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Multiple solutions of semilinear elliptic systems on the Heisenberg group

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Abstract

In this paper, a class of semilinear elliptic systems which have a strong resonance at the first eigenvalue on the Heisenberg group is considered. Under certain assumptions, by virtue of the variational methods, the multiple weak solutions of the systems are obtained.

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1 Introduction

Let $\mathbb{H}^N$ be the space $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$ equipped with the following group operation:

$$\eta \circ \eta' = (x,y,t) \circ (x',y',t') = \left(x + x', y + y', t + t' + 2\left(x' \cdot y - x \cdot y'\right)\right),$$

where '$\cdot$' denotes the usual inner-product in $\mathbb{R}^N$. This operation endows $\mathbb{H}^N$ with the structure of a Lie group. The vector fields $X_1, \ldots, X_N$, $Y_1, \ldots, Y_N$, T, given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \qquad T = \frac{\partial}{\partial t},$$

form a basis for the tangent space at $\eta = (x,y,t)$.

Definition 1.1 The Heisenberg Laplacian is by definition

$$\Delta_{\mathrm{H}} = \sum_{j=1}^{N} \left(X_j^2 + Y_j^2\right),$$

and let $\nabla_{\mathrm{H}} u$ denote the $2N$-vector $(X_1 u, \ldots, X_N u, Y_1 u, \ldots, Y_N u)$.

Definition 1.2 The space $S_0^{1,2}(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ in the norm

$$\|u\|^2_{S_0^{1,2}} = \int_\Omega \sum_{j=1}^{N} \left(|X_j u|^2 + |Y_j u|^2\right) = \int_\Omega |\nabla_{\mathrm{H}} u|^2.$$

Some existence and nonexistence for the semilinear equations or systems on the Heisenberg group have been studied by Garofalo, Lanconelli and Niu, see [1, 2], *etc.*

In this paper, we study the problems on the existence and multiplicity of solutions for the system

$$\begin{cases} -\Delta_{\mathrm{H}}\binom{u}{v} = \lambda_1\begin{pmatrix} a(x) & b(x) \\ b(x) & d(x)\end{pmatrix}\binom{u}{v} - \binom{f(x,u,v)}{g(x,u,v)}, & x \in \Omega, \\ u = v = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{H}^N$ is a bounded smooth domain, $a, b, d \in C^0(\overline{\Omega}, \mathbb{R})$ and $f, g \in C^1(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$. Moreover, we assume that there is some function $F(x, u, v) \in C^2(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$ such that $\nabla F = \binom{f}{g}$. Here ∇F denotes the gradient in the variable u and v, *i.e.*, $\frac{\partial F}{\partial u} = f$, $\frac{\partial F}{\partial v} = g$.

In fact, the condition in $\mathbb{R}^N$ was studied by da Silva; we can see [3]. In this paper we study the problem on the Heisenberg group $\mathbb{H}^N$. The elliptic problems at resonance have been studied by many authors; see [4–7].

We use the variation methods to solve problem (1.1). Finding weak solutions of (1.1) in $E = S_0^{1,2}(\Omega) \times S_0^{1,2}(\Omega)$ is equivalent to finding critical points of the C^2 functional given by

$$I(h) = \frac{1}{2}\|h\|^2 - \frac{1}{2}\int_\Omega \langle Ah, h\rangle + \int_\Omega F(x, h), \tag{1.2}$$

where

$$h \in E, \quad h = \begin{pmatrix} h^{(1)} \\ h^{(2)} \end{pmatrix}, \qquad \|h\|^2 = \int_\Omega \left|\nabla_H h^{(1)}\right|^2 + \left|\nabla_H h^{(2)}\right|^2,$$

and $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^2$.

