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Some results concerning the solution mappings of mixed variational inequality problems

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Abstract

This paper shows some continuities of mappings between the space of mixed variational inequality problems and the graph space of their solution mappings. The space of mixed variational inequality problems is homeomorphic to the graph of a continuous mapping. These generalize the results in the corresponding references.

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Keywords: mixed variational inequality; monotone; homeomorphism

1 Introduction and preliminaries

Variational inequalities have become important methods to analyze many linear and nonlinear problems; see [1] and [2], such as linear complementarity problems in [3], convex optimization in [4], imaging problems in [5], *etc.*

Classical variational inequalities were introduced by Hartman and Stampacchia in the 1960s; see [6] and [7]. They are of the form suitable to find u^* such that $(u-u^*)^T f(u^*) \geq 0$, $\forall u \in K$. The spaces consisting of nonlinear problems may have some interesting topological properties: some nonlinear problem spaces are homeomorphic to the graph spaces of their solution mappings, such as bimatrix games in [8], normal game problems in [9, 10], game trees in [11], classical variational inequality problems [12], *etc.*

Duvaut and Lions considered a kind of mixed variational inequality, see [13], which added a function to a classical variational inequality. The stability, algorithm, and generalization of this kind of variational inequality has been studied in many forms [14–19] and applied to many fields; see [20].

Let K be a compact convex subset of $\mathbb{R}^n$. We consider the following mixed variational inequality problem: to find a point $u^* \in K$ such that u^* satisfies

$$\theta(u) - \theta\left(u^*\right) + \left(u-u^*\right)^T f\left(u^*\right) \geq 0, \quad \forall u \in K, \tag{1}$$

where $f: K \to \mathbb{R}^n$ is a mapping and $\theta: K \to \mathbb{R}$ is a real function. This was first introduced in [13]. When $f \equiv 0$ on K, this leads us to find a maximizer $u^* \in K$ of the function θ on K.

Definition 1.1 A mapping $f: K \to \mathbb{R}^n$ is said to be monotone if $(f(x)-f(y))^T(x-y) \geq 0$, $\forall x, y \in K$; f is said to be strictly monotone if, f is monotone and if $(f(x)-f(y))^T(x-y) = 0$, then $x = y$; f is strongly monotone if there exists a constant c such that $(f(x)-f(y))^T(x-y) \geq c\|x-y\|^2$, $\forall x, y \in K$.

If f is monotone and θ is convex, we call the above variational inequality (1) a mixed monotone variational inequality problem. Denote by M the set as follows:

$$M = \left\{ (\theta,f): \begin{array}{l} \theta : K \to \mathbb{R} \text{ is continuous and convex;} \\ f : K \to \mathbb{R}^n \text{ is continuous and monotone} \end{array} \right\}.$$

For any two $(\theta_1,f_1), (\theta_2,f_2) \in M$, define the metric ρ between (θ_1,f_1) and (θ_2,f_2) as

$$\rho\big((\theta_1,f_1),(\theta_2,f_2)\big) = \sup_{x\in K}\big\|f_1(x)-f_2(x)\big\| + \sup_{x\in K}\big|\theta_1(x)-\theta_2(x)\big|.$$

Then M is a metric space. For convenience, we denote $\rho((\theta_n,f_n),(\theta_0,f_0)) \to 0$ by $(\theta_n,f_n) \xrightarrow{\rho} (\theta_0,f_0)$. A point u^* is a solution of the mixed monotone variational inequality problem $(\theta,f) \in M$; we write it as $u^* \in V(\theta,f)$. Then a set-valued mapping V from M to K is defined. It is well known that, for each $(\theta,f) \in M$, we have $V(\theta,f)$ is nonempty. If f is strictly monotone, then $V(\theta,f)$ is a singleton set; for example, $f + I$ is strictly monotone, where I is the identity mapping on K, and if f is strongly monotone, then f is strictly monotone.

Denote by N the graph of the set-valued mapping V, that is,

$$N = \big\{(\theta,f,u) \in M \times K \mid u \in V(\theta,f)\big\}.$$

For each $(\theta,f,u^*) \in N$, let

$$\phi\big(\theta,f,u^*\big) = (\theta,f - C_{u^*}),$$

where C_x denotes the constant mapping with $C_x(u) = x$, $\forall u \in K$. Clearly, $C_{u^*} \in M$ is monotone, then ϕ is mapping from N to M. For each $(\theta,f) \in M$, define $\psi(\theta,f)$ such that

$$\psi(\theta,f) = (\theta,f + C_{\bar{u}},\bar{u}),$$

where $\bar{u}$ is the unique solution of the problem $(\theta,f + I)$.

