $L^2(\Omega)$.

By letting $\delta = 0$ in the definition of $\Phi_\delta(\cdot)$, we can define $\Phi_0(\cdot)$ as

$$\Phi_0(x) = \begin{cases} \frac{1}{(1-n)(2-n)} x^{2-n} - \frac{1}{1-n} x + \frac{1}{2-n} & \text{if } n \in [0, 2), n \neq 1; \\ x \ln x - x + 1 & \text{if } n = 1. \end{cases}$$

Lemma 7 *In the sense of $\mathcal{D}'(0,T)$, there exists a constant $C_0 > 0$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \Phi_0(\overline{u}) \,\mathrm{d}x + C_0 \int_{\Omega} |\Delta\overline{u}|^2 \,\mathrm{d}x + \nu \int_{\Omega} |\overline{u}_t|^2 \,\mathrm{d}x \leq A \int_{\Omega} \frac{|\nabla\overline{u}|^2}{(\overline{u}+\varepsilon)^{n+\alpha}(1+\varepsilon|\nabla\overline{u}|^2)} \,\mathrm{d}x.$$

Proof From the idea of (23) and the L^p-estimate, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \Phi(u(x,t)) \,\mathrm{d}x + C_0 \int_{\Omega} |\Delta u|^2 \,\mathrm{d}x + \nu \int_{\Omega} |u_t|^2 \,\mathrm{d}x$$

$$\leq \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \Phi(u(x,t)) \,\mathrm{d}x + \int_{\Omega} |w|^2 \,\mathrm{d}x + \nu \int_{\Omega} |u_t|^2 \,\mathrm{d}x$$

$$= A \int_{\Omega} \frac{|\nabla u|^2}{(u_+ + \varepsilon)^{n+\alpha}(1+\varepsilon|\nabla u|^2)} \,\mathrm{d}x. \tag{40}$$

Since $u \to \overline{u}$ in $C(\overline{Q}_T)$ as $\delta \to 0$, we have

$$-\int_0^T \phi'(t) \int_{\Omega} \Phi(u) \,\mathrm{d}x \,\mathrm{d}t \to -\int_0^T \phi'(t) \int_{\Omega} \Phi_0(\overline{u}) \,\mathrm{d}x \,\mathrm{d}t \tag{41}$$

for any nonnegative function $\phi \in \mathcal{D}'(0,T)$. By applying the limit $\Delta u \rightharpoonup \Delta\overline{u}$ in $L^2(Q_T)$ as $\delta \to 0$, one has

$$\liminf_{\delta \to 0} \int_0^T \phi(t) \int_{\Omega} |\Delta u|^2 \,\mathrm{d}x \,\mathrm{d}t \geq \int_0^T \phi(t) \int_{\Omega} |\Delta\overline{u}|^2 \,\mathrm{d}x \,\mathrm{d}t. \tag{42}$$

Finally, it is easy to check that

$$A \iint_{Q_T} \frac{|\nabla u|^2 \phi(t)}{(u_+ + \varepsilon)^{n+\alpha}(1+\varepsilon|\nabla u|^2)} \,\mathrm{d}x \,\mathrm{d}t$$

$$\to A \iint_{Q_T} \frac{|\nabla\overline{u}|^2 \phi(t)}{(\overline{u}+\varepsilon)^{n+\alpha}(1+\varepsilon|\nabla\overline{u}|^2)} \,\mathrm{d}x \,\mathrm{d}t. \tag{43}$$

Equations (40)-(43) give the result of this lemma. □

Lemma 8 *If one of the following conditions holds*:
  (I) $\int_{\Omega} \Phi_0(w_0) \,\mathrm{d}x < \infty$, $A \leq 0$, *and*
  (II) $\int_{\Omega} \Phi_0(w_0) \,\mathrm{d}x < \infty$, $\alpha \leq 1 - n$, $n < 1$, *one has* $\overline{u} \in L^2(0,T;H_{\mathrm{per}}^2(\Omega))$, $\overline{w}, \overline{u}_t \in L^2(Q_T)$
    independent of ε.

