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Nonlocal boundary value problems with resonant or non-resonant conditions

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Abstract
We study solvability of nonlocal boundary value problems for second-order differential equations with resonance or non-resonance. The method of proof relies on Schauder's fixed point theorem. Some examples are presented to illustrate the main results.

MSC: 34B05

Keywords: nonlocal boundary value problems; resonance; non-resonance; Schauder's fixed point theorem

1 Introduction

In this paper, we investigate the existence of solutions for the following boundary value problem:

$$\begin{cases} -x''(t) = g(x(t)) - f(t,x(t)), & t \in J, \\ x'(0) = 0, & x(1) = \sum_{i=1}^{k} a_i x(\eta_i), \end{cases} \tag{1.1}$$

where $J = [0,1]$, $0 < a_i \leq 1$ for $1 \leq i \leq k$, $0 < \eta_1 < \eta_2 < \cdots < \eta_k < 1$, $g : \mathbb{R} \to \mathbb{R}$, $f : J \times \mathbb{R} \to \mathbb{R}$.

Nonlocal boundary value problems, studied by Il'in and Moiseev [1], have been addressed by many authors; see, for example, [2–9] and references therein. In the related literature, (1.1) is called resonance when $\sum_{i=1}^{k} a_i = 1$, and non-resonance when $\sum_{i=1}^{k} a_i \neq 1$. For the boundary value problems at resonance, researchers usually use the continuity method or nonlinear alternative, which involves a complicated *a priori* estimate for the solution set; see [7, 10–12]. However, it is very difficult to obtain a related estimate for general differential equations. Here we list only a classical result about nonlocal boundary value problems at resonance of the form

$$\begin{cases} -x''(t) = h(t,x(t),x'(t)) + e(t), & t \in J, \\ x'(0) = 0, & x(1) = \sum_{i=1}^{k} a_i x(\eta_i), \end{cases} \tag{1.2}$$

where $h : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $e : J \to \mathbb{R}$ is continuous and $a_i > 0$ for $1 \leq i \leq k$, $\sum_{i=1}^{k} a_i = 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_k < 1$.

Theorem 1.1 [13] *Suppose that there are two constants $M, \delta > 0$ such that*

(A1) $x[h(t,x,0) + e(t)] > \delta$ *for any* $|x| > M$, $t \in J$;

(A2) *there exist constants* $L_1, L_2 : L_1 > M$, $L_2 < -M$ *such that*

$$h(t,x,L_1) + e(t) \geq 0, \quad \forall (t,x) \in J \times [-M,M],$$
$$h(t,x,L_2) + e(t) \leq 0, \quad \forall (t,x) \in J \times [-M,M];$$

(A3) *for* $(t,x,p) \in J \times [-M,M] \times [L_2,L_1]$,

$$\left|h(t,x,p) + e(t)\right| \leq \frac{M}{1-\eta_1}.$$

Then (1.2) *has at least one solution.*

For (1.1), condition (A2) in Theorem 1.1 implies that $f(t,x) \equiv g(x)$ for $(t,x) \in J \times [-M,M]$. It follows that (1.1) has infinitely many solutions ($x \equiv C \in [-M,M]$ is the solution of (1.1)). At this point, Theorem 1.1 has little significance for (1.1). Moreover, there are few papers considering multiple results at resonance. For the case with non-resonance, there is an extensive literature; see [14–17] and the references therein.

The main purpose of this article is to discuss the existence of solutions of equation (1.1) by means of Schauder's fixed point theorem. We only need to consider the behavior of g and f on some closed sets. Consequently, information on the location of the solution is obtained and multiple results are obtained if g and f satisfy the given conditions on distinct regions. Our approach is valid for the cases at resonance or non-resonance. In addition, some of our conditions are easily certified (see Corollaries 3.1 and 3.2).

The paper is organized as follows. Section 2 introduces an important lemma. Section 3 is devoted to the existence results of (1.1). In Section 4 we extend some results of Section 3 to the general boundary conditions.

2 Preliminaries

In this section, we consider the following boundary value problem for the linear differential equation:

$$\begin{cases} -x''(t) + px(t) = h(t), & t \in J, \\ x'(0) = 0, & x(1) = \sum_{i=1}^{k} a_i x(\eta_i), \end{cases} \tag{2.1}$$

where $p > 0$, $0 < \alpha_i \leq 1$, $0 < \eta_i < 1$ for $1 \leq i \leq k$, $0 < a := \sum_{i=1}^{k} a_i \leq 1$ and $h \in C(J,\mathbb{R})$.

