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A Clifford algebra associated to generalized Fibonacci quaternions

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Abstract

In this paper, using the construction of Clifford algebras, we associate to the set of generalized Fibonacci quaternions a quaternion algebra A (*i.e.*, a Clifford algebra of dimension four). Indeed, for the generalized quaternion algebra $\mathbb{H}(\beta_1, \beta_2)$, denoting $E(\beta_1, \beta_2) = \frac{1}{5}[1 + \beta_1 + 2\beta_2 + 5\beta_1\beta_2 + \alpha(\beta_1 + 3\beta_2 + 8\beta_1\beta_2)]$, if $E(\beta_1, \beta_2) > 0$, therefore the algebra A is split. If $E(\beta_1, \beta_2) < 0$, then the algebra A is a division algebra. In this way, we provide a nice algorithm to obtain a division quaternion algebra starting from a quaternion non-division algebra and *vice versa*.

MSC: 11E88; 11B39

Keywords: Clifford algebras; generalized Fibonacci quaternions

1 Introduction

In 1878, WK Clifford discovered Clifford algebras. These algebras generalize the real numbers, complex numbers and quaternions (see [1]).

The theory of Clifford algebras is intimately connected with the theory of quadratic forms. In the following, we will consider K to be a field of characteristic not two. Let (V, q) be a K-vector space equipped with a nondegenerate quadratic form over the field K. A *Clifford algebra* for (V, q) is a K-algebra C with a linear map $i : V \to C$ satisfying the property

$$i(x)^2 = q(x) \cdot 1_C, \quad \forall x \in V,$$

such that for any K-algebra A and any K linear map $\gamma : V \to A$ with $\gamma^2(x) = q(x) \cdot 1_A$, $\forall x \in V$, there exists a unique K-algebra morphism $\gamma' : C \to A$ with $\gamma = \gamma' \circ i$.

Such an algebra can be constructed using the tensor algebra associated to a vector space V. Let $T(V) = K \oplus V \oplus (V \otimes V) \oplus \cdots$ be the tensor algebra associated to the vector space V, and let $\mathcal{J}$ be the two-sided ideal of $T(V)$ generated by all elements of the form $x \otimes x - q(x) \cdot 1$ for all $x \in V$. The associated Clifford algebra is the factor algebra $C(V, q) = T(V)/\mathcal{J}$ (see [2, 3]).

Theorem 1.1 (Poincaré-Birkhoff-Witt [2, p.44]) *If $\{e_1, e_2, \ldots, e_n\}$ is a basis of V, then the set $\{1, e_{j_1}e_{j_2}\cdots e_{j_s}, 1 \leq s \leq n, 1 \leq j_1 < j_2 < \cdots < j_s \leq n\}$ is a basis in $C(V, q)$.*

The most important Clifford algebras are those defined over real and complex vector spaces equipped with nondegenerate quadratic forms. Every nondegenerate quadratic form over a real vector space is equivalent to the following standard diagonal form:

$$q(x) = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_s^2,$$

where $n = r + s$ is the dimension of the vector space. The pair of integers (r, s) is called *the signature* of the quadratic form. The real vector space with this quadratic form is usually denoted by $\mathbb{R}_{r,s}$ and the Clifford algebra on $\mathbb{R}_{r,s}$ is denoted by $\mathrm{Cl}_{r,s}(\mathbb{R})$. For other details about Clifford algebras, the reader is referred to [4–6] and [7].

Example 1.2

(i) For $p = q = 0$, we have $\mathrm{Cl}_{0,0}(K) \simeq K$.

(ii) For $p = 0$, $q = 1$, it results that $\mathrm{Cl}_{0,1}(K)$ is a two-dimensional algebra generated by a single vector e_1 such that $e_1^2 = -1$, and therefore $\mathrm{Cl}_{0,1}(K) \simeq K(e_1)$. For $K = \mathbb{R}$, it follows that $\mathrm{Cl}_{0,1}(\mathbb{R}) \simeq \mathbb{C}$.

(iii) For $p = 0$, $q = 2$, the algebra $\mathrm{Cl}_{0,2}(K)$ is a four-dimensional algebra spanned by the set $\{1, e_1, e_2, e_1e_2\}$. Since $e_1^2 = e_2^2 = (e_1e_2)^2 = -1$ and $e_1e_2 = -e_2e_1$, we obtain that this algebra is isomorphic to the division quaternions algebra $\mathbb{H}$.

