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# Some differential subordinations using Ruscheweyh derivative and Sălăgean operator

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## Abstract

In the present paper we study the operator defined by using the Ruscheweyh derivative  $R^n f(z)$  and the Sălăgean operator  $S^n f(z)$ , denoted by  $L_\alpha^n : \mathcal{A} \rightarrow \mathcal{A}$ ,  $L_\alpha^n f(z) = (1 - \alpha)R^n f(z) + \alpha S^n f(z)$ ,  $z \in U$ , where  $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \dots, z \in U\}$  is the class of normalized analytic functions with  $\mathcal{A}_1 = \mathcal{A}$ . We obtain several differential subordinations regarding the operator  $L_\alpha^n$ .

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**Keywords:** differential subordination; convex function; best dominant; differential operator; Sălăgean operator; Ruscheweyh derivative

## 1 Introduction

Denote by  $U$  the unit disc of the complex plane,  $U = \{z \in \mathbb{C} : |z| < 1\}$ , and by  $\mathcal{H}(U)$  the space of holomorphic functions in  $U$ . Let  $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \dots, z \in U\}$  with  $\mathcal{A}_1 = \mathcal{A}$  and  $\mathcal{H}[a, n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, z \in U\}$  for  $a \in \mathbb{C}$  and  $n \in \mathbb{N}$ . Denote by  $K = \{f \in \mathcal{A} : \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, z \in U\}$  the class of normalized convex functions in  $U$ .

If  $f$  and  $g$  are analytic functions in  $U$ , we say that  $f$  is subordinate to  $g$ , written  $f \prec g$ , if there is a function  $w$  analytic in  $U$ , with  $w(0) = 0$ ,  $|w(z)| < 1$ , for all  $z \in U$ , such that  $f(z) = g(w(z))$  for all  $z \in U$ . If  $g$  is univalent, then  $f \prec g$  if and only if  $f(0) = g(0)$  and  $f(U) \subseteq g(U)$ .

Let  $\psi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$  and let  $h$  be a univalent function in  $U$ . If  $p$  is analytic in  $U$  and satisfies the (second-order) differential subordination

$$\psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z), \quad z \in U, \quad (1.1)$$

then  $p$  is called a solution of the differential subordination. The univalent function  $q$  is called a dominant of the solutions of the differential subordination, or more simply a dominant, if  $p \prec q$  for all  $p$  satisfying (1.1).

A dominant  $\tilde{q}$  that satisfies  $\tilde{q} \prec q$  for all dominants  $q$  of (1.1) is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of  $U$ .

**Definition 1.1** (Sălăgean [1]) For  $f \in \mathcal{A}$ ,  $n \in \mathbb{N}$ , the operator  $S^n$  is defined by  $S^n : \mathcal{A} \rightarrow \mathcal{A}$ ,

$$S^0 f(z) = f(z),$$

$$S^1 f(z) = zf'(z),$$

...

$$S^{n+1}f(z) = z(S^n f(z))', \quad z \in U.$$

**Remark 1.1** If  $f \in \mathcal{A}$ ,  $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ , then  $S^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j$ ,  $z \in U$ .

**Definition 1.2** (Ruscheweyh [2]) For  $f \in \mathcal{A}$ ,  $n \in \mathbb{N}$ , the operator  $R^n$  is defined by  $R^n : \mathcal{A} \rightarrow \mathcal{A}$ ,

$$R^0 f(z) = f(z),$$

$$R^1 f(z) = z f'(z),$$

...

$$(n+1)R^{n+1}f(z) = z(R^n f(z))' + nR^n f(z), \quad z \in U.$$

**Remark 1.2** If  $f \in \mathcal{A}$ ,  $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ , then  $R^n f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^n a_j z^j$ ,  $z \in U$ .