Lemma 2.1

(1) *Boundary value problem* (2.1) *has a unique solution* $x_h \in C^2(J,\mathbb{R})$.
(2) *If* $h \equiv C \in R$ *on* J *and* $a = 1$, *then* $x_h \equiv C/p$ *on* J.
(3) *If* $h(t) \geq 0$ *for all* $t \in J$, *then* $x_h \geq 0$ *on* J; *if* $h(t) \leq 0$ *for all* $t \in J$, *then* $x_h \leq 0$ *on* J.
(4) *If* $|h(t)| \leq C$ $(C > 0)$ *on* J, *then* $|x_h| \leq C/p$ *on* J.
(5) *Define an operator* $A : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ *by* $A(h) = x_h$, *where* $\|h\| = \max_{t\in J} |h(t)|$; *then* A *is completely continuous.*

Proof (1) Any solution of the differential equation $-x''(t) + px(t) = h(t)$ can be written as

$$x(t) = c_1 e^{t\sqrt{p}} + c_2 e^{-t\sqrt{p}} + \varphi(t),$$

where c_1, c_2 are constants and $\varphi \in C^2(t,\mathbb{R})$ is a particular solution of $-x''(t) + px(t) = h(t)$. From the boundary conditions, we obtain that

$$\begin{cases} c_1[e^{\sqrt{p}} - \sum_{i=1}^{k} a_i e^{\sqrt{p}\eta_i}] + c_2[e^{-\sqrt{p}} - \sum_{i=1}^{k} a_i e^{-\sqrt{p}\eta_i}] = \sum_{i=1}^{k} a_i \varphi(\eta_i) - \varphi(1), \\ \sqrt{p}(c_1 - c_2) = -\varphi'(0). \end{cases} \tag{2.2}$$

Since the above system has a unique solution (c_1, c_2), (2.1) has a unique solution $x_h \in C^2(t,\mathbb{R})$.

(2) The conclusion is obvious.

(3) Here we only prove the case of $h \geq 0$. We consider two cases.

Case 3.1 Assume that $x_h(t) \leq 0$ for all $t \in J$. We show that $x_h(t) \equiv 0$, $t \in J$. From (2.1), we obtain that

$$x_h''(t) = px_h(t) - h(t) \leq 0, \quad \forall t \in J,$$

which implies that x_h' is nonincreasing on J. Noting $x_h'(0) = 0$, we obtain that x_h is nonincreasing. Thus $x_h(1) = \min_{t\in J} x_h(t) \leq 0$.

Since x_h is continuous, by the intermediate value theorem, there exists $\theta \in (0,1)$ such that $ax_h(\theta) = \sum_{i=1}^{m} a_i x_h(\eta_i)$.

If $a = 1$, one can obtain from the monotonicity of x_h that $x_h(t) \equiv x_h(1)$ on $[\theta, 1]$. Hence,

$$0 \geq px(1) = -x_h''(t) + px_h(t) = h(t) \geq 0, \quad t \in [\theta, 1],$$

which implies that $x_h(1) = 0$. Hence, $x_h(t) \equiv 0$ on J.

If $0 < a < 1$, then $x_h(1) = \sum_{i=1}^{m} a_i x_h(\eta_i) = ax_h(\theta) \geq ax_h(1)$, which implies that $x_h(1) = 0$. Hence, $x_h(t) \equiv 0$ on J.

Case 3.2 There exist $t_1, t_2 \in J$ such that $x_h(t_1) < 0$ and $x_h(t_2) > 0$. We assume that $t_1 < t_2$. Otherwise, $x_h(t) \leq 0$ for all $t \in [t_1, 1]$. Similar to Case 3.1, one can show that $x_h \equiv 0$ for $t \in [t_1, 1]$, which is impossible.