(iv) For $p = 1$, $q = 1$ or $p = 2$, $q = 0$, we obtain the algebra $\mathrm{Cl}_{1,1}(K) \simeq \mathrm{Cl}_{2,0}(K)$ which is isomorphic with a split (*i.e.*, nondivision) quaternion algebra [8].

2 Preliminaries

Let $\mathbb{H}(\beta_1, \beta_2)$ be a generalized real quaternion algebra, the algebra of the elements of the form $a = a_1 \cdot 1 + a_2e_2 + a_3e_3 + a_4e_4$, where $a_i \in \mathbb{R}$, $i \in \{1, 2, 3, 4\}$, and the elements of the basis $\{1, e_2, e_3, e_4\}$ satisfy the following multiplication table:

$\cdot$	1	e_2	e_3	e_4
1	1	e_2	e_3	e_4
e_2	e_2	$-\beta_1$	e_4	$-\beta_1 e_3$
e_3	e_3	$-e_4$	$-\beta_2$	$\beta_2 e_2$
e_4	e_4	$\beta_1 e_3$	$-\beta_2 e_2$	$-\beta_1\beta_2$

We denote by $\mathbf{n}(a)$ the norm of a real quaternion a. The norm of a generalized quaternion has the following expression $\mathbf{n}(a) = a_1^2 + \beta_1 a_2^2 + \beta_2 a_3^2 + \beta_1\beta_2 a_4^2$. For $\beta_1 = \beta_2 = 1$, we obtain the real division algebra $\mathbb{H}$, with the basis $\{1, i, j, k\}$, where $i^2 = j^2 = k^2 = -1$ and $ij = -ji$, $ik = -ki$, $jk = -kj$.

Proposition 2.1 ([3, Proposition 1.1]) *The quaternion algebra $\mathbb{H}(\beta_1, \beta_2)$ is isomorphic to quaternion algebra $\mathbb{H}(x^2\beta_1, y^2\beta_2)$, where $x, y \in K^*$.*

The quaternion algebra $\mathbb{H}(\beta_1, \beta_2)$ with $\beta_1, \beta_2 \in K^*$ is either a division algebra or is isomorphic to $\mathbb{H}(-1, -1) \simeq \mathcal{M}_2(K)$ [3].

For other details about the quaternions, the reader is referred, for example, to [3, 9, 10].

The Fibonacci numbers were introduced by *Leonardo of Pisa* (1170-1240) in his book *Liber abbaci*, book published in 1202 AD (see [11, pp.1, 3]). This name is attached to the following sequence of numbers:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots,$$

with the nth term given by the formula

$$f_n = f_{n-1} + f_{n-2}, \quad n \geq 2,$$

where $f_0 = 0, f_1 = 1$.

In [12], the author generalized Fibonacci numbers and gave many properties of them:

$$h_n = h_{n-1} + h_{n-2}, \quad n \geq 2,$$

where $h_0 = p, h_1 = q$, with p, q being arbitrary integers. In the same paper [12, relation (7)], the following relation between Fibonacci numbers and generalized Fibonacci numbers was obtained:

$$h_{n+1} = pf_n + qf_{n+1}. \tag{2.1}$$

For the generalized real quaternion algebra, the Fibonacci quaternions and generalized Fibonacci quaternions are defined in the same way:

$$F_n = f_n \cdot 1 + f_{n+1}e_2 + f_{n+2}e_3 + f_{n+3}e_4,$$

for the nth Fibonacci quaternions and

$$H_n = h_n \cdot 1 + h_{n+1}e_2 + h_{n+2}e_3 + h_{n+3}e_4 = pF_n + qF_{n+1}, \tag{2.2}$$

for the nth generalized Fibonacci quaternions.

In the following, we will denote the nth generalized Fibonacci number and the nth generalized Fibonacci quaternion element by $h_n^{p,q}$, respectively $H_n^{p,q}$. In this way, we emphasize the starting integers p and q.