We introduce the eigenvalue problem with weights. Let us denote by $\mathcal{A}$ the set of all continuous, cooperative and symmetric matrices A of order 2, given by

$$A(x) = \begin{pmatrix} a(x) & b(x) \\ b(x) & d(x) \end{pmatrix},$$

where the functions $a, b, d \in C(\overline{\Omega}, \mathbb{R})$ satisfy the following conditions:

(A_1) $A(x)$ is cooperative, that is, $b(x) \geq 0$.
(A_2) There is an $x_0 \in \Omega$ such that $a(x_0) > 0$ or $d(x_0) > 0$.

Given $A \in \mathcal{A}(\Omega)$, consider the weighted eigenvalue problem

$$\begin{cases} -\Delta_{\mathrm{H}}\binom{h^{(1)}}{h^{(2)}} = \lambda A(x)\binom{h^{(1)}}{h^{(2)}}, & \text{in } \Omega, \\ h^{(1)} = h^{(2)} = 0, \quad \text{on } \partial\Omega, \end{cases}$$

if $A \in \mathcal{A}(\Omega)$. By virtue of the spectral theory for compact operators, we obtain the sequence of eigenvalues

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$$

such that $\lambda_k \to +\infty$ as $k \to \infty$; see [6, 8, 9]. Here, each eigenvalue λ_k, $k \geq 1$ has finite multiplicity, and we have

$$\frac{1}{\lambda_k} = \sup\left\{\int_\Omega \langle Ah, h\rangle, \|h\| = 1, h \in V_{k-1}^\perp\right\},$$

where $V_k = \operatorname{span}\{\Phi_1, \dots, \Phi_k\}$ with $k \geq 1$.

Remark 1.1

(1) $E = V_k \oplus V_k^\perp$ for $k \geq 1$.

(2) The following variational inequalities hold:

$$\|h\|^2 \leq \lambda_k \int_\Omega \langle Ah, h\rangle, \quad \forall h \in V_k, k \geq 1, \tag{1.3}$$

$$\|h\|^2 \geq \lambda_{k+1} \int_\Omega \langle Ah, h\rangle, \quad \forall h \in V_k^\perp, k \geq 0. \tag{1.4}$$

The variational inequalities will be used in the next section. We would like to mention that the Φ_1 is positive in Ω. In the paper, without loss of generality, we assume that $\lambda_1 = 1$.

We now state the assumptions and the main results in this paper. Firstly, we define the following functions:

$$\begin{cases} T^+ = \liminf_{(u,v)\to(\infty,\infty)} F(x,u,v), & S^+ = \limsup_{(u,v)\to(\infty,\infty)} F(x,u,v), \\ T^- = \liminf_{(u,v)\to(-\infty,-\infty)} F(x,u,v), & S^- = \limsup_{(u,v)\to(-\infty,-\infty)} F(x,u,v). \end{cases} \tag{1.5}$$

The above functions belong to $L^1(\Omega)$ and the limits are taken a.e. and uniformly in $x \in \Omega$.

Now we make the following basic hypotheses:

(E_0) There exists $k \in C(\overline{\Omega})$ such that

$$\lim_{|h|\to\infty} \nabla F(x,h) = 0, \qquad \left|F(x,h)\right| \leq k(x), \quad \text{a.e. } x \in \Omega, \forall h \in \mathbb{R}^2.$$

(E_1) $F(x,h) \geq \frac{1}{2}(1-\lambda_2)\langle Ah, h\rangle + b_1|\Omega|^{-1}$, $b_1 \geq 0$, $\forall (x,h) \in \Omega \times \mathbb{R}^2$.

(E_2) $\langle Ah, h\rangle \geq 0$, $\forall (x,h) \in \Omega \times \mathbb{R}^2$.

(E_3) There exist $\alpha \in (0,1)$ and $\delta > 0$ such that

$$F(x,h) \geq \frac{1-\alpha}{2}\langle Ah, h\rangle, \quad \forall x \in \Omega \text{ and } |z| < \delta.$$

(E_4) $\int_\Omega S^+ \leq 0$ and $\int_\Omega S^- \leq 0$.