If there is $\varepsilon > 0$ such that $x_h(t) \leq 0$ for $t \in (0, \varepsilon)$, it is easy to check that $x_h(t) \leq 0$ for all $t \in J$, a contradiction. Since $x_h(1) \geq \sum_{\{i:x_h(\eta_i)<0\}} a_i x_h(\eta_i) \geq \min\{x_h(\eta_i) : x_h(\eta_i) < 0\}$, there exists $r \in (0,1)$ such that $x_h(r) = \min_{t\in J} x_h(t)$. Noting that $x_h''(r) \geq 0$ and $x_h(r) < 0$, we obtain that

$$0 > px_h(r) = h(r) + x_h''(r) \geq 0,$$

which is a contradiction.

From Cases 3.1 and 3.2, one can easily obtain that $x_h \geq 0$ for all $t \in J$.

(4) Since $-C \leq h \leq C$, using the conclusion of (3), we have

$$x_{h-C} \leq 0, \qquad x_{h+C} \geq 0, \quad t \in J.$$

Noting that $x_{h-C} = x_h - x_C$, $x_{h+C} = x_h - x_{-C}$, we obtain that

$$x_{-C} \leq x_h \leq x_C, \quad t \in J. \tag{2.3}$$

If $a = 1$, then $x_C = C/p$ and $x_{-C} = -C/p$. Thus $|x_h| \leq C/p$ on J.

Assume that $a < 1$. There exist $t_1, t_2 \in [0,1)$ such that

$$x_C(t_1) = \max_{t\in J} x_C(t), \qquad x_{-C}(t_2) = \min_{t\in J} x_{-C}(t).$$

Noting that $x_C''(t_1) \le 0$ and $x_{-C}''(t_2) \ge 0$, we have

$$0 \le px_C(t_1) = C + x_C''(t_1) \le C, \tag{2.4}$$

and

$$0 \ge px_{-C}(t_2) = -C + x_{-C}''(t_2) \ge -C. \tag{2.5}$$

From (2.3), (2.4) and (2.5), we obtain that $|x_h| \le C/p$ on J.

(5) Let $h_n \to h$ in $C(J,\mathbb{R})$. For any $\varepsilon > 0$, there exists $N(\varepsilon) > 0$ such that

$$\left|h_n(t) - h(t)\right| < \varepsilon, \quad \forall t \in J, \forall n > N(\varepsilon).$$

Noting that $A(h_n - h) = x_{h_n} - x_h$, using the conclusion of (4), we obtain that

$$|x_{h_n} - x_h| \le \varepsilon/p, \quad \forall t \in J, \forall n > N(\varepsilon),$$

which implies that A is continuous. Let D be a bounded set in $C(J,\mathbb{R})$. Then there is $M_1 > 0$ such that $\|h\| \le M_1$ for all $h \in D$. From the conclusion of (4), we obtain that $|A(h)| \le M_1/p$, which implies that $A(D)$ is a uniformly bounded set. Since $A(h)$, h are bounded for $h \in D$ and

$$(Ah)'(t) = p\int_0^t (Ah)(s)\,ds - \int_0^t h(s)\,ds,$$

there exists $M_2 > 0$ such that

$$\left|(Ah)'(t)\right| \le M_2, \quad t \in J, h \in D,$$

which implies that $A(D)$ is equicontinuous. It follows that $A(D)$ is relatively compact in $C(J,\mathbb{R})$ and A is a completely continuous operator. The proof is complete. □

Remark 2.1 Let $h \equiv 1$ and

$$\alpha_p^{h=1} = \max_{t\in J} px_h(t), \qquad \beta_p^{h=1} = \min_{t\in J} px_h(t).$$

By a direct computation, one can obtain that $0 < \beta_p^{h=1} \le \alpha_p^{h=1} \le 1$ and

$$\alpha_p^{h=1} = 1 - \frac{2(1-a)}{e^{\sqrt{p}} + e^{-\sqrt{p}} - \sum_{i=1}^k a_i[e^{\sqrt{p}\eta_i} + e^{-\sqrt{p}\eta_i}]},$$
$$\beta_p^{h=1} = 1 - \frac{(1-a)(e^{\sqrt{p}} + e^{-\sqrt{p}})}{e^{\sqrt{p}} + e^{-\sqrt{p}} - \sum_{i=1}^k a_i[e^{\sqrt{p}\eta_i} + e^{-\sqrt{p}\eta_i}]}.$$

The following well-known Schauder fixed point theorem is crucial in our arguments.

Lemma 2.2 [18] *Let X be a Banach space and $D \subset X$ be closed and convex. Assume that $T : D \to D$ is a completely continuous map; then T has a fixed point in D.*

3 Main results

The following theorem is the main result of the paper.