**Definition 1.3** ([3]) Let  $\alpha \geq 0$ ,  $n \in \mathbb{N}$ . Denote by  $L_\alpha^n$  the operator given by  $L_\alpha^n : \mathcal{A} \rightarrow \mathcal{A}$ ,

$$L_\alpha^n f(z) = (1 - \alpha)R^n f(z) + \alpha S^n f(z), \quad z \in U.$$

**Remark 1.3** If  $f \in \mathcal{A}$ ,  $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ , then  $L_\alpha^n f(z) = z + \sum_{j=2}^{\infty} (\alpha j^n + (1 - \alpha)C_{n+j-1}^n) a_j z^j$ ,  $z \in U$ .

This operator was studied also in [3–5].

**Lemma 1.1** (Hallenbeck and Ruscheweyh [6, Th. 3.1.6, p.71]) *Let  $h$  be a convex function with  $h(0) = a$ , and let  $\gamma \in \mathbb{C} \setminus \{0\}$  be a complex number with  $\operatorname{Re} \gamma \geq 0$ . If  $p \in \mathcal{H}[a, n]$  and*

$$p(z) + \frac{1}{\gamma} z p'(z) \prec h(z), \quad z \in U,$$

then

$$p(z) \prec g(z) \prec h(z), \quad z \in U,$$

where  $g(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) t^{\gamma/n-1} dt$ ,  $z \in U$ .

**Lemma 1.2** (Miller and Mocanu [6]) *Let  $g$  be a convex function in  $U$  and let  $h(z) = g(z) + n\alpha z g'(z)$ , for  $z \in U$ , where  $\alpha > 0$  and  $n$  is a positive integer.*

*If  $p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \dots$ ,  $z \in U$ , is holomorphic in  $U$  and*

$$p(z) + \alpha z p'(z) \prec h(z), \quad z \in U,$$

then

$$p(z) \prec g(z), \quad z \in U,$$

and this result is sharp.

## 2 Main results

**Theorem 2.1** *Let  $g$  be a convex function,  $g(0) = 1$  and let  $h$  be the function  $h(z) = g(z) + \frac{z}{\delta} g'(z)$ ,  $z \in U$ .*

If  $\alpha, \delta \geq 0, n \in \mathbb{N}, f \in \mathcal{A}$  and satisfies the differential subordination

$$\left(\frac{L_\alpha^n f(z)}{z}\right)^{\delta-1} (L_\alpha^n f(z))' < h(z), \quad z \in U, \tag{2.1}$$

then

$$\left(\frac{L_\alpha^n f(z)}{z}\right)^\delta < g(z), \quad z \in U,$$

and this result is sharp.

*Proof* By using the properties of the operator  $L_\alpha^n$ , we have

$$L_\alpha^n f(z) = z + \sum_{j=2}^{\infty} (\alpha j^n + (1-\alpha)C_{n+j-1}^n) a_j z^j, \quad z \in U.$$

Consider  $p(z) = \left(\frac{L_\alpha^n f(z)}{z}\right)^\delta = \left(\frac{z + \sum_{j=2}^{\infty} (\alpha j^n + (1-\alpha)C_{n+j-1}^n) a_j z^j}{z}\right)^\delta = 1 + p_\delta z^\delta + p_{\delta+1} z^{\delta+1} + \dots, z \in U.$

We deduce that  $p \in \mathcal{H}[1, \delta].$

Differentiating, we obtain  $\left(\frac{L_\alpha^n f(z)}{z}\right)^{\delta-1} (L_\alpha^n f(z))' = p(z) + \frac{1}{\delta} z p'(z), z \in U.$

Then (2.1) becomes

$$p(z) + \frac{1}{\delta} z p'(z) < h(z) = g(z) + \frac{z}{\delta} g'(z), \quad z \in U.$$

By using Lemma 1.2, we have

$$p(z) < g(z), \quad z \in U, \quad \text{i.e.,} \quad \left(\frac{L_\alpha^n f(z)}{z}\right)^\delta < g(z), \quad z \in U. \quad \square$$

**Theorem 2.2** Let  $h$  be a holomorphic function which satisfies the inequality  $\text{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}, z \in U,$  and  $h(0) = 1.$

If  $\alpha, \delta \geq 0, n \in \mathbb{N}, f \in \mathcal{A}$  and satisfies the differential subordination

$$\left(\frac{L_\alpha^n f(z)}{z}\right)^{\delta-1} (L_\alpha^n f(z))' < h(z), \quad z \in U, \tag{2.2}$$

then

$$\left(\frac{L_\alpha^n f(z)}{z}\right)^\delta < q(z), \quad z \in U,$$

where  $q(z) = \frac{\delta}{z^\delta} \int_0^z h(t) t^{\delta-1} dt.$  The function  $q$  is convex and it is the best dominant.