2 Main results

Theorem 2.1 *ψ is a continuous mapping from M to N. ϕ is continuous on N.*

Proof For each $(\theta,f) \in M$, since $\psi(\theta,f) = (\theta,f + C_{\bar{u}},\bar{u})$, where $\bar{u} \in V(\theta,f + I)$, we have $\theta(u) - \theta(\bar{u}) + (u-\bar{u})^T(f(\bar{u}) + \bar{u}) \geq 0$, $\forall u \in K$, that is, $\bar{u} \in V(\theta,f + C_{\bar{u}})$. ψ maps M onto N.

Next, we prove that ψ is continuous on M. Let $\{(\theta_n,f_n)\}_{n=1}^{\infty} \subset M$ with $(\theta_n,f_n) \xrightarrow{\rho} (\theta_0,f_0)$. We need to show that $(\theta_n,f_n + C_{\bar{u}_n}) \xrightarrow{\rho} (\theta_0,f_0 + C_{\bar{u}_0})$ and $\|\bar{u}_n - \bar{u}_0\| \to 0$, where $\bar{u}_n \in V(\theta_n,f_n + I)$ and $\bar{u}_0 \in V(\theta_0,f_0 + I)$. Since

$$\theta_n(u) - \theta_n(\bar{u}_n) + (u-\bar{u}_n)^T\big(f_n(\bar{u}_n) + \bar{u}_n\big) \geq 0, \quad \forall u \in K$$

and

$$\theta_0(u) - \theta_0(\bar{u}_0) + (u-\bar{u}_0)^T\big(f_0(\bar{u}_0) + \bar{u}_0\big) \geq 0, \quad \forall u \in K,$$

we have

$$\theta_n(\bar{u}_0) - \theta_n(\bar{u}_n) + (\bar{u}_0 - \bar{u}_n)^T \big(f_n(\bar{u}_n) + \bar{u}_n\big) \geq 0 \tag{2}$$

and

$$\theta_0(\bar{u}_n) - \theta_0(\bar{u}_0) + (\bar{u}_n - \bar{u}_0)^T \big(f_0(\bar{u}_0) + \bar{u}_0\big) \geq 0. \tag{3}$$

Adding equation (2) and equation (3), it follows that

$$\theta_n(\bar{u}_0) - \theta_0(\bar{u}_0) + \theta_0(\bar{u}_n) - \theta_n(\bar{u}_n) + (\bar{u}_0 - \bar{u}_n)^T \big(f_n(\bar{u}_n) - f_0(\bar{u}_0) + \bar{u}_n - \bar{u}_0\big) \geq 0.$$

Then

$$\theta_n(\bar{u}_0) - \theta_0(\bar{u}_0) + \theta_0(\bar{u}_n) - \theta_n(\bar{u}_n) + (\bar{u}_0 - \bar{u}_n)^T \big(f_n(\bar{u}_n) - f_0(\bar{u}_0)\big) \geq \|\bar{u}_0 - \bar{u}_n\|.$$

This is equivalent to the following inequality:

$$\begin{aligned}&\theta_n(\bar{u}_0) - \theta_0(\bar{u}_0) + \theta_0(\bar{u}_n) - \theta_n(\bar{u}_n)\\ &\qquad - (\bar{u}_0 - \bar{u}_n)^T \big(f_0(\bar{u}_0) - f_0(\bar{u}_n) + f_0(\bar{u}_n) - f_n(\bar{u}_n)\big)\\ &\quad \geq \|\bar{u}_0 - \bar{u}_n\|.\end{aligned}$$

By transposition, we have

$$\begin{aligned}&\theta_n(\bar{u}_0) - \theta_0(\bar{u}_0) + \theta_0(\bar{u}_n) - \theta_n(\bar{u}_n) - (\bar{u}_0 - \bar{u}_n)^T \big(f_0(\bar{u}_n) - f_n(\bar{u}_n)\big)\\ &\quad \geq \|\bar{u}_0 - \bar{u}_n\| + (\bar{u}_0 - \bar{u}_n)^T \big(f_0(\bar{u}_0) - f_0(\bar{u}_n)\big).\end{aligned}$$

Since f_0 is monotone, we have $(\bar{u}_0 - \bar{u}_n)^T (f_0(\bar{u}_0) - f_0(\bar{u}_n)) \geq 0$. Then

$$\theta_n(\bar{u}_0) - \theta_0(\bar{u}_0) + \theta_0(\bar{u}_n) - \theta_n(\bar{u}_n) - (\bar{u}_0 - \bar{u}_n)^T \big(f_0(\bar{u}_n) - f_n(\bar{u}_n)\big) \geq \|\bar{u}_0 - \bar{u}_n\|.$$