Theorem 3.1 *Assume that there exist constants $M > m$, $p > 0$ such that $g \in C([m,M],\mathbb{R})$, $f \in C(J \times [m,M],\mathbb{R})$, $pu + g(u)$ is nondecreasing in $u \in [m,M]$ and $\alpha_p^{h=1} m \le \beta_p^{h=1} M$. Further suppose that*

$$g(M) - \frac{(1-\alpha_p^{h=1})pM}{\alpha_p^{h=1}} \le f(t,u) \le g(m) - \frac{(1-\beta_p^{h=1})pm}{\beta_p^{h=1}}, \quad \forall (t,u) \in J \times [m,M]. \tag{3.1}$$

Then (1.1) *has at least one solution x with $m \le x \le M$.*

Proof From Lemma 2.1, if x is a solution of (1.1), x satisfies

$$x = (A \circ H)x,$$

where $A \circ H$ is composition of A and H defined as $(A \circ H)x = A(Hx)$, and the operator H is defined in $C(J,\mathbb{R})$ as

$$(Hx)(t) = px(t) + g\bigl(x(t)\bigr) - f\bigl(t,x(t)\bigr).$$

Note that

$$\begin{cases} -[(A \circ H)x]''(t) + p[(A \circ H)x](t) = (Hx)(t), & t \in J, \\ [(A \circ H)x]'(0) = 0, & [(A \circ H)x](1) = \sum_{i=1}^{k} a_i[(A \circ H)x](\eta_i). \end{cases} \tag{3.2}$$

Obviously, a fixed point of $A \circ H$ is a solution of (1.1). Set $\Omega = \{x \in C(J,\mathbb{R}) : m \le x(t) \le M, t \in J\}$. Since the function $pu + g(u)$ is nondecreasing in $u \in [m,M]$, we obtain that for $x \in \Omega$,

$$pm + g(m) \le px(t) + g\bigl(x(t)\bigr) \le pM + g(M).$$

Using (3.1), we have

$$\frac{pm}{\beta_p^{h=1}} \le (Hx)(t) = px(t) + g\bigl(x(t)\bigr) - f\bigl(t,x(t)\bigr) \le \frac{pM}{\alpha_p^{h=1}}$$

for any $x \in \Omega$. Since $\beta_p^{h=1} \le pA(1) \le \alpha_p^{h=1}$, and A is one nondecreasing operator, we obtain that for $x \in \Omega$,

$$m \le A\left(\frac{pm}{\beta_p^{h=1}}\right) \le (A \circ H)x \le A\left(\frac{pM}{\alpha_p^{h=1}}\right) = \frac{pM}{\alpha_p^{h=1}} A(1) \le M.$$

Hence, $(A \circ H)(\Omega) \subset \Omega$.

Also, the fact that A is completely continuous and H is continuous gives that $A \circ H : \Omega \to \Omega$ is a continuous, compact map. By Lemma 2.2, $A \circ H$ has at least one fixed point in Ω. The proof is complete. □

Remark 3.1 In Theorem 3.1, the condition that $pu + g(u)$ is nondecreasing in $u \in [m, M]$ can been replaced by the weaker condition

$$pm + g(m) \leq pu + g(u) \leq pM + g(M), \quad \forall u \in [m, M].$$

Corollary 3.1 *Assume that* $a = 1$ *and the following condition holds*:

(H_1) *There exist constants* $m < M$ *such that* $g \in C^1([m, M], \mathbb{R}), f \in C(J \times [m, M], \mathbb{R})$, *and*

$$g(M) \leq f(t, u) \leq g(m), \quad \forall (t, u) \in J \times [m, M]. \tag{3.3}$$

Then (1.1) *has at least one solution* x *with* $m \leq x \leq M$.

Proof Since $g \in C^1([m, M], \mathbb{R})$, there exists $p > 0$ such that the function $pu + g(u)$ is nondecreasing in $u \in [m, M]$. When $a = 1$, $\alpha_p^{h=1} = \beta_p^{h=1} = 1$. We directly apply Theorem 3.1 and this ends the proof. □

Corollary 3.2 *Assume that* $0 < a < 1$ *and the following condition holds*:

(H_2) *There exists constant* $M > 0$ *such that* $g \in C^1([0, M], \mathbb{R}), f \in C(J \times [0, M], \mathbb{R})$, *and*

$$g(M) \leq f(t, u) \leq g(0), \quad \forall (t, u) \in J \times [0, M]. \tag{3.4}$$

Then (1.1) *has at least one solution* x *with* $0 \leq x \leq M$.