It is known that the expression for the nth term of a Fibonacci element is

$$f_n = \frac{1}{\sqrt{5}}\left[\alpha^n - \beta^n\right] = \frac{\alpha^n}{\sqrt{5}}\left[1 - \frac{\beta^n}{\alpha^n}\right], \tag{2.3}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

From the above, we obtain the following limit:

$$\begin{aligned} \lim_{n\to\infty} \mathbf{n}(F_n) &= \lim_{n\to\infty}\left(f_n^2 + \beta_1 f_{n+1}^2 + \beta_2 f_{n+2}^2 + \beta_1\beta_2 f_{n+3}^2\right) \\ &= \lim_{n\to\infty}\left(\frac{\alpha^{2n}}{5} + \beta_1\frac{\alpha^{2n+2}}{5} + \beta_2\frac{\alpha^{2n+4}}{5} + \beta_1\beta_2\frac{\alpha^{2n+6}}{5}\right) \\ &= \operatorname{sgn} E(\beta_1, \beta_2) \cdot \infty, \end{aligned}$$

where $E(\beta_1, \beta_2) = \frac{1}{5}[1 + \beta_1 + 2\beta_2 + 5\beta_1\beta_2 + \alpha(\beta_1 + 3\beta_2 + 8\beta_1\beta_2)]$, since $\alpha^2 = \alpha + 1$ (see [13]).

If $E(\beta_1, \beta_2) > 0$, there exists a number $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$, we have $\mathbf{n}(F_n) > 0$. In the same way, if $E(\beta_1, \beta_2) < 0$, there exists a number $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$, we have $\mathbf{n}(F_n) < 0$. Therefore, for all $\beta_1, \beta_2 \in \mathbb{R}$ with $E(\beta_1, \beta_2) \neq 0$, in the algebra $\mathbb{H}(\beta_1, \beta_2)$ there is a natural number $n_0 = \max\{n_1, n_2\}$ such that $\mathbf{n}(F_n) \neq 0$. Hence F_n is an invertible element for all $n \geq n_0$. Using the same arguments, we can compute the following limit:

$$\lim_{n\to\infty} \left(\mathbf{n}\left(H_n^{p,q}\right)\right) = \lim_{n\to\infty} \left(h_n^2 + \beta_1 h_{n+1}^2 + \beta_2 h_{n+2}^2 + \beta_1\beta_2 h_{n+3}^2\right) = \operatorname{sgn} E'(\beta_1, \beta_2) \cdot \infty,$$

where $E'(\beta_1, \beta_2) = \frac{1}{5}(p + \alpha q)^2 E(\beta_1, \beta_2)$, if $E'(\beta_1, \beta_2) \neq 0$ (see [13]).

Therefore, for all $\beta_1, \beta_2 \in \mathbb{R}$ with $E'(\beta_1, \beta_2) \neq 0$, in the algebra $\mathbb{H}(\beta_1, \beta_2)$ there exists a natural number n_0' such that $\mathbf{n}(H_n^{p,q}) \neq 0$, hence $H_n^{p,q}$ is an invertible element for all $n \geq n_0'$.

Theorem 2.2 ([13, Theorem 2.6]) *For all $\beta_1, \beta_2 \in \mathbb{R}$ with $E'(\beta_1, \beta_2) \neq 0$, there exists a natural number n' such that for all $n \geq n'$, Fibonacci elements F_n and generalized Fibonacci elements $H_n^{p,q}$ are invertible elements in the algebra $\mathbb{H}(\beta_1, \beta_2)$.*

Theorem 2.3 ([13, Theorem 2.1]) *The set $\mathcal{H}_n = \{H_n^{p,q} / p, q \in \mathbb{Z}, n \geq m, m \in \mathbb{N}\} \cup \{0\}$ is a $\mathbb{Z}$-module.*

3 Main results

Remark 3.1 We remark that the $\mathbb{Z}$-module from Theorem 2.3 is a free $\mathbb{Z}$-module of rank two. Indeed, $\varphi : \mathbb{Z} \times \mathbb{Z} \to \mathcal{H}_n$, $\varphi((p, q)) = H_n^{p,q}$ is a $\mathbb{Z}$-module isomorphism and $\{\varphi(1, 0) = F_n, \varphi(0, 1) = F_{n+1}\}$ is a basis in $\mathcal{H}_n$.

Remark 3.2 By extension of scalars, we obtain that $\mathbb{R} \otimes_{\mathbb{Z}} \mathcal{H}_n$ is an $\mathbb{R}$-vector space of dimension two. A basis is $\{\overline{e}_1 = 1 \otimes F_n, \overline{e}_2 = 1 \otimes F_{n+1}\}$. We have that $\mathbb{R} \otimes_{\mathbb{Z}} \mathcal{H}_n$ is an isomorphic with the $\mathbb{R}$-vector space $\mathcal{H}_n^{\mathbb{R}} = \{H_n^{p,q} / p, q \in \mathbb{R}\} \cup \{0\}$. A basis in $\mathcal{H}_n^{\mathbb{R}}$ is $\{F_n, F_{n+1}\}$.