Therefore, we have

$$\begin{aligned}\|\bar{u}_0 - \bar{u}_n\| &\leq \big|\theta_n(\bar{u}_0) - \theta_0(\bar{u}_0)\big| + \big|\theta_0(\bar{u}_n) - \theta_n(\bar{u}_n)\big| + \big|(\bar{u}_0 - \bar{u}_n)^T \big(f_0(\bar{u}_n) - f_n(\bar{u}_n)\big)\big|\\ &\leq \big|\theta_n(\bar{u}_0) - \theta_0(\bar{u}_0)\big| + \big|\theta_0(\bar{u}_n) - \theta_n(\bar{u}_n)\big| + \|\bar{u}_0 - \bar{u}_n\| \big\|f_0(\bar{u}_n) - f_n(\bar{u}_n)\big\|.\end{aligned}$$

Then

$$\big(1 - \big\|f_0(\bar{u}_n) - f_n(\bar{u}_n)\big\|\big)\|\bar{u}_0 - \bar{u}_n\| \leq \big|\theta_n(\bar{u}_0) - \theta_0(\bar{u}_0)\big| + \big|\theta_0(\bar{u}_n) - \theta_n(\bar{u}_n)\big|.$$

Since $\theta_n \to \theta_0$ and $\max_{u\in K} \|f_0(u) - f_n(u)\| \to 0$, we have $\|\bar{u}_0 - \bar{u}_n\| \to 0$. Hence, it can be checked that

$$\begin{aligned}&\big\|f_n(u) + C_{\bar{u}_n}(u) - f_0(u) - C_{\bar{u}_0}(u)\big\|\\ &\quad \leq \big\|f_n(u) - f_0(u)\big\| + \big\|C_{\bar{u}_n}(u) - C_{\bar{u}_0}(u)\big\|\\ &\quad = \big\|f_n(u) - f_0(u)\big\| + \|\bar{u}_n - \bar{u}_0\| \to 0.\end{aligned}$$

Then $(\theta_n, f_n + C_{\bar{u}_n}) \xrightarrow{\rho} (\theta_0, f_0 + C_{\bar{u}_0})$.

For the part with ϕ is continuous on N, let $\{(\theta_n,f_n,u_n)\}_{n=1}^{\infty} \subset N$ with $(\theta_n,f_n) \xrightarrow{\rho} (\theta_0,f_0)$ and $\|u_n - u_0\| \to 0$; we show that $\phi(\theta_n,f_n,u_n) \xrightarrow{\rho} \phi(\theta_0,f_0,u_0)$. In fact, for each $u \in K$, we have

$$\begin{aligned}
&\big\|\theta_n(u) - \theta_0(u)\big\| + \big\|f_n(u) - f_0(u) - u_n + u_0\big\| \\
&\quad \le \big\|\theta_n(u) - \theta_0(u)\big\| + \big\|f_n(u) - f_0(u)\big\| + \|u_n - u_0\| \to 0,
\end{aligned}$$

then $\phi(\theta_n,f_n,u_n) \xrightarrow{\rho} \phi(\theta_0,f_0,u_0)$. □

Let X be the set of (θ,f) satisfying: (i) $\theta : K \to \mathbb{R}$ is proper convex lower semi-continuous; (ii) $f : K \to \mathbb{R}^n$ is continuous and monotone. Then each variational inequality $(\theta,f) \in X$ has a solution (see [21]) and further if f is strictly monotone, then the corresponding solution is unique. Then (X,ρ) is a metric space. From Theorem 2.1, there can be obtained some stability results for solution sets of mixed monotone variational inequalities.

Corollary 2.1 *The set-valued mapping V is upper semi-continuous from M to 2^K.*

Proof From the proof of Theorem 2.1, $\psi : X \to N$ is continuous, noting that $M \subset X$ is closed, then N, the graph of V, is closed. Therefore, V is upper semi-continuous. □

Remark 2.1 By Corollary 2.1, V is upper semi-continuous on M. Then, when $f_0 + \varepsilon_n I \to f_0$ as positive $\varepsilon_n \to 0$ and $\theta_n \to \theta_0$, there is a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $\{x_n\} = V(\theta_n,f_0 + \varepsilon_n I)$ such that $x_{n_k} \to x_0 \in V(\theta_0,f_0)$. Alternatively, from Theorem 2.1, let $\varepsilon_n > 0$, if we define $\psi_{\varepsilon_n}(\theta,f) = (\theta,f + \varepsilon_n C_{\bar{u}_n},\bar{u}_n)$ for each $(\theta,f) \in M$, where $\{\bar{u}_n\} = V(\theta,f + \varepsilon_n I)$, then ψ_{ε_n} is also a continuous mapping from M to N. Note that $\bar{u}_n \in V(\theta,f + \varepsilon_n C_{\bar{u}_n})$, therefore, if $\varepsilon_n \to 0$, we assert that there exists a subsequence $\{\bar{u}_{n_k}\}$ of $\{\bar{u}_n\}$ with $\bar{u}_{n_k} \to u_0 \in V(\theta,f)$.