Proof Since $g \in C^1([0, M], \mathbb{R})$, there exists $p > 0$ such that the function $pu + g(u)$ is nondecreasing in $u \in [0, M]$. Condition (3.1) is satisfied if (3.4) holds. The proof is complete. □

Example 3.1 Consider the differential equation

$$\begin{cases} -x''(t) = \sin x + t^2 e^{-x(t)}, & t \in J, \\ x'(0) = 0, \quad x(1) = x(\eta), & \eta \in (0, 1). \end{cases} \tag{3.5}$$

Let $m_n = (2n + 0.5)\pi$, $M_n = (2n + 1.5)\pi$, $g(u) = \sin u$, $f(t, u) = -t^2 e^{-u}$ and n be a positive integer. For any $t \in J$ and $u \in [m_n, M_n]$,

$$g(M_n) = -1 \leq f(t, u) \leq 1 = g(m_n).$$

Hence, by Corollary 3.1, (3.5) has a solution $m_n \leq x \leq M_n$. Since n is an arbitrary positive integer, (3.5) has infinitely many solutions.

Example 3.2 Consider the differential equation

$$\begin{cases} -x''(t) = 2 - e^{x(t)} + x^2(t), & t \in J, \\ x'(0) = 0, \quad x(1) = \frac{1}{2} x(\eta), & \eta \in (0, 1). \end{cases} \tag{3.6}$$

Using Corollary 3.2, we obtain that (3.6) has at least a nonnegative solution $\overline{x}$. Moreover, it is not difficult to show that $\overline{x} \in (0,5)$ for any $t \in J$.

4 Generalization

In this section, we extend some results in the previous section to the following equation:

$$\begin{cases} -x''(t) = g(x(t)) - f(t, x(t)), & t \in J, \\ U_1 x = 0, \qquad U_2 x = 0, \end{cases} \tag{4.1}$$

where U_1 and U_2 are linear operators defined as

$$U_1 x = \sum_{i=1}^{n_1} a_i x(\lambda_i) + \sum_{j=1}^{n_2} b_j x'(\mu_j), \qquad U_2 x = \sum_{s=1}^{m_1} c_s x(\nu_s) + \sum_{l=1}^{m_2} d_l x'(\kappa_l),$$

where $a_i, b_j, c_s, d_l \in \mathbb{R}$ and $0 < \lambda_i, \mu_j, \nu_s, \kappa_l \le 1$ for $1 \le i \le n_1$, $1 \le j \le n_2$, $1 \le s \le m_1$, $1 \le l \le m_2$.

Consider the boundary value problem for the linear differential equation:

$$\begin{cases} -x''(t) + px(t) = h(t), & t \in J, \\ U_1 x = 0, \qquad U_2 x = 0, \end{cases} \tag{4.2}$$

where $p > 0$ is sufficiently large, $h : J \to \mathbb{R}$. We introduce the following assumptions.

(P_1) The condition $h \in C(J, \mathbb{R})$ implies that boundary value problem (4.2) has a unique solution $x_h \in C^2(J, \mathbb{R})$.
(P_2) The condition $h \equiv C \in \mathbb{R}$ implies that $x_h \equiv C/p$ on J.
(P_3) The condition $h \in C(J, \mathbb{R}) : h \ge 0$ $(t \in J)$ implies that $x_h \ge 0$ on J.
(P_4) The condition $h \in C(J, \mathbb{R}) : |h(t)| \le C$ $(C > 0)$ implies that $|x_h| \le C/p$ on J.

We say that the boundary condition $U_1 x = U_2 x = 0$ satisfies P_{123} if (4.2) satisfies conditions (P_1), (P_2), (P_3), and P_{134} if (4.2) satisfies conditions (P_1), (P_3), (P_4).