Let $T(\mathcal{H}_n^{\mathbb{R}})$ be the tensor algebra associated to the $\mathbb{R}$-vector space $\mathcal{H}_n^{\mathbb{R}}$, and let $C(\mathcal{H}_n^{\mathbb{R}})$ be the Clifford algebra associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$. From Theorem 1.1, it results that this algebra has dimension four.

Case 1: $\mathbb{H}(\beta_1, \beta_2)$ is a division algebra

Remark 3.3 Since in this case $E(\beta_1, \beta_2) > 0$ for all $n \geq n'$ (as in Theorem 2.2), then $\mathcal{H}_n^{\mathbb{R}}$ is an Euclidean vector space. Indeed, let $z, w \in \mathcal{H}_n^{\mathbb{R}}$, $z = x_1 F_n + x_2 F_{n+1}$, $w = y_1 F_n + y_2 F_{n+1}$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$. The inner product is defined as follows:

$$\langle z, w\rangle = x_1 y_1 \mathbf{n}(F_n) + x_2 y_2 \mathbf{n}(F_{n+1}).$$

We remark that all properties of the inner product are fulfilled. Indeed, since for all $n \geq n'$ we have $\mathbf{n}(F_n) > 0$ and $\mathbf{n}(F_{n+1}) > 0$, it results that $\langle z, z\rangle = x_1^2 \mathbf{n}(F_n) + x_2^2 \mathbf{n}(F_{n+1}) = 0$ if and only if $x_1 = x_2 = 0$, therefore $z = 0$.

On $\mathcal{H}_n^{\mathbb{R}}$ with the basis $\{F_n, F_{n+1}\}$, we define the following quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}} : \mathcal{H}_n^{\mathbb{R}} \to \mathbb{R}$:

$$q_{\mathcal{H}_n^{\mathbb{R}}}(x_1 F_n + x_2 F_{n+1}) = \mathbf{n}(F_n) x_1^2 + \mathbf{n}(F_{n+1}) x_2^2.$$

Let $Q_{\mathcal{H}_n^{\mathbb{R}}}$ be a bilinear form associated to the quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$,

$$\begin{aligned} Q_{\mathcal{H}_n^{\mathbb{R}}}(x, y) &= \frac{1}{2}\left(q_{\mathcal{H}_n^{\mathbb{R}}}(x + y) - q_{\mathcal{H}_n^{\mathbb{R}}}(x) - q_{\mathcal{H}_n^{\mathbb{R}}}(y)\right) \\ &= \mathbf{n}(F_n) x_1 y_1 + \mathbf{n}(F_{n+1}) x_2 y_2. \end{aligned}$$

The matrix associated to the quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is

$$A = \begin{pmatrix} \mathbf{n}(F_n) & 0 \\ 0 & \mathbf{n}(F_{n+1}) \end{pmatrix}.$$

We remark that $\det A = \mathbf{n}(F_n)\mathbf{n}(F_{n+1}) > 0$ for all $n \geq n'$. Since $E(\beta_1, \beta_2) > 0$, therefore $\mathbf{n}(F_n) > 0$ for $n > n'$. We obtain that the quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is positive definite and the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic to $\mathrm{Cl}_{2,0}(K)$ which is isomorphic to a split quaternion algebra.

From the above results and using Proposition 2.1, we obtain the following theorem.

Theorem 3.4 *If* $\mathbb{H}(\beta_1, \beta_2)$ *is a division algebra, there is a natural number* n' *such that for all* $n \geq n'$, *the Clifford algebra associated to the real vector space* $\mathcal{H}_n^{\mathbb{R}}$ *is isomorphic with the split quaternion algebra* $\mathbb{H}(-1, -1)$.