(E_5) There exists $t_0 \in \mathbb{R}$ such that

$$\int_\Omega F(x, t_0\Phi_1) < \min\left\{\int_\Omega T^+, \int_\Omega T^-\right\}.$$

(E_6) There are $t_1^- < 0$ and $t_1^+ > 0$ such that

$$\int_\Omega F\bigl(x, t_1^\pm \Phi_1\bigr) < \min\left\{\int_\Omega T^+, \int_\Omega T^-\right\}.$$

We can prove that the associated functional J has the saddle geometry. Actually, we have the following results.

Theorem 1.1 *Let $\Omega \subseteq \mathbb{H}^N$ be a bounded smooth domain, $a(x), b(x), d(x) \in C^0(\overline{\Omega}, \mathbb{R})$ and $f(x,u,v), g(x,u,v) \in C^1(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$. Assume that there is some function $F(x,u,v) \in C^2(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$ such that $\frac{\partial F}{\partial u} = f$, $\frac{\partial F}{\partial v} = g$. Furthermore, if the conditions* (E_0), (E_1), (E_2) *are satisfied, problem* (1.1) *has at least one solution $z_1 \in E$.*

Remark 1.2 For the hypotheses $\nabla F(x,0,0) \equiv 0$ and $F(x,0,0) \equiv 0$, problem (1.1) admits the trivial solution $(u,v) = 0$. In this case, the main point is to assure the existence of non-trivial solutions.

Theorem 1.2 *Let $\Omega \subseteq \mathbb{H}^N$ be a bounded smooth domain, $a(x), b(x), d(x) \in C^0(\overline{\Omega}, \mathbb{R})$ and $f(x,u,v), g(x,u,v) \in C^1(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$. Assume that there is some function $F(x,u,v) \in C^2(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$ such that $\frac{\partial F}{\partial u} = f$, $\frac{\partial F}{\partial v} = g$. Furthermore, if the conditions* (E_0), (E_2), (E_3), (E_4) *and* (E_5) *are satisfied, then problem* (1.1) *has at least two nontrivial solutions.*

Theorem 1.3 *Let $\Omega \subseteq \mathbb{H}^N$ be a bounded smooth domain, $a(x), b(x), d(x) \in C^0(\overline{\Omega}, \mathbb{R})$ and $f(x,u,v), g(x,u,v) \in C^1(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$. Assume that there is some function $F(x,u,v) \in C^2(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$ such that $\frac{\partial F}{\partial u} = f$, $\frac{\partial F}{\partial v} = g$. Furthermore, if the conditions (E_0),* (E_1), (E_2), (E_3), (E_4) *and* (E_6) *are satisfied, then problem* (1.1) *has at least three nontrivial solutions.*

2 Preliminaries and fundamental lemmas

In this section, we prove some lemmas needed in the proof of our main theorems.

We first introduce the Folland-Stein embedding theorem (see [10]) as follows.

Lemma 2.1 *Let $\Omega \subseteq \mathbb{H}^N$ be a bounded domain and let $Q = 2N + 2$. Then $S_0^{1,2}(\Omega)$ compactly embedding in $L^p(\Omega)$, where $2 \leq p < \frac{2Q}{Q-2}$.*

To establish Lemmas 2.7 and 2.8, we introduce the following corollary of the Ekeland variation principle (see [11]).

Lemma 2.2 *X is a metric space, $I \in C^1(X, \mathbb{R})$ is bounded from below, which satisfies the $(PS)_c$ condition, then $c = \inf_{x \in X} E(x)$ is a critical value of E.*

Next, we describe some results under the geometry for the functional I.

Lemma 2.3 *Under hypotheses* (E_0) *and* (E_1), *the functional I has the following saddle geometry*:

(L3-1) *$I(h) \to \infty$ if $\|h\| \to \infty$ with $h \in V_1^{\perp}$.*
(L3-2) *There is $\alpha \in \mathbb{R}$ such that $I(h) \leq \alpha$, $\forall z \in V_1$.*
(L3-3) *$I(h) \geq b_1$, $\forall z \in V_1^{\perp}$.*

Proof (L3-1). From (1.2), (1.4) we have

$$I(h) \geq \frac{1}{2}\left(1 - \frac{1}{\lambda_2}\right)\|h\|^2 + \int_{\Omega} F(x,h), \quad h \in V_1^{\perp}.$$

Using (E_0), we have $J(h) \to \infty$, as $\|h\| \to \infty$.