Theorem 2.2 *The mappings $\phi \circ \psi$ and $\psi \circ \phi$ are identity mappings on M and N, respectively.*

Proof For each $(\theta,f) \in M$, from the definition of ψ and ϕ, we have

$$\begin{aligned}
\phi \circ \psi(\theta,f) &= \phi(\theta,f + C_{\bar{u}},\bar{u}) \\
&= (\theta,f + C_{\bar{u}} - C_{\bar{u}},\bar{u}) \\
&= (\theta,f),
\end{aligned}$$

where $\bar{u} \in V(\theta,f + I)$.

For each $(\theta,f,u^*) \in N$, we have

$$\begin{aligned}
\psi \circ \phi\big(\theta,f,u^*\big) &= \psi(\theta,f - C_{u^*}) \\
&= (\theta,f - C_{u^*} + C_{\bar{u}},\bar{u}),
\end{aligned}$$

where $\bar{u} \in V(\theta,f - C_{u^*} + I)$. Then

$$\theta(u) - \theta(\bar{u}) + (u - \bar{u})^T\big(f(\bar{u}) - u^* + \bar{u}\big) \ge 0, \quad \forall u \in K.$$

Particularly, we have

$$\theta(u^*) - \theta(\bar{u}) + (u^* - \bar{u})^T (f(\bar{u}) - u^* + \bar{u}) \geq 0. \tag{4}$$

Note that $(\theta, f, u^*) \in N$, we have

$$\theta(\bar{u}) - \theta(u^*) + (\bar{u} - u^*)^T f(u^*) \geq 0. \tag{5}$$

Then, by adding equation (4) and equation (5), we obtain

$$(u^* - \bar{u})^T (f(\bar{u}) - f(u^*) + \bar{u} - u^*) \geq 0,$$

hence,

$$(u^* - \bar{u})^T (f(u^*) - f(\bar{u})) + (u^* - \bar{u})^T (u^* - \bar{u}) \leq 0.$$

From the monotonicity of f, we get $\|u^* - \bar{u}\| \leq 0$. Therefore, $u^* = \bar{u}$, hence, $\psi \circ \phi(\theta, f, u^*) = (\theta, f, u^*)$. □

Theorem 2.3 *The spaces N and M are homeomorphic.*

Proof It follows immediately from Theorems 2.1 and 2.2. □

Remark 2.2 The homeomorphism results were shown between game spaces and the graphs of solutions mappings in games [9–11]. For any population game $F : X \to \mathbb{R}^n$, each Nash equilibrium point x^* of the game F is equivalent to x^* being a solution of variational inequality $(y - x^*)^T F(x^*) \leq 0$; see [22]. A population game F belongs to the class of stable population games when F is monotone, then, from Theorem 2.3, we can assert that the space of stable population games is homeomorphic to the graph space of their solution mappings. Theorem 2.3 generalizes the homeomorphism result for classical variational inequalities in [12].

From Theorem 2.3, N is homeomorphic to the space M. Furthermore, we see that N is homeomorphic to the graph of a continuous mapping from M to K.

Let π be the projection from N to K and $\tilde{K}$ be a compact convex subset of $\mathbb{R}^n$ with $K \subset \operatorname{int}(\bar{f})$. Then there exists a retraction r from $\tilde{K}$ to K, that is, $r(x) = x$, $\forall x \in K$. From Urysohn lemma, there is a continuous mapping s from $\tilde{K}$ to the closed interval $[0,1]$ such that $s(x) = 1$, $\forall x \in K$, and $s(x) = 0$, $\forall x \in \mathrm{Bd}(K_\varepsilon)$, where $\mathrm{Bd}(K_\varepsilon)$ denotes the boundary of $\tilde{K}$.

Define two mappings α, β from $[0,1] \times M \times \tilde{K}$ to $M \times \tilde{K}$ such that, for each $(t, \theta, f, \tilde{x}) \in [0,1] \times M \times \tilde{K}$,

$$\alpha_t = \alpha(t, \theta, f, \tilde{x}) = (\theta, \bar{f}, \tilde{x}) \quad \text{with } \bar{f} = f - s(\tilde{x}) t C_x$$

and

$$\beta_t = \beta(t, \theta, \bar{f}, \tilde{x}) = (\theta, f, \tilde{x}) \quad \text{with } f = \bar{f} + s(\tilde{x}) t C_x,$$

where $x = r(\tilde{x})$.

Theorem 2.4 *Let $h = \pi \circ \psi : M \to K$. Then*: (i) *for each $t \in [0,1]$, $\beta_t \circ \alpha_t$ and $\alpha_t \circ \beta_t$ are identity mappings on $M \times \tilde{K}$*; (ii) *for each $t \in [0,1]$, $\tilde{x} \in \mathrm{Bd}(\tilde{K})$, $\alpha_t(\cdot,\cdot,\tilde{x})$ is a constant mapping*; (iii) *α_0 is an identity on $M \times \tilde{K}$*; *α_1 is a homeomorphism between N and the graph of h with its inverse β_1*.