Case 2: $\mathbb{H}(\beta_1, \beta_2)$ is not a division algebra

Remark 3.5 (i) If $E(\beta_1, \beta_2) > 0$, then $\mathcal{H}_n^{\mathbb{R}}$ is an Euclidean vector space, for all $n \geq n'$, as in Theorem 2.2. Indeed, let $z, w \in \mathcal{H}_n^{\mathbb{R}}$, $z = x_1F_n + x_2F_{n+1}$, $w = y_1F_n + y_2F_{n+1}$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$. The inner product is defined as follows:

$$\langle z, w\rangle = x_1y_1\mathbf{n}(F_n) + x_2y_2\mathbf{n}(F_{n+1}).$$

(ii) If $E(\beta_1, \beta_2) < 0$, then $\mathcal{H}_n^{\mathbb{R}}$ is also an Euclidean vector space, for all $n \geq n'$, as in Theorem 2.2. Indeed, let $z, w \in \mathcal{H}_n^{\mathbb{R}}$, $z = x_1F_n + x_2F_{n+1}$, $w = y_1F_n + y_2F_{n+1}$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$. The inner product is defined as follows:

$$\langle z, w\rangle = -x_1y_1\mathbf{n}(F_n) - x_2y_2\mathbf{n}(F_{n+1}).$$

We have $\langle z, z\rangle = -x_1^2\mathbf{n}(F_n) - x_2^2\mathbf{n}(F_{n+1})$, and since for all $n \geq n'$ we have $\mathbf{n}(F_n) < 0$ and $\mathbf{n}(F_{n+1}) < 0$, it results that $\langle z, z\rangle = -x_1^2\mathbf{n}(F_n) - x_2^2\mathbf{n}(F_{n+1}) = 0$ if and only if $x_1 = x_2 = 0$, therefore $z = 0$.

On $\mathcal{H}_n^{\mathbb{R}}$ with the basis $\{F_n, F_{n+1}\}$,we define the following quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}} : \mathcal{H}_n^{\mathbb{R}} \to \mathbb{R}$:

$$q_{\mathcal{H}_n^{\mathbb{R}}}(x_1F_n+x_2F_{n+1}) = q_{\mathcal{H}_n^{\mathbb{R}}}(x_1F_n + x_2F_{n+1}) = \mathbf{n}(F_n)x_1^2 + \mathbf{n}(F_{n+1})x_2^2.$$

Let $Q_{\mathcal{H}_n^{\mathbb{R}}}$ be a bilinear form associated to the quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$,

$$\begin{aligned} Q_{\mathcal{H}_n^{\mathbb{R}}}(x, y) &= \frac{1}{2}\big(q_{\mathcal{H}_n^{\mathbb{R}}}(x + y) - q_{\mathcal{H}_n^{\mathbb{R}}}(x) - q_{\mathcal{H}_n^{\mathbb{R}}}(y)\big) \\ &= \mathbf{n}(F_n)x_1y_1 + \mathbf{n}(F_{n+1})x_2y_2. \end{aligned}$$

The matrix associated to the quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is

$$A = \begin{pmatrix} \mathbf{n}(F_n) & 0 \\ 0 & \mathbf{n}(F_{n+1}) \end{pmatrix}.$$

We remark that $\det A = \mathbf{n}(F_n)\mathbf{n}(F_{n+1}) > 0$ for all $n \geq n'$.

If $E(\beta_1, \beta_2) > 0$, therefore $\mathbf{n}(F_n) > 0$ for $n > n'$. We obtain that the quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is positive definite and the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic with $\mathrm{Cl}_{2,0}(K)$ which is isomorphic to a split quaternion algebra.

If $E(\beta_1, \beta_2) < 0$, therefore $\mathbf{n}(F_n) < 0$ for $n > n'$. Then the quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is negative definite and the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic with $\mathrm{Cl}_{0,2}(K)$ which is isomorphic to the quaternion division algebra $\mathbb{H}$.

From the above results and using Proposition 2.1, we obtain the following theorem.

Theorem 3.6 *If $\mathbb{H}(\beta_1, \beta_2)$ is not a division algebra, there is a natural number n' such that for all $n \geq n'$, if $E(\beta_1, \beta_2) > 0$, then the Clifford algebra associated to the real vector space $\mathcal{H}_n^{\mathbb{R}}$ is isomorphic with the split quaternion algebra $\mathbb{H}(-1,-1)$. If $E(\beta_1, \beta_2) < 0$, then the Clifford algebra associated to the real vector space $\mathcal{H}_n^{\mathbb{R}}$ is isomorphic to the division quaternion algebra $\mathbb{H}(1,1)$.*

Example 3.7 (1) For $\beta_1 = 1$, $\beta_2 = -1$, we obtain the split quaternion algebra $\mathbb{H}(1,-1)$. In this case, we have $E(\beta_1, \beta_2) = \frac{1}{5}[-5 - 10\alpha] < 0$ and, for $n' = 0$, we obtain $\mathbf{n}(F_n) = f_n^2 + f_{n+1}^2 - f_{n+2}^2 - f_{n+3}^2 < 0$, $\mathbf{n}(F_{n+1}) = f_{n+1}^2 + f_{n+2}^2 - f_{n+3}^2 - f_{n+4}^2 < 0$ for all $n \geq 0$. The quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is negative definite, therefore the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic to $\mathrm{Cl}_{0,2}(K)$ which is isomorphic to the quaternion division algebra $\mathbb{H}(1,1)$.