Proof By Lemma 7 and the condition (I), we can prove the result easily.
  If the condition (II) holds, Lemma 1 and Lemma 7 give

$$\int_{\Omega} \Phi_0(\overline{u}) \,\mathrm{d}x + C_0 \int_{\Omega} |\Delta\overline{u}|^2 \,\mathrm{d}x + \nu \iint_{Q_T} |\overline{u}_t|^2 \,\mathrm{d}x \,\mathrm{d}t$$

$$\leq \int_{\Omega} \Phi_0(\overline{u}_0) \,\mathrm{d}x + |A| \iint_{Q_T} \frac{|\nabla\overline{u}|^2}{(\overline{u}+\varepsilon)(1+\varepsilon|\nabla\overline{u}|^2)} \,\mathrm{d}x \,\mathrm{d}t$$

$$\leq \int_{\Omega} \Phi_0(\overline{u}_0) \,\mathrm{d}x + |A| \iint_{Q_T} \frac{|\nabla\overline{u}|^2}{(\overline{u}+\varepsilon)^{\alpha+n}} \,\mathrm{d}x \,\mathrm{d}t$$

$$\leq \int_{\Omega} \Phi_0(\overline{u}_0) \, dx + |A| \left(\iint_{Q_T} \frac{|\nabla \overline{u}|^4}{(\overline{u} + \varepsilon)^2} \, dx \, dt \right)^{\frac{1}{2}} \left(\iint_{Q_T} (\overline{u} + \varepsilon)^{2(1-(\alpha+n))} \, dx \, dt \right)^{\frac{1}{2}}$$

$$\leq \int_{\Omega} \Phi_0(\overline{u}_0) \, dx + C \left(\int_0^T \|\overline{u}\|_{H^2(\Omega)} \, dt \right)^{\frac{1}{2}} \left(\iint_{Q_T} (\overline{u} + \varepsilon) \, dx \, dt \right)^{1-(\alpha+n)}$$

$$\leq \frac{C_0}{2} \iint_{Q_T} |\Delta \overline{u}|^2 \, dx \, dt + C, \tag{44}$$

which yields $\Delta \overline{u} \in L^2(Q_T)$. Applying the second equation of Proposition 1, we get $\overline{w} \in L^2(Q_T)$. $\qquad\square$

Now we are in the position to prove Theorem 1.

Proof of Theorem 1 By Lemma 8, we can show the existence of two functions $u \geq 0$ and w such that, as $\varepsilon \to 0$,

$$\overline{u} \rightharpoonup u \quad \text{in } L^2\big(0, T; H_{\mathrm{per}}^2(\Omega)\big); \tag{45}$$

$$\overline{u}_t \rightharpoonup u_t \quad \text{in } L^2(Q_T); \tag{46}$$

$$\overline{w} \rightharpoonup w \quad \text{in } L^2(Q_T); \tag{47}$$

$$\overline{u} \to u \quad \text{in } C\big([0, T]; H_{\mathrm{per}}^1(\Omega)\big); \tag{48}$$

$$\overline{u} \to u \quad \text{in } L^2\big(0, T; H_{\mathrm{per}}^1(\Omega)\big); \tag{49}$$

$$\overline{u} \to u, \qquad \nabla \overline{u} \to \nabla u \quad \text{a.e. in } Q_T. \tag{50}$$

Furthermore, Lemma 3 yields

$$\|\overline{u}\|_{C([0,T];H_{\mathrm{per}}^s(\Omega))} \leq C; \tag{51}$$

$$\|u\|_{C([0,T];H_{\mathrm{per}}^s(\Omega))} \leq C \tag{52}$$

for $\frac{3}{2} < s < 2$. By the Sobolev embedding theorem with the case $N \leq 3$, we have $\|\overline{u}\|_{L^\infty(Q_T)} \leq C$ and $\|u\|_{L^\infty(Q_T)} \leq C$.

Step 1. By using (51)-(52) and Vitali's theorem, we get $\overline{u}^n \to u^n$ in $L^q(Q_T)$ for any $q > 0$ and thus one has

$$\iint_{Q_T} \overline{u}^n \overline{w} \Delta \phi \, dx \, dt \to \iint_{Q_T} u^n w \Delta \phi \, dx \, dt \tag{53}$$

as $\varepsilon \to 0$ for any test function $\phi \in C^\infty([0, T]; C_{\mathrm{per}}^2(\overline{\Omega}))$.

Step 2. In this step, we will prove the limit $\overline{u}^{n-1} \nabla \overline{u} \to u^{n-1} \nabla u$ in $L^2(Q_T)$.