(L3-2). By simple calculation, we get

$$I(h) = \int_\Omega F(x,h), \quad h \in V_1.$$

By using (E_0), we have

$$I(h) = \int_\Omega F(x,h) \le \int_\Omega k(x).$$

So, we choose $\alpha = \int_\Omega k(x)$.

(L3-3). By (E_1) and the variational inequality (1.4), we have

$$\begin{aligned} I(h) &= \frac{1}{2}\|h\|^2 - \frac{1}{2}\int_\Omega \langle Ah,h\rangle + \int_\Omega F(x,h) \\ &\ge \frac{1}{2}\|h\|^2 - \frac{\lambda_2}{2}\int_\Omega \langle Ah,h\rangle + b_1 \\ &\ge b_1, \quad \forall z \in V_1^\perp, \end{aligned}$$

the proof of this lemma is completed. □

Next, we prove the Palais-Smale conditions at some levels for the functional I. We recall that $I: E \to R$ is said to satisfy the Palais-Smale conditions at the level $c \in \mathbb{R}$ ($(PS)_c$ in short) if any sequence $\{h_n\} \subseteq E$ such that

$$I(h_n) \to c, \qquad I'(h_n) \to 0,$$

as $n \to \infty$, possesses a convergent subsequence in E. Moreover, we say that I satisfies the (PS) conditions when we have $(PS)_c$ for all $c \in \mathbb{R}$.

Lemma 2.4 *Assume that the condition* (E_0) *holds. Then the functional I has the* $(PS)_c$ *conditions whenever* $c < \min\{\int_\Omega T^+, \int_\Omega T^-\}$ *or* $c > \max\{\int_\Omega S^+, \int_\Omega S^-\}$.

Proof We only prove the condition for all $c < \min\{\int_\Omega T^+, \int_\Omega T^-\}$. For the case $c > \max\{\int_\Omega S^+, \int_\Omega S^-\}$, we can use similar methods.

1. Boundedness of the (PS) sequence.

The proof is by contradiction. Suppose that there exists a $(PS)_c$ unbounded sequence $\{h_n\} \in E$ such that $c < \min\{\int_\Omega T^+, \int_\Omega T^-\}$. For the ease of notation and without loss of generality, we assume that

$\|h_n\| \to \infty$,

$I(h_n) \to c$,

$I'(h_n) \to 0$, $n \to \infty$.

We define $\overline{h}_n = \frac{h_n}{\|h_n\|}$, hence there is an $\overline{h} \in E$ with the following properties:

$\overline{h}_n \rightharpoonup \overline{h}$ in E,

$\overline{h}_n \to \overline{h}$ in $L^p(\Omega) \times L^p(\Omega)$, where $2 \le p < 2^*$ and $2^* = \frac{2N+2}{N}$,

$\overline{h}_n \to \overline{h}$ a.e. in Ω.

For any $\Phi \in E$, obviously $\frac{I'(h_n)\Phi}{\|h_n\|} \to 0$. By simple calculation, it is easy to obtain

$$I'(h_n)\Phi = \int_\Omega \langle \nabla_{\mathrm{H}} h_n, \nabla_H \Phi\rangle - \int_\Omega \langle A h_n, \Phi\rangle + \int_\Omega \bigl\langle \nabla F(x, h_n), \Phi\bigr\rangle,$$

where $h_n = \binom{h_n^{(1)}}{h_n^{(2)}}$, $\Phi = \binom{\Phi^{(1)}}{\Phi^{(2)}}$. We have

$$\frac{I'(h_n)\Phi}{\|h_n\|} = \int_\Omega \langle \nabla_{\mathrm{H}} \overline{h}_n, \nabla_{\mathrm{H}} \Phi\rangle - \int_\Omega \langle A\overline{h}_n, \Phi\rangle + \frac{\int_\Omega \langle \nabla F(x, h_n), \Phi\rangle}{\|h_n\|} \to 0 \quad \text{as } n \to \infty.$$