(2) For $\beta_1 = -2$, $\beta_2 = -3$, we obtain the split quaternion algebra $\mathbb{H}(-2,-3)$. In this case, we have $E(\beta_1, \beta_2) = \frac{1}{5}[23 + 43\alpha] > 0$. For $n' = 0$, we obtain $\mathbf{n}(F_n) = f_n^2 - f_{n+1}^2 - f_{n+2}^2 + f_{n+3}^2 > 0$, $\mathbf{n}(F_{n+1}) = f_{n+1}^2 - f_{n+2}^2 - f_{n+3}^2 + f_{n+4}^2 > 0$ for all $n \geq 0$. The quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is positive definite, therefore the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic to $\mathrm{Cl}_{2,0}(K)$ which is isomorphic to the split quaternion algebra $\mathbb{H}(-1,-1)$.

(3) For $\beta_1 = 2$, $\beta_2 = -3$, we obtain the split quaternion algebra $\mathbb{H}(2,-3)$. In this case, we have $E(\beta_1, \beta_2) = \frac{1}{5}[-33 - 44\alpha] < 0$. For $n' = 0$, we obtain $\mathbf{n}(F_n) = f_n^2 + 2f_{n+1}^2 - 3f_{n+2}^2 - 6f_{n+3}^2 < 0$, $\mathbf{n}(F_{n+1}) = f_{n+1}^2 + 2f_{n+2}^2 - 3f_{n+3}^2 - 6f_{n+4}^2 > 0$ for all $n \geq 0$. The quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is negative definite, therefore the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic to $\mathrm{Cl}_{0,2}(K)$ which is isomorphic to the division quaternion algebra $\mathbb{H}(1,-1)$.

(4) For $\beta_1 = \beta_2 = -\frac{1}{2}$, we obtain the split quaternion algebra $\mathbb{H}(-\frac{1}{2}, -\frac{1}{2})$. Therefore $E(\beta_1, \beta_2) = \frac{3}{20} > 0$, and for $n' = 1$ we obtain $\mathbf{n}(F_n) > 0$ and $\mathbf{n}(F_{n+1}) > 0$. The quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is positive definite, therefore the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic to $\mathrm{Cl}_{2,0}(K)$ which is isomorphic to the split quaternion algebra $\mathbb{H}(-1,-1)$.

The algorithm

(1) Let $\mathbb{H}(\beta_1, \beta_2)$ be a quaternion algebra, $\alpha = \frac{1+\sqrt{5}}{2}$ and $E(\beta_1, \beta_2) = \frac{1}{5}[1 + \beta_1 + 2\beta_2 + 5\beta_1\beta_2 + \alpha(\beta_1 + 3\beta_2 + 8\beta_1\beta_2)]$.

(2) Let V be the $\mathbb{R}$-vector space $\mathcal{H}_n^{\mathbb{R}} = \{H_n^{p,q}/p, q \in \mathbb{R}\} \cup \{0\}$.

(3) If $E(\beta_1, \beta_2) > 0$, then the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic to $\mathrm{Cl}_{2,0}(K)$ which is isomorphic to the split quaternion algebra $\mathbb{H}(-1,-1)$.

(4) If $E(\beta_1,\beta_2)<0$, then the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic to $\mathrm{Cl}_{0,2}(K)$ which is isomorphic to the division quaternion algebra $\mathbb{H}(1,1)$.

4 Conclusions

In this paper, we have extended the $\mathbb{Z}$-module of the generalized Fibonacci quaternions to a real vector space $\mathcal{H}_n^{\mathbb{R}}$. We have proved that the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic to a split quaternion algebra or to a division algebra if $E(\beta_1,\beta_2)=\frac{1}{5}[1+\beta_1+2\beta_2+5\beta_1\beta_2+\alpha(\beta_1+3\beta_2+8\beta_1\beta_2)]$ is positive or negative. We also have given an algorithm which allows us to find a division quaternion algebra starting from a split quaternion algebra and *vice versa*.

Competing interests
The author declares that she has no competing interests.

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