Proof (i) For each $(\theta,f,\tilde{x}) \in M \times \tilde{K}$, we have $\beta_t \circ \alpha_t(\theta,f,\tilde{x}) = \beta_t(\theta,f - s(\tilde{x})tC_x,\tilde{x}) = (\theta,f,\tilde{x})$ and $\alpha_t \circ \beta_t(\theta,f,\tilde{x}) = \alpha_t(\theta,f + s(\tilde{x})tC_x,\tilde{x}) = (\theta,f,\tilde{x})$.

(ii) For each $\tilde{x} \in \mathrm{Bd}(\tilde{K})$ and $(\theta,f,\tilde{x}) \in M \times \tilde{K}$, $\alpha_t(\theta,f,\tilde{x}) = (\theta,f - s(\tilde{x})tC_x,\tilde{x})$. Noting that $\tilde{x} \in \mathrm{Bd}(\tilde{K})$, we have $s(\tilde{x}) = 0$, then $\alpha_t(\theta,f,\tilde{x}) = (\theta,f,\tilde{x})$.

(iii) It is clear that α_0 is an identity on $M \times \tilde{K}$. Next, for each $(\theta,f,\tilde{x}) \in N$, we have $\tilde{x} \in V(\theta,f)$, then $\tilde{x} \in K$. Hence $s(\tilde{x}) = 1$ and $r(\tilde{x}) = x = \tilde{x}$. Therefore, $\alpha_1(\theta,f,\tilde{x}) = (\theta,f - C_x,\tilde{x}) = (\phi(\theta,f,x),x)$. One needs to show that $x = \pi \circ \psi(\phi(\theta,f,x))$. Since $\psi \circ \phi$ is an identity on N, we have $\pi \circ \psi(\phi(\theta,f,x)) = \pi(\theta,f,x) = x$. Conversely, for each point $(\theta,f,\tilde{x})$ on the graph of the h, we need to show $(\theta,f,\tilde{x}) \in \alpha_1(N)$. Since $\tilde{x} = \pi \circ \psi(\theta,f)$, we have $\psi(\theta,f) = (\theta,f + C_{\tilde{x}},\tilde{x})$, then $(\theta,f + C_{\tilde{x}},\tilde{x}) \in N$. Hence, $\alpha_1(\theta,f + C_{\tilde{x}},\tilde{x}) = (\theta,\hat{f},\tilde{x})$ with $\hat{f} = f + C_{\tilde{x}} - s(\tilde{x})C_x$ and $x = r(\tilde{x})$. Noting that $(\theta,f + C_{\tilde{x}},\tilde{x}) \in N$, we have $\tilde{x} \in K$, then $x = r(\tilde{x}) = \tilde{x}$ and $s(\tilde{x}) = 1$. Therefore, $\hat{f} = f$. The proof is completed. □

Remark 2.3 From Theorems 2.3 and 2.4, the graph of set-valued mapping V is homeomorphic to M and the graph of a continuous mapping. Easily, we see that the graph of a continuous mapping can be homeomorphic to the graph of a constant mapping. Therefore, N and M are all homeomorphic to the graph of a constant mapping. Like (un)knots and Nash dynamics [23], this may contribute to variational dynamics like [24].

In the following part, we generalize the results (Theorems 2.1-2.3) to Hilbert spaces in relation to linear mappings.

Let K be a compact convex subset in a Hilbert space $(X, \langle\cdot,\cdot\rangle)$, where $\langle\cdot,\cdot\rangle$ represents the inner product on X. Denote by X^* the dual space of X (all continuous linear mappings on X). A mapping T from K to X^* is said to be monotone if $(T(u) - T(v), u - v) \geq 0$, $\forall u, v \in K$, where $(\cdot,\cdot)$ is the pairing of X^* and X. A monotone mapping T is called strictly monotone if $(T(u) - T(v), u - v) = 0$ implies $u = v$.

Let $T : K \to X^*$ and $f \in X^*$, we consider the mixed variational inequality problem: to find a $u \in K$ such that

$$\bigl(T(u), v - u\bigr) + \theta(v) - \theta(u) \geq (f, v - u), \quad \forall v \in K. \tag{6}$$

Denote by M' the set

$$M' = \left\{ (\theta, T, f): \begin{array}{l} \theta : K \to \mathbb{R} \text{ is continuous and convex;} \\ T : K \to X^* \text{ is continuous on } K; \\ f \in X^* \end{array} \right\}.$$

For each (θ, T, f), it is well known that this kind of mixed variational inequality problem with (θ, T, f) has a solution; if T is strictly monotone, then the solution is unique; denote by $V'(\theta, T, f)$ the set of all solutions of the problem (θ, T, f), then a set-valued mapping V' from M' to K is well defined.