From the convergence of $\{\overline{h}_n\}$, we have

$$\int_\Omega \langle \nabla_{\mathrm{H}} \overline{h}, \nabla_{\mathrm{H}} \Phi\rangle = \int_\Omega \langle A\overline{h}, \Phi\rangle.$$

We see that $\lambda_1 = 1$, and by the definition of λ_1, we obtain that $\overline{h} = \pm\Phi_1$. So, we suppose initially that $\overline{h} = \Phi_1$. Because Φ_1 is positive, *i.e.*, $\Phi_1^{(1)} > 0$, $\Phi_1^{(2)} > 0$, it is obvious that $h_n^{(1)} \to \infty$, $h_n^{(2)} \to \infty$, $\forall x \in \Omega$ as $n \to \infty$.

Hence, we can take $h_n = t_n \Phi_1 + \omega_n$, where $\{t_n\} \in \mathbb{R}$, $\{\omega_n\} \in V_1^\perp$, and we have

$$\begin{aligned} I(h_n) &= \frac{1}{2}\|t_n\Phi_1 + \omega_n\|^2 - \frac{1}{2}\int_\Omega \bigl\langle A(t_n\Phi_1 + \omega_n), t_n\Phi_1 + \omega_n\bigr\rangle + \int_\Omega F(x, h_n) \\ &= \frac{1}{2}\|\omega_n\|^2 - \int_\Omega \langle A\omega_n, \omega_n\rangle + \int_\Omega F(x, h_n). \end{aligned}$$

Using (1.4), we obtain

$$I(h_n) \geq \frac{1}{2}\left(1 - \frac{1}{\lambda_2}\right)\|\omega_n\|^2 + \int_\Omega F(x, h_n). \tag{2.1}$$

Since $I(h_n) \to c$, it is easy to obtain that the sequence $\{\omega_n\}$ is bounded. On the other hand, because of $\|h_n\| \to \infty$, on a subsequence $|t_n| \to \infty$, without loss of generality, we assume $t_n \to -\infty$.

Now, using Hölder's inequality and (E_0), we have

$$\left|\int_\Omega \nabla F(x, h_n)\omega_n\right| \leq C\left(\int_\Omega \bigl|\nabla F(x, h_n)\bigr|^2\right)^{\frac{1}{2}}.$$

Thus, applying the dominated convergence theorem, we conclude that

$$\lim_{n\to\infty} \int_\Omega \nabla F(x, h_n)\omega_n = 0. \tag{2.2}$$

On the other hand,

$$\begin{aligned} I'(h_n)\omega_n &= \int_\Omega \langle \nabla_H h_n, \nabla_H \omega_n\rangle - \int_\Omega \langle A h_n, \omega_n\rangle + \int_\Omega \bigl\langle \nabla F(x, h_n), \omega_n\bigr\rangle \\ &= \|\omega_n\|^2 - \int_\Omega \langle A h_n, \omega_n\rangle + \int_\Omega \bigl\langle \nabla F(x, h_n), \omega_n\bigr\rangle. \end{aligned}$$

Using (2.2), (1.4), we obtain

$$\left(1-\frac{1}{\lambda_2}\right)\|\omega_n\|^2 \le \left|I'(h_n)\omega_n\right| + \left|\int_\Omega \bigl\langle \nabla F(x,h_n),\omega_n\bigr\rangle\right| \to 0$$

as $n\to\infty$. Therefore, by variational inequalities (1.3) and (1.4), we obtain that

$$\|\omega_n\|^2 - \int_\Omega \langle Ah_n,\omega_n\rangle \to 0 \quad \text{as } n\to\infty.$$