*Proof* Let

$$\begin{aligned} p(z) &= \left(\frac{L_\alpha^n f(z)}{z}\right)^\delta = \left(\frac{z + \sum_{j=2}^{\infty} (\alpha j^n + (1-\alpha)C_{n+j-1}^n) a_j z^j}{z}\right)^\delta \\ &= \left(1 + \sum_{j=2}^{\infty} (\alpha j^n + (1-\alpha)C_{n+j-1}^n) a_j z^{j-1}\right)^\delta = 1 + \sum_{j=\delta+1}^{\infty} p_j z^{j-1} \end{aligned}$$

for  $z \in U, p \in \mathcal{H}[1, \delta].$

Differentiating, we obtain  $(\frac{L_\alpha^n f(z)}{z})^{\delta-1} (L_\alpha^n f(z))' = p(z) + \frac{1}{\delta} z p'(z)$ ,  $z \in U$ , and (2.2) becomes

$$p(z) + \frac{1}{\delta} z p'(z) < h(z), \quad z \in U.$$

Using Lemma 1.1, we have

$$p(z) < q(z), \quad z \in U, \quad \text{i.e.,} \quad \left(\frac{L_\alpha^n f(z)}{z}\right)^\delta < q(z) = \frac{\delta}{z^\delta} \int_0^z h(t) t^{\delta-1} dt, \quad z \in U,$$

and  $q$  is the best dominant. □

**Corollary 2.3** Let  $h(z) = \frac{1+(2\beta-1)z}{1+z}$  be a convex function in  $U$ , where  $0 \leq \beta < 1$ .

If  $\alpha, \delta \geq 0$ ,  $n \in \mathbb{N}$ ,  $f \in \mathcal{A}$  and satisfies the differential subordination

$$\left(\frac{L_\alpha^n f(z)}{z}\right)^{\delta-1} (L_\alpha^n f(z))' < h(z), \quad z \in U, \tag{2.3}$$

then

$$\left(\frac{L_\alpha^n f(z)}{z}\right)^\delta < q(z), \quad z \in U,$$

where  $q$  is given by  $q(z) = (2\beta - 1) + \frac{2(1-\beta)\delta}{z^\delta} \int_0^z \frac{t^{\delta-1}}{1+t} dt$ ,  $z \in U$ . The function  $q$  is convex and it is the best dominant.

*Proof* Following the same steps as in the proof of Theorem 2.2 and considering  $p(z) = (\frac{L_\alpha^n f(z)}{z})^\delta$ , the differential subordination (2.3) becomes

$$p(z) + \frac{z}{\delta} p'(z) < h(z) = \frac{1 + (2\beta - 1)z}{1 + z}, \quad z \in U.$$

By using Lemma 1.1 for  $\gamma = \delta$ , we have  $p(z) < q(z)$ , i.e.,

$$\begin{aligned} \left(\frac{L_\alpha^n f(z)}{z}\right)^\delta < q(z) &= \frac{\delta}{z^\delta} \int_0^z h(t) t^{\delta-1} dt \\ &= \frac{\delta}{z^\delta} \int_0^z t^{\delta-1} \frac{1 + (2\beta - 1)t}{1 + t} dt = \frac{\delta}{z^\delta} \int_0^z \left[ (2\beta - 1)t^{\delta-1} + 2(1 - \beta) \frac{t^{\delta-1}}{1 + t} \right] dt \\ &= (2\beta - 1) + \frac{2(1 - \beta)\delta}{z^\delta} \int_0^z \frac{t^{\delta-1}}{1 + t} dt, \quad z \in U. \end{aligned} \quad \square$$

**Remark 2.1** For  $n = 1$ ,  $\alpha = 2$ ,  $\delta = 1$ , we obtain the same example as in [7, Example 2.2.1, p.26].

**Theorem 2.4** Let  $g$  be a convex function such that  $g(0) = 1$  and let  $h$  be the function  $h(z) = g(z) + \frac{z}{\gamma} g'(z)$ ,  $z \in U$ , where  $\gamma > 0$ .