For two $\alpha_1 = (\theta_1, T_1, f_1), \alpha_2 = (\theta_2, T_2, f_2) \in M'$, measure the metric between them by

$$\rho'(\alpha_1, \alpha_2) = \sup_{x\in K}\bigl|\theta_1(x) - \theta_2(x)\bigr| + \sup_{x\in K}\bigl\| T_1(x) - T_2(x)\bigr\| + \|f_1 - f_2\|,$$

where, for each $g_1, g_2 \in X^*$,

$$\|g_1 - g_2\| = \sup_{\|y\|=1, y\in X}\bigl|(g_1, y) - (g_2, y)\bigr|.$$

Let N' be the graph of V', that is,

$$N' = \bigl\{(\theta, T, f, u) \in M' \times K \mid u \in V'(\theta, T, f)\bigr\}.$$

Define a mapping $\phi' : N' \to M'$ such that, for each $(\theta, T, f, u) \in N'$,

$$\phi'(\theta, T, f, u) = (\theta, T - l_u, f),$$

where $l_u : K \to X^*$ is a mapping with $l_u(x) = \langle u, \cdot\rangle$, $\forall x \in K$. For each $T : K \to X^*$, let $R_T : K \to X^*$ such that, for each $x \in K$,

$$\bigl(R_T(x), z\bigr) = \bigl(T(x), z\bigr) + \langle x, z\rangle, \quad \forall z \in X.$$

Then we can check that R_T is strictly monotone. For each $(\theta, T, f) \in M'$, define a mapping ψ' on M' such that

$$\psi'(\theta, T, f) = (\theta, T + l_u, f, u),$$

where u is the unique solution of the problem (θ, R_T, f), that is, $V'(\theta, R_T, f) = \{u\}$.

Theorem 2.5 *ψ' is a continuous mapping from M' to N'.*

Proof For each $(\theta, T, f) \in M'$, we have $\psi'(\theta, T, f) = (\theta, T + l_u, u)$, where $u \in V'(\theta, R_T, f)$. Then

$$\begin{aligned}
&\theta(v) - \theta(u) + \bigl(T(u), v - u\bigr) + \langle u, v - u\rangle \\
&\quad = \theta(v) - \theta(u) + \bigl(T(u) + l_u, v - u\bigr) \\
&\quad \ge (f, v - u), \quad \forall v \in K,
\end{aligned}$$

it follows that $u \in V'(T + l_u)$, that is, ψ' maps M' onto N'.

For the part that ψ' is continuous on M': let $\{(\theta_n, T_n, f_n)\}_{n=1}^{\infty} \subset M'$ with $(\theta_n, T_n, f_n) \xrightarrow{\rho'} (\theta_0, T_0, f_0) \in M'$. It is sufficient to show that $\psi'(\theta_n, T_n, f_n) \to \psi'(\theta_0, T_0, f_0)$, that is, $(\theta_n, T_n + l_{u_n}, f_n) \xrightarrow{\rho'} (\theta_0, T_0 + l_{u_0}, f_0)$ and $\|u_n - u_0\| \to 0$, where $u_n \in V'(\theta_n, R_{T_n}, f_n)$ and $u_0 \in V'(\theta_0, R_{T_0}, f_0)$, this leads to the fact that, for each $x \in K$,

$$\theta_n(v) - \theta_n(u_n) + \bigl(T_n(u_n), v - u_n\bigr) + \langle u_n, v - u_n\rangle \ge (f_n, v - u_n), \quad \forall v \in K$$

and

$$\theta_0(v) - \theta_0(u_0) + \bigl(T_0(u_0), v - u_0\bigr) + \langle u_0, v - u_0\rangle \geq (f_0, v - u_0), \quad \forall v \in K.$$

Particularly, we have

$$\theta_n(u_0) - \theta_n(u_n) + \bigl(T_n(u_n), u_0 - u_n\bigr) + \langle u_n, u_0 - u_n\rangle \geq (f_n, u_0 - u_n) \tag{7}$$

and

$$\theta_0(u_n) - \theta_0(u_0) + \bigl(T_0(u_0), u_n - u_0\bigr) + \langle u_0, u_n - u_0\rangle \geq (f_0, u_n - u_0). \tag{8}$$

Adding equation (7) and equation (8), we get

$$\begin{aligned} &\theta_n(u_0) - \theta_0(u_0) + \theta_0(u_n) - \theta_n(u_n) + \bigl(T_n(u_n) - T_0(u_0), u_0 - u_n\bigr) \\ &\quad \geq \|u_n - u_0\| + (f_n - f_0, u_0 - u_n). \end{aligned}$$