Consequently, by virtue of Fatou's lemma and (E_0), we have

$$\begin{aligned} c &= \lim_{n\to\infty}\left(\frac{1}{2}\|\omega_n\|^2 - \int_\Omega \langle A\omega_n,\omega_n\rangle + \int_\Omega F(x,h_n)\right) = \liminf_{n\to\infty}\int_\Omega F(x,t_n\Phi_1+\omega_n) \\ &\ge \int_\Omega \liminf_{n\to\infty} F(x,t_n\Phi_1+\omega_n) = \int_\Omega T^-, \end{aligned}$$

which contradicts the condition $c<\min\{\int_\Omega T^+,\int_\Omega T^-\}$. Hence, the $(PS)_c$ sequence is bounded.

2. Various convergence of $\{h_n\}$.

Since $\{h_n\}$ is a bounded sequence, there is an $h\in E$ with the following properties:

$h_n \rightharpoonup h$ in E,

$h_n \to h$ in $L^p(\Omega)\times L^p(\Omega)$, where $2\le p<2^*$ and $2^*=\frac{2N+2}{N}$,

$h_n\to h$ a.e. in Ω.

3. $\{h_n\}$ convergence to h in E.

From the definition of $(PS)_c$ sequence, we have, as $n\to\infty$,

$$\begin{aligned} I'(h_n)h &= \int_\Omega \langle \nabla_{\mathrm H}h_n,\nabla_H h\rangle - \int_\Omega \langle Ah_n,h\rangle + \int_\Omega \bigl\langle \nabla F(x,h_n),h\bigr\rangle \to 0, \\ I'(h_n)h_n &= \int_\Omega |\nabla_{\mathrm H}h_n|^2 - \int_\Omega \langle Ah_n,h_n\rangle + \int_\Omega \bigl\langle \nabla F(x,h_n),h_n\bigr\rangle \to 0. \end{aligned}$$

By Fatou's lemma and the above convergence of $\{h_n\}$, it is easy to show that

$$\begin{aligned} &\int_\Omega \langle Ah_n,h\rangle \to \int_\Omega \langle Ah,h\rangle, \\ &\int_\Omega \bigl\langle \nabla F(x,h_n),h\bigr\rangle \to \int_\Omega \bigl\langle \nabla F(x,h),h\bigr\rangle, \\ &\int_\Omega \langle Ah_n,h_n\rangle \to \int_\Omega \langle Ah,h\rangle, \\ &\int_\Omega \bigl\langle \nabla F(x,h_n),h_n\bigr\rangle \to \int_\Omega \bigl\langle \nabla F(x,h),h\bigr\rangle \end{aligned}$$

as $n\to\infty$. Hence, we have

$$\int_\Omega \langle \nabla_{\mathrm H}h_n,\nabla_{\mathrm H}h\rangle \to \int_\Omega \langle Ah,h\rangle - \int_\Omega \bigl\langle \nabla F(x,h),h\bigr\rangle \quad \text{as } n\to\infty, \tag{2.3}$$

$$\int_\Omega |\nabla_{\mathrm H}h_n|^2 \to \int_\Omega \langle Ah,h\rangle - \int_\Omega \bigl\langle \nabla F(x,h),h\bigr\rangle \quad \text{as } n\to\infty. \tag{2.4}$$

By weak convergence, we have

$$\int_\Omega \langle \nabla_{\mathrm{H}} h_n, \nabla_{\mathrm{H}} h\rangle \to \int_\Omega \langle \nabla_{\mathrm{H}} h, \nabla_{\mathrm{H}} h\rangle \quad \text{as } n \to \infty. \tag{2.5}$$

Using (2.3), (2.4) and (2.5), by simple calculation, we obtain

$$\int_\Omega |\nabla_{\mathrm{H}} h_n - \nabla_{\mathrm{H}} h|^2 \to 0 \quad \text{as } n \to \infty.$$