If  $\alpha \geq 0$ ,  $n \in \mathbb{N}$ ,  $f \in \mathcal{A}$  and the differential subordination

$$\frac{(\gamma + 1)z}{\gamma} \frac{L_\alpha^n f(z)}{(L_\alpha^{n+1} f(z))^2} + \frac{z^2}{\gamma} \frac{L_\alpha^n f(z)}{(L_\alpha^{n+1} f(z))^2} \left[ \frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} - 2 \frac{(L_\alpha^{n+1} f(z))'}{L_\alpha^{n+1} f(z)} \right] < h(z), \quad z \in U, \tag{2.4}$$

holds, then

$$z \frac{L_{\alpha}^n f(z)}{(L_{\alpha}^{n+1} f(z))^2} < g(z), \quad z \in U,$$

and this result is sharp.

*Proof* For  $f \in \mathcal{A}$ ,  $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ , we have  $L_{\alpha}^n f(z) = z + \sum_{j=2}^{\infty} (\alpha)^j + (1 - \alpha) C_{n+j-1}^n a_j z^j$ ,  $z \in U$ .

Consider  $p(z) = z \frac{L_{\alpha}^n f(z)}{(L_{\alpha}^{n+1} f(z))^2}$  and we obtain  $p(z) + \frac{z}{\gamma} p'(z) = \frac{(\gamma+1)z}{\gamma} \frac{L_{\alpha}^n f(z)}{(L_{\alpha}^{n+1} f(z))^2} + \frac{z^2}{\gamma} \frac{L_{\alpha}^n f(z)}{(L_{\alpha}^{n+1} f(z))^2} \times \left[ \frac{(L_{\alpha}^n f(z))'}{L_{\alpha}^n f(z)} - 2 \frac{(L_{\alpha}^{n+1} f(z))'}{L_{\alpha}^{n+1} f(z)} \right]$ .

Relation (2.4) becomes

$$p(z) + \frac{z}{\gamma} p'(z) < h(z) = g(z) + \gamma g'(z), \quad z \in U.$$

By using Lemma 1.2, we have

$$p(z) < g(z), \quad z \in U, \quad \text{i.e.,} \quad z \frac{L_{\alpha}^n f(z)}{(L_{\alpha}^{n+1} f(z))^2} < g(z), \quad z \in U. \quad \square$$

**Theorem 2.5** Let  $h$  be a holomorphic function which satisfies the inequality  $\operatorname{Re}(1 + \frac{zh'(z)}{h(z)}) > -\frac{1}{2}$ ,  $z \in U$ , and  $h(0) = 1$ .

If  $\alpha \geq 0$ ,  $\gamma \in \mathbb{C} \setminus \{0\}$  is a complex number with  $\operatorname{Re} \gamma \geq 0$ ,  $n \in \mathbb{N}$ ,  $f \in \mathcal{A}$  and satisfies the differential subordination

$$\frac{(\gamma+1)z}{\gamma} \frac{L_{\alpha}^n f(z)}{(L_{\alpha}^{n+1} f(z))^2} + \frac{z^2}{\gamma} \frac{L_{\alpha}^n f(z)}{(L_{\alpha}^{n+1} f(z))^2} \left[ \frac{(L_{\alpha}^n f(z))'}{L_{\alpha}^n f(z)} - 2 \frac{(L_{\alpha}^{n+1} f(z))'}{L_{\alpha}^{n+1} f(z)} \right] < h(z), \quad z \in U, \quad (2.5)$$

then

$$z \frac{L_{\alpha}^n f(z)}{(L_{\alpha}^{n+1} f(z))^2} < q(z), \quad z \in U,$$

where  $q(z) = \frac{\gamma}{z^{\gamma}} \int_0^z h(t) t^{\gamma-1} dt$ . The function  $q$  is convex and it is the best dominant.