This is equivalent to the following:

$$\begin{aligned} &\theta_n(u_0) - \theta_0(u_0) + \theta_0(u_n) - \theta_n(u_n) + \bigl((T_n - T_0)(u_n) + T_0(u_n - u_0), u_0 - u_n\bigr) \\ &\quad \geq \|u_n - u_0\| + (f_n - f_0, u_0 - u_n). \end{aligned}$$

Since T_0 is monotone, we have $(T_0(u_n - u_0), u_0 - u_n) \leq 0$. Then

$$\begin{aligned} &\theta_n(u_0) - \theta_0(u_0) + \theta_0(u_n) - \theta_n(u_n) \\ &\qquad + \bigl((T_n - T_0)(u_n), u_0 - u_n\bigr) - (f_n - f_0, u_0 - u_n) \\ &\quad \geq \|u_n - u_0\|. \end{aligned}$$

Hence,

$$\begin{aligned} &\|u_n - u_0\| \\ &\quad \leq \theta_n(u_0) - \theta_0(u_0) + \theta_0(u_n) - \theta_n(u_n) \\ &\qquad + \|u_n - u_0\|\left((T_n - T_0)(u_n), \frac{u_0 - u_n}{\|u_0 - u_n\|}\right) - \|u_n - u_0\|\left(f_n - f_0, \frac{u_0 - u_n}{\|u_0 - u_n\|}\right), \end{aligned}$$

thus,

$$\begin{aligned} \|u_n - u_0\| \leq{}& 2\sup_{x\in K}\bigl|\theta_n(x) - \theta_0(x)\bigr| \\ &+ \|u_n - u_0\|\sup_{x\in K}\bigl\|(T_n - T_0)(x)\bigr\| + \|u_n - u_0\|\|f_n - f_0\|, \end{aligned}$$

consequently, we obtain

$$\Bigl(1 - \sup_{x\in K}\bigl\|(T_n - T_0)(x)\bigr\| - \|f_n - f_0\|\Bigr)\|u_n - u_0\| \leq 2\sup_{x\in K}\bigl|\theta_n(x) - \theta_0(x)\bigr|.$$

Since $(\theta_n, T_n, f_n) \xrightarrow{\rho'} (\theta_0, T_0, f_0)$, we have $\|u_n - u_0\| \to 0$.

In addition, for each $x \in K$,

$$\begin{aligned}
&\left|\theta_n(x) - \theta_0(x)\right| + \left\|(T_n + l_{u_n})(x) - (T_0 + l_{u_0})(x)\right\| + \|f_n - f_0\| \\
&\quad \le \left|\theta_n(x) - \theta_0(x)\right| + \left\|(T_n - T_0)(x)\right\| + \left\|(l_{u_n} - l_{u_0})(x)\right\| + \|f_n - f_0\| \\
&\quad \le \left|\theta_n(x) - \theta_0(x)\right| + \left\|(T_n - T_0)(x)\right\| + \|u_n - u_0\| + \|f_n - f_0\| \to 0.
\end{aligned}$$

Note that $\|u_n - u_0\| \to 0$ and $(\theta_n, T_n, f_n) \xrightarrow{\rho'} (\theta_0, T_0, f_0)$, we have $(\theta_n, T_n + l_{u_n}, f_n) \xrightarrow{\rho'} (\theta_0, T_0 + l_{u_0}, f_0)$. The proof is completed. □

Theorem 2.6 *ϕ' is a continuous mapping from N' to M'.*

Proof Let $\{(\theta_n, T_n, f_n, u_n)\}_{n=1}^{\infty} \subset N'$ with $(\theta_n, T_n, f_n, u_n) \to (\theta_0, T_0, f_0, u_0) \in N'$, that is, $(\theta_n, T_n, f_n) \xrightarrow{\rho'} (\theta_0, T_0, f_0)$, and $\|u_n - u_0\| \to 0$. We need to show that $(\theta_n, T_n - l_{u_n}, f_n) \xrightarrow{\rho'} (\theta_0, T_0 - l_{u_0}, f_0)$. For each $x \in K$, we have

$$\begin{aligned}
&\rho'\big((\theta_n, T_n - l_{u_n}, f_n), (\theta_0, T_0 - l_{u_0}, f_0)\big) \\
&\quad = \sup_{x\in K}\left|\theta_n(x) - \theta_0(x)\right| + \sup_{x\in K}\left\|(T_n - T_0)(x) - (l_{u_n} - l_{u_0})(x)\right\| + \|f_n - f_0\| \\
&\quad \le \sup_{x\in K}\left|\theta_n(x) - \theta_0(x)\right| + \sup_{x\in K}\left\|(T_n - T_0)(x)\right\| + \|f_n - f_0\| \\
&\qquad + \sup_{x\in K}\left\|(l_{u_n} - l_{u_0})(x)\right\| \\
&\quad = \rho'\big((\theta_n, T_n, f_n), (\theta_0, T_0, f_0)\big) + \sup_{\|y\|=1, y\in X}\left|\langle u_n - u_0, y\rangle\right| \\
&\quad \le \rho'\big((\theta_n, T_n, f_n), (\theta_0, T_0, f_0)\big) + \sup_{\|y\|=1, y\in X}\|u_n - u_0\|\|y\| \\
&\quad = \rho'\big((\theta_n, T_n, f_n), (\theta_0, T_0, f_0)\big) + \|u_n - u_0\| \to 0,
\end{aligned}$$