First of all, the Bernis inequality yields $\iint_{Q_T} |\frac{\nabla \overline{u}}{\sqrt{\overline{u}}}|^4 \, dx \, dt \leq C$ and then we have

$$\iint_{\Delta_0} \overline{u}^{n-1} |\nabla \overline{u}|^2 \, dx \, dt = \iint_{\Delta_0} \overline{u}^{2n-1} \frac{|\nabla \overline{u}|^2}{\sqrt{\overline{u}}} \, dx \, dt$$

$$\leq C \left(\iint_{\Delta_0} \overline{u}^{4n-2} \, dx \, dt \right)^{\frac{1}{2}} \to 0 \tag{54}$$

as $\varepsilon \to 0$ with $\Delta_0 = \{(x,t) \in Q_T | u(x,t) = 0\}$. On the other hand, it is easy to get

$$\frac{\nabla \overline{u}}{\sqrt{\overline{u}}} \to \frac{\nabla u}{\sqrt{u}} \quad \text{a.e. in } Q_T \backslash \Delta_0$$

as $\varepsilon \to 0$. By Vitali's theorem, we have

$$\overline{u}^{n-1} \nabla \overline{u} \to u^{n-1} \nabla u \quad \text{in } L^2(Q_T \backslash \Delta_0). \tag{55}$$

Hence, we have

$$\overline{u}^{n-1} \nabla \overline{u} \to u^{n-1} \nabla u \quad \text{in } L^2(Q_T), \tag{56}$$

where we define $u^{n-1} \nabla u = 0$ on Δ_0.

Step 3. In this step, we prove the limit $F_\varepsilon = \frac{\overline{u}^n \nabla \overline{u}}{(\overline{u}+\varepsilon)^{n+\alpha}(1+\varepsilon|\nabla\overline{u}|^2)} \to u^{-\alpha} \nabla u$ in $L^2(Q_T)$.

If $\alpha \le \frac{1}{2}$, we have

$$\iint_{\Delta_0} |F_\varepsilon|^2 \, dx \, dt \le \iint_{\Delta_0} \overline{u}^{1-2\alpha} \frac{|\nabla \overline{u}|^2}{\overline{u}} \, dx \, dt$$

$$\le C \left(\iint_{\Delta_0} \overline{u}^{2-4\alpha} \, dx \, dt \right)^{\frac{1}{2}} \to 0 \tag{57}$$

as $\varepsilon \to 0$. Beside, it is easy to show $F_\varepsilon \to u^{-\alpha} \nabla u$ a.e. in $Q_T \backslash \Delta_0$ and Vitali's theorem yields

$$\iint_{\Delta_0} |F_\varepsilon - u^{-\alpha} \nabla u|^2 \, dx \, dt \to 0 \tag{58}$$

as $\varepsilon \to 0$. By (57)-(58), we have

$$F_\varepsilon \to u^{-\alpha} \nabla u \quad \text{in } L^2(Q_T), \tag{59}$$

where we define $u^{-\alpha} \nabla u = 0$ on Δ_0.

As $\varepsilon \to 0$, the convergence (56) and (46)-(47) give $\iint_{Q_T} \overline{u}_t \phi \, dx \, dt \to \iint_{Q_T} u_t \phi \, dx \, dt$ and $\iint_{Q_T} \overline{u}^{n-1} \nabla \overline{u} w \nabla \phi \, dx \, dt \to \iint_{Q_T} u^{n-1} \nabla u w \nabla \phi \, dx \, dt$. Step 3 yields

$$\iint_{Q_T} \frac{\overline{u}^n \nabla \overline{u} \nabla \phi}{(\overline{u}+\varepsilon)^{n+\alpha}(1+\varepsilon|\nabla\overline{u}|^2)} \, dx \, dt \to \iint_{Q_T} u^{-\alpha} \nabla u \nabla \phi \, dx \, dt.$$

Now we can take the limit $\varepsilon \to 0$ in the equality

$$\iint_{Q_T} \overline{u}_t \phi \, dx \, dt + \iint_{Q_T} \overline{u}^n \overline{w} \Delta \phi \, dx \, dt$$

$$+ n \iint_{Q_T} \overline{u}^{n-1} \nabla \overline{u} w \nabla \phi \, dx \, dt - A \iint_{Q_T} \frac{\overline{u}^n \nabla \overline{u} \nabla \phi}{(\overline{u}+\varepsilon)^{n+\alpha}(1+\varepsilon|\nabla\overline{u}|^2)} \, dx \, dt = 0,$$

$$\iint_{Q_T} \overline{w} \phi \, dx \, dt = -\iint_{Q_T} \Delta \overline{u} \phi \, dx \, dt + \nu \iint_{Q_T} \overline{u}_t \phi \, dx \, dt$$

for any test function $\phi \in C([0,T]; C^2_{\mathrm{per}}(\overline{\Omega}))$. For the initial value, this holds in the sense of $u \in C([0,T]; H^1_{\mathrm{per}}(\Omega))$. □

Authors' contributions
XP and BL completed the main study. MP and YW verified the calculation. All authors read and approved the final manuscript.

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