Theorem 4.1

(1) *Assume that the boundary condition* $U_1 x = U_2 x = 0$ *satisfies* P_{123} *and* (H_1) *holds. Then* (4.1) *has at least one solution* x *with* $m \le x \le M$.
(2) *Assume that the boundary condition* $U_1 x = U_2 x = 0$ *satisfies* P_{134} *and* (H_2) *holds. Then* (4.1) *has at least one solution* x *with* $0 \le x \le M$.

The proof of Theorem 4.1 is similar to that of Theorem 3.1 and we omit it.

Remark 4.1 The solution obtained in Corollary 3.2 or (2) of Theorem 4.1 may be trivial. Further suppose that

$$g(0) - f(t, 0) \not\equiv 0, \quad t \in J.$$

Then the solution obtained is nonnegative and nontrivial.

Remark 4.2 The boundary condition $U_1 x = U_2 x = 0$ satisfies P_{134} if it satisfies P_{123}.

Remark 4.3 Consider the two-point boundary conditions:

$$x'(0) = x'(1) = 0, \tag{4.3}$$

$$x(0) = x(1), \qquad x'(0) = x'(1), \tag{4.4}$$

$$x(0) = x(1) = 0, \tag{4.5}$$

$$x'(0) = x(1) = 0, \tag{4.6}$$

$$x(0) = x'(1) = 0. \tag{4.7}$$

One can easily check that boundary conditions (4.3), (4.4) satisfy P_{123}, and conditions (4.5), (4.6), (4.7) satisfy P_{134}.

Next, we consider the boundary conditions

$$\begin{aligned} U_1x &:= \alpha x(0) - \beta x'(0) - \sum_{i=1}^{k} b_i x(\eta_i) = 0, \\ U_2x &:= \lambda x(1) + \mu x'(1) - \sum_{j=1}^{n} c_j x(\xi_j) = 0, \end{aligned} \tag{4.8}$$

where α, β, λ, μ, b_i $(1 \le i \le k)$, c_j $(1 \le j \le n)$ are constants and $\alpha, \lambda \in (0, +\infty)$, β, μ, b_i, $c_j \in [0, +\infty)$, $\eta_i, \xi_j \in (0, 1)$.

Theorem 4.2 *Set* $b = \sum_{i=1}^{k} b_i$, $c = \sum_{j=1}^{n} c_j$.

(1) *If* $b = \alpha$, $c = \lambda$, *then the boundary condition* $U_1x = U_2x = 0$ *satisfies* P_{123}.

(2) *If* $b \in [0, \alpha]$, $c \in [0, \lambda)$ *or* $b \in [0, \alpha)$, $c \in [0, \lambda]$, *then the boundary condition* $U_1x = U_2x = 0$ *satisfies* P_{134}.

Proof Without loss of generality, we assume that $m = n = 1$. For any sufficiently large $p > 0$ and $h \in C(J, \mathbb{R})$, the linear differential equation

$$\begin{cases} -x''(t) + px(t) = h(t), & t \in J, \\ U_1x = 0, \qquad U_2x = 0 \end{cases} \tag{4.9}$$

has a unique solution $x_h \in C^2(J, \mathbb{R})$.

Suppose that $h \ge 0$ on J; if $x_h \ge 0$ is not true, by the maximum principle, we get that $x_h(0) = \min_{t \in J} x_h(t) < 0$ or $x_h(1) = \min_{t \in J} x(t) < 0$. If $x_h(0) = \min_{t \in J} x_h(t) < 0$, then $x_h'(0) \ge 0$. From the boundary conditions, we have

$$0 > x_h(0) = \frac{\beta}{\alpha} x_h'(0) + \frac{b_1}{\alpha} x_h(\eta_1) \ge \frac{b_1}{\alpha} x_h(\eta_1) \ge x_h(\eta_1).$$

By the maximum principle, we obtain that $x_h(\eta_1) \ge 0$, which is a contradiction. If $x_h(1) = \min_{t \in J} x_h(t) < 0$, then $x_h'(1) \le 0$. From the boundary conditions, we have

$$0 > x_h(1) = \frac{c_1}{\lambda} x_h(\xi_1) - \frac{\mu}{\lambda} x_h'(1) \ge \frac{c_1}{\lambda} x_h(\xi_1) \ge x_h(\xi_1).$$

By the maximum principle, we obtain that $x_h(\xi_1) \ge 0$, a contradiction.

If $h \equiv C$ and $b = \alpha$, $c = \lambda$, then $x_h \equiv C/p$.