then $\phi'(\theta_n, T_n - l_{u_n}, f_n) \to \phi'(\theta_0, T_0 - l_{u_0}, f_0)$. □

Theorem 2.7 *$\phi' \circ \psi'$ and $\psi' \circ \phi'$ are identity mappings on M' and N', respectively.*

Proof (a) $\phi' \circ \psi'$ is an identity mapping on M'. For each $(\theta, T, f) \in M'$, by Theorem 2.5, we have

$$\phi' \circ \psi'(\theta, T, f) = \phi'(\theta, T + l_u, f, u) = (\theta, T + l_u - l_u, f) = (\theta, T, f).$$

(b) $\psi' \circ \phi'$ is an identity mapping on N'. For each $(\theta, T, f, u^*) \in N'$, we have

$$\psi' \circ \phi'\big(\theta, T, f, u^*\big) = \psi'(\theta, T - l_{u^*}, f) = (\theta, T - l_{u^*} + l_{\bar{u}}, f, \bar{u}),$$

where $\bar{u} \in V'(\theta, R_{T - l_{u^*}}, f)$, and $R_{T-l_{u^*}}$ means that, for each $x \in K$,

$$\big(R_{T-l_{u^*}}(x), z\big) = \big(T(x), z\big) - \big\langle u^*, z\big\rangle + \langle x, z\rangle, \quad \forall z \in X.$$

Since ψ' maps M' onto N', we have $\bar{u} \in V'(\theta, T - l_{u^*} + l_{\bar{u}}, f)$, then

$$\theta(v) - \theta(\bar{u}) + \big(T(\bar{u}) - l_{u^*}(\bar{u}) + l_{\bar{u}}(\bar{u}), v - \bar{u}\big) \geq (f, v - \bar{u}), \quad \forall v \in K.$$

Hence,

$$\theta(v) - \theta(\bar{u}) + \big((T - l_{u^*} + l_{\bar{u}})(\bar{u}), v - \bar{u}\big) \geq (f, v - \bar{u}), \quad \forall v \in K. \tag{9}$$

Let $v = u^*$ in (9), then

$$\theta\big(u^*\big) - \theta(\bar{u}) + \big((T - l_{u^*} + l_{\bar{u}})(\bar{u}), u^* - \bar{u}\big) \geq \big(f, u^* - \bar{u}\big). \tag{10}$$

Since $u^* \in V'(\theta, T, f)$, we have

$$\theta(\bar{u}) - \theta\big(u^*\big) + \big(T\big(u^*\big), \bar{u} - u^*\big) \geq \big(f, \bar{u} - u^*\big). \tag{11}$$

Add $\langle \bar{u} - u^*, \bar{u} - u^* \rangle$ to the left-hand side of equation (11),

$$\theta(\bar{u}) - \theta\big(u^*\big) + \big(T\big(u^*\big), \bar{u} - u^*\big) + \big\langle \bar{u} - u^*, \bar{u} - u^*\big\rangle \geq \big(f, \bar{u} - u^*\big),$$

that is,

$$\theta(\bar{u}) - \theta\big(u^*\big) + \big((T - l_{u^*} + l_{\bar{u}})\big(u^*\big), \bar{u} - u^*\big) \geq \big(f, \bar{u} - u^*\big). \tag{12}$$

Adding equation (10) and equation (12), we get

$$\big((T - l_{u^*} + l_{\bar{u}})\big(\bar{u} - u^*\big), u^* - \bar{u}\big) \geq 0.$$

Note that $T - l_{u^*} + l_{\bar{u}}$ is monotone, we have

$$\big((T - l_{u^*} + l_{\bar{u}})\big(\bar{u} - u^*\big), \bar{u} - u^*\big) = 0.$$

Furthermore, T is strictly monotone, it follows that $\bar{u} = u^*$. □

Theorem 2.8 *The spaces M' and N' are homeomorphic.*

Proof It follows from Theorems 2.5, 2.6, and 2.7. □

Remark 2.4 Theorem 2.8 generalizes the homeomorphism result for the variational inequalities to find a point $u \in K$ such that $(T(u), v - u) \geq 0$, $\forall v \in K$, in [12].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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