The proof is completed. □

Lemma 2.5 *Suppose that* (E_0) *and* (E_3) *are satisfied. Then the origin is a local minimum for the functional I.*

Proof Using (E_3), we can choose $p \in (2, 2^*)$ and a constant $C > 0$ such that

$$F(x,h) \ge \frac{1-\alpha}{2}\langle Ah, h\rangle - C|h|^p, \quad \forall (x,h) \in \Omega \times \mathbb{R}^2.$$

Consequently, we have

$$\begin{aligned} I(h) &= \frac{1}{2}\|h\|^2 - \frac{1}{2}\int_\Omega \langle Ah, h\rangle + \int_\Omega F(x,h) \ge \frac{1}{2}(1-\alpha)\|h\|^2 - C\int_\Omega |h|^p \\ &\ge \frac{1}{2}(1-\alpha)\|h\|^2 - C\|h\|^p \ge \frac{1}{4}(1-\alpha)\|h\|^2, \quad \|h\| < \rho, \end{aligned}$$

where ρ is small enough and $0 < \rho < t_0$, t_0 is provided by (E_5). Therefore the proof has been completed. □

To complete the mountain pass geometry, we prove the following result.

Lemma 2.6 *Let the hypotheses* (E_0), (E_4) *and* (E_5) *hold. Then there exist* $h_0 \in E$ *and* $\rho > 0$ *such that* $I(h_0) < 0$ *and* $\|h_0\| > \rho$.

Proof Using (E_2) and (E_5), we take $h_0 = t_0\Phi_1$, where t_0 is provided by (E_5). Thus, we obtain

$$\begin{aligned} I(t_0\Phi_1) &= \frac{1}{2}\|t_0\Phi_1\|^2 - \frac{1}{2}\int_\Omega \bigl\langle A(t_0\Phi_1), t_0\Phi_1\bigr\rangle + \int_\Omega F(x, t_0\Phi_1), \\ &= \int_\Omega F(x, t_0\Phi_1) < \min\left\{\int_\Omega T^+, \int_\Omega T^-\right\} < \max\left\{\int_\Omega S^+, \int_\Omega S^-\right\} \le 0, \end{aligned}$$

and $\|t_0\Phi_1\| = t_0$. If we take $0 < \rho < t_0$, then the conclusion follows. □

Lemma 2.7 *Under hypotheses* (E_0), (E_4) *and* (E_5), *problem* (1.1) *has at least one nontrivial solution* $h_0 \in E$. *Moreover,* h_0 *has negative energy, i.e.,* $J(h_0) < 0$.

Proof By (E_0) and (1.4), we obtain

$$I(h) = \frac{1}{2}\|h\|^2 - \frac{1}{2}\int_\Omega \langle Ah, h\rangle + \int_\Omega F(x,h) \ge \int_\Omega F(x,h) \ge -\int_\Omega k(x).$$

Therefore, the functional I is bounded below. In this case, we would like to mention that I has the $(PS)_c$ conditions with $c = \inf\{I(h) : h \in E\}$. For seeing this, by Lemma 2.4, we take $t_0 \in \mathbb{R}$ provided by (E_5) we can obtain

$$c \leq I(t\Phi_1) = \int_\Omega F(x, t\Phi_1) < \min\left\{\int_\Omega T^+, \int_\Omega T^-\right\} \leq 0.$$

Consequently, applying Lemma 2.2, we have one critical point $h_0 \in E$ such that $I(h_0) = \inf\{I(h) : h \in E\} \leq I(t\Phi_1) < 0$. The proof of this lemma is completed. □

To prove Theorem 1.3, we establish the following lemma.