Now suppose that $0 \le h \le C$ on J and $h \not\equiv 0$.

If there is $\theta \in (0,1)$ such that $x_h(\theta) = \max_{t\in J} x(t) > 0$, noting that $x_h''(\theta) \le 0$, we have

$$0 \le px_h(\theta) \le -x_h''(\theta) + px_h(\theta) = h(\theta) \le C.$$

If $x_h(0) = \max_{t\in J} x(t) > 0$, from the boundary conditions, we obtain that $b = \alpha$, $\beta x_h'(0) = 0$, which implies that $x_h(\eta_1) = x_h(0) = \max_{t\in J} x(t)$. The case has been discussed. If $x_h(1) = \max_{t\in J} x(t) > 0$, from the boundary conditions, we obtain that $c = \lambda$, $\mu x_h'(1) = 0$, which implies that $x_h(\xi_1) = x_h(1) = \max_{t\in J} x(t)$. The case has also been discussed. The proof is complete. □

Example 4.1 Consider the differential equation

$$\begin{cases} -x''(t) = \sin^{-1} x(t) - x^{\lambda}(t) + tx(t), & t \in J, \\ x(0) = x(\eta), & x(1) = x(\xi), \end{cases} \tag{4.10}$$

where $\lambda > 0$, $0 < \eta, \xi < 1$ are constants.

Let $g(u) = \sin^{-1} u$ and $f(t,u) = u^{\lambda} - tu$. Set $m_n = 2n\pi + (4n\pi)^{-\lambda-1}$, $M_n = 2n\pi + 0.5\pi$. If n is a sufficiently large, positive integer, then for any $t \in J$, $u \in [m_n, M_n]$,

$$1 = g(M_n) \le f(t,u) \le g(m_n) \approx (4n\pi)^{\lambda+1}.$$

By Theorems 4.1 and 4.2, (4.10) has a solution $m_n \le x \le M_n$. Hence, (4.10) has infinitely many solutions.

Example 4.2 Consider the differential equation

$$\begin{cases} -x''(t) + x^{\mu}(t)\cos x(t) = t^2, & t \in J, \\ x(0) - x'(0) = x(\xi_1), & x(1) = \frac{1}{2}x(\xi_2) + \lambda x(\xi_3), \end{cases} \tag{4.11}$$

where $\mu > 0$, $0 < \xi_1, \xi_2, \xi_3 < 1$, $0 \le \lambda \le 0.5$ are constants.

In fact, $g(u) = -u^{\mu}\cos u$ and $f(t) = -t^2$. The boundary conditions in (4.11) satisfy P_{134} for $\lambda \in [0, 0.5]$ and P_{123} for $\lambda = 0.5$.

(1) Equation (4.11) has a solution $0 \le \widetilde{x} \le 1$ and $\widetilde{x}(t) > 0$, $t \in (0,1]$ for all $0 \le \lambda \le 0.5$. Set $M = 1$, then $g(M) = -1 \le f(t) \le g(0)$ for all $t \in J$. By Theorems 4.1 and 4.2, (4.11) has a solution $0 \le \widetilde{x} \le 1$. Now we show that $\widetilde{x}(t) > 0$ for $t \in (0,1]$. Assume that there exists $r \in (0,1)$ with $\widetilde{x}(r) = 0$. Since $\widetilde{x}(t) \ge 0$, $\widetilde{x}(r)$ is minimum value and $\widetilde{x}'(r) = 0$, $\widetilde{x}''(r) \ge 0$. On the other hand, $\widetilde{x}''(r) = \widetilde{x}^{\mu}(r)\cos\widetilde{x}(r) - r^2 = -r^2 < 0$, a contradiction. If $\widetilde{x}(1) = 0$, then $\widetilde{x}(\xi_2) = 0$. This is impossible.

(2) Equation (4.11) has infinitely many solutions for $\lambda = 0.5$. Set $M_n = 2n\pi + 2\pi$, $m_n = 2n\pi + 1.5\pi$, where $n > 0$ is an integer. Since $g(M_n) = -M_n^{\mu} \le f(t) \le g(m_n) = m_n^{\mu}$ for all $t \in J$, (4.11) has a solution $m_n \le x \le M_n$. Hence, (4.11) has infinitely many solutions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Authors typed, read and approved the final draft.

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