*Proof* Let  $p(z) = z \frac{L_{\alpha}^n f(z)}{(L_{\alpha}^{n+1} f(z))^2}$ ,  $z \in U$ ,  $p \in \mathcal{H}[1, 1]$ . Differentiating, we obtain  $p(z) + \frac{z}{\gamma} p'(z) = \frac{(\gamma+1)z}{\gamma} \frac{L_{\alpha}^n f(z)}{(L_{\alpha}^{n+1} f(z))^2} + \frac{z^2}{\gamma} \frac{L_{\alpha}^n f(z)}{(L_{\alpha}^{n+1} f(z))^2} \left[ \frac{(L_{\alpha}^n f(z))'}{L_{\alpha}^n f(z)} - 2 \frac{(L_{\alpha}^{n+1} f(z))'}{L_{\alpha}^{n+1} f(z)} \right]$ ,  $z \in U$ , and (2.5) becomes

$$p(z) + \frac{z}{\gamma} p'(z) < h(z), \quad z \in U.$$

Using Lemma 1.1, we have

$$p(z) < q(z), \quad z \in U, \quad \text{i.e.,} \quad z \frac{L_{\alpha}^n f(z)}{(L_{\alpha}^{n+1} f(z))^2} < q(z) = \frac{\gamma}{z^{\gamma}} \int_0^z h(t) t^{\gamma-1} dt, \quad z \in U,$$

and  $q$  is the best dominant. □

**Theorem 2.6** Let  $g$  be a convex function such that  $g(0) = 1$  and let  $h$  be the function  $h(z) = g(z) + \frac{z}{\gamma}g'(z)$ ,  $z \in U$ , where  $\gamma > 0$ .

If  $\alpha \geq 0$ ,  $n \in \mathbb{N}$ ,  $f \in \mathcal{A}$  and the differential subordination

$$\frac{(\gamma + 2)z^2}{\gamma} \frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} + \frac{z^3}{\gamma} \left[ \frac{(L_\alpha^n f(z))''}{L_\alpha^n f(z)} - \left( \frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} \right)^2 \right] < h(z), \quad z \in U, \tag{2.6}$$

holds, then

$$z^2 \frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} < g(z), \quad z \in U.$$

This result is sharp.

*Proof* Let  $p(z) = z^2 \frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)}$ . We deduce that  $p \in \mathcal{H}[0, 1]$ .

Differentiating, we obtain  $p(z) + \frac{z}{\gamma}p'(z) = \frac{(\gamma+2)z^2}{\gamma} \frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} + \frac{z^3}{\gamma} \left[ \frac{(L_\alpha^n f(z))''}{L_\alpha^n f(z)} - \left( \frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} \right)^2 \right]$ ,  $z \in U$ .

Using the notation in (2.6), the differential subordination becomes

$$p(z) + \frac{1}{\gamma}zp'(z) < h(z) = g(z) + \frac{z}{\gamma}g'(z).$$

By using Lemma 1.2, we have

$$p(z) < g(z), \quad z \in U, \quad \text{i.e.,} \quad z^2 \frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} < g(z), \quad z \in U,$$

and this result is sharp. □

**Theorem 2.7** Let  $h$  be a holomorphic function which satisfies the inequality  $\operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$ ,  $z \in U$ , and  $h(0) = 1$ .

If  $\alpha \geq 0$ ,  $\gamma \in \mathbb{C} \setminus \{0\}$  is a complex number with  $\operatorname{Re} \gamma \geq 0$ ,  $n \in \mathbb{N}$ ,  $f \in \mathcal{A}$  and satisfies the differential subordination

$$\frac{(\gamma + 2)z^2}{\gamma} \frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} + \frac{z^3}{\gamma} \left[ \frac{(L_\alpha^n f(z))''}{L_\alpha^n f(z)} - \left( \frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} \right)^2 \right] < h(z), \quad z \in U, \tag{2.7}$$

then

$$z^2 \frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} < q(z), \quad z \in U,$$

where  $q(z) = \frac{\gamma}{z^\gamma} \int_0^z h(t)t^{\gamma-1} dt$ . The function  $q$  is convex and it is the best dominant.