Lemma 2.8 *Assume that the conditions* (E_0), (E_1), (E_4) *and* (E_6) *hold. Then problem* (1.1) *has at least two nontrivial solutions with negative energy.*

Proof Define

$$M^+ = \{t\Phi_1 + \omega, t \geq 0, \omega \in V_1^\perp\}, \qquad M^- = \{t\Phi_1 + \omega, t \leq 0, \omega \in V_1^\perp\}.$$

We have $\partial M^+ = \partial M^- = V_1^\perp$. Hence, we minimize the functional I restricted to M^+ and M^-.

Firstly, we consider the functionals $I^\pm = I|_{M^\pm}$. Using Lemma 2.4, $I^\pm$ possesses the $(PS)_c$ conditions whenever $c < \min\{\int_\Omega T^+, \int_\Omega T^-\}$. Therefore, we obtain that $I^\pm$ satisfies the $(PS)_c$ conditions with $c^\pm = \inf\{I^\pm(h) : h \in M^\pm\}$.

In this way, by using Lemma 2.2 for the functional $I^\pm$, we obtain two critical points which we denote by h_0^+ and h_0^-, respectively. Thus, we have $c^+ = I^+(h_0^+) = \inf_{h\in M^+}\{I(h)\}$ and $c^- = I^-(h_0^-) = \inf_{h\in M^-}\{I(h)\}$.

Moreover, we affirm that h_0^+ and h_0^- are nonzero critical points. To see this, from (E_4) and (E_6), we obtain that

$$I^\pm(h_0^\pm) \leq I^\pm(t_1^\pm\Phi_1) = \int_\Omega F(x, t_1^\pm\Phi_1) < \min\left\{\int_\Omega T^+, \int_\Omega T^-\right\} \leq 0,$$

and I restricted to $V_1^\perp$ is nonnegative. More specifically, given $\omega \in V_1^\perp$, using (L3-3) in Lemma 2.3, we have

$$I(\omega) \geq b_1 \geq 0. \tag{2.6}$$

Next, we prove that h_0^+ and h_0^- are distinct. The proof of this affirmation is by contradiction. If $h_0^+ = h_0^-$, then $h_0^+ = h_0^- \in V_1^\perp$. Using (2.6), we obtain $I(h_0^+) < 0 \leq I(h_0^+)$. Therefore, we have a contradiction. Consequently, we get $h_0^+ \neq h_0^-$. Thus problem (1.1) has at least two nontrivial solutions. Moreover, these solutions have negative energy. □

3 Proof of main theorems

In this section, we prove Theorem 1.1, Theorem 1.2 and Theorem 1.3.

Proof of Theorem 1.1 From Lemma 2.4, the functional I satisfies the $(PS)_c$ conditions for some levels $c \in \mathbb{R}$. Set $E = V_1 \oplus V_1^\perp$, where $V_1 = \operatorname{span}\{\Phi_1\}$. Using Lemma 2.3, we get that the functional I satisfies the saddle point geometry (see [12], Theorem 1.11). This implies that I has one critical point $h_1 \in E$. Theorem 1.1 is proved. □

Proof of Theorem 1.2 From Lemma 2.5 and Lemma 2.6, we know that the functional I satisfies the geometric conditions of the mountain pass theorem. Moreover, the functional I satisfies the $(PS)_c$ conditions for all $c \geq 0$. Thus, we have a solution $h_2 \in E$ given by the mountain pass theorem. Obviously, the solution h_2 satisfies $I(h_2) > 0$.

On the other hand, by Lemma 2.7, we get another solution h_0 and $I(h_0) < 0$. It follows that problem (1.1) has at least two nontrivial solutions. The proof is completed. □

Proof of Theorem 1.3 Since the conditions (E_0), (E_3), (E_4) and (E_5) imply that Lemma 2.5 and Lemma 2.6 hold. Thus, we have one solution h_2 which satisfies $I(h_2) > 0$.

On the other hand, using Lemma 2.8, we obtain two distinct critical points $h_0^{\pm}$ such that $I(h_0^{\pm}) < 0$. Therefore, we obtain that problem (1.1) has at least three nontrivial solutions. The proof is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
We declare that all authors collaborated and dedicated the same amount of time in order to perform this article.

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