*Proof* Let  $p(z) = z^2 \frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)}$ ,  $z \in U$ ,  $p \in \mathcal{H}[0, 1]$ .

Differentiating, we obtain  $p(z) + \frac{z}{\gamma}p'(z) = \frac{(\gamma+2)z^2}{\gamma} \frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} + \frac{z^3}{\gamma} \left[ \frac{(L_\alpha^n f(z))''}{L_\alpha^n f(z)} - \left( \frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} \right)^2 \right]$ ,  $z \in U$ , and (2.7) becomes

$$p(z) + \frac{1}{\gamma}zp'(z) < h(z), \quad z \in U.$$

Using Lemma 1.1, we have

$$p(z) \prec q(z), \quad z \in U, \quad \text{i.e.,} \quad z^2 \frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} \prec q(z) = \frac{\gamma}{z^\gamma} \int_0^z h(t)t^{\gamma-1} dt, \quad z \in U,$$

and  $q$  is the best dominant. □

**Theorem 2.8** *Let  $g$  be a convex function such that  $g(0) = 1$  and let  $h$  be the function  $h(z) = g(z) + zg'(z)$ ,  $z \in U$ .*

*If  $\alpha \geq 0$ ,  $n \in \mathbb{N}$ ,  $f \in \mathcal{A}$  and the differential subordination*

$$1 - \frac{L_\alpha^n f(z) \cdot (L_\alpha^n f(z))''}{[(L_\alpha^n f(z))']^2} \prec h(z), \quad z \in U, \tag{2.8}$$

*holds, then*

$$\frac{L_\alpha^n f(z)}{z(L_\alpha^n f(z))'} \prec g(z), \quad z \in U.$$

*This result is sharp.*

*Proof* Let  $p(z) = \frac{L_\alpha^n f(z)}{z(L_\alpha^n f(z))'}$ . We deduce that  $p \in \mathcal{H}[1, 1]$ .

Differentiating, we obtain  $1 - \frac{L_\alpha^n f(z) \cdot (L_\alpha^n f(z))''}{[(L_\alpha^n f(z))']^2} = p(z) + zp'(z)$ ,  $z \in U$ .

Using the notation in (2.8), the differential subordination becomes

$$p(z) + zp'(z) \prec h(z) = g(z) + zg'(z).$$

By using Lemma 1.2, we have

$$p(z) \prec g(z), \quad z \in U, \quad \text{i.e.,} \quad \frac{L_\alpha^n f(z)}{z(L_\alpha^n f(z))'} \prec g(z), \quad z \in U,$$

and this result is sharp. □

**Theorem 2.9** *Let  $h$  be a holomorphic function which satisfies the inequality  $\operatorname{Re}(1 + \frac{zh''(z)}{h'(z)}) > -\frac{1}{2}$ ,  $z \in U$ , and  $h(0) = 1$ .*

*If  $\alpha \geq 0$ ,  $n \in \mathbb{N}$ ,  $f \in \mathcal{A}$  and satisfies the differential subordination*

$$1 - \frac{L_\alpha^n f(z) \cdot (L_\alpha^n f(z))''}{[(L_\alpha^n f(z))']^2} \prec h(z), \quad z \in U, \tag{2.9}$$

*then*

$$\frac{L_\alpha^n f(z)}{z(L_\alpha^n f(z))'} \prec q(z), \quad z \in U,$$

*where  $q(z) = \frac{1}{z} \int_0^z h(t) dt$ . The function  $q$  is convex and it is the best dominant.*

*Proof* Let  $p(z) = \frac{L_\alpha^n f(z)}{z(L_\alpha^n f(z))'}$ ,  $z \in U$ ,  $p \in \mathcal{H}[0, 1]$ .

Differentiating, we obtain  $1 - \frac{L_{\omega}^n f(z) \cdot (L_{\omega}^n f(z))'}{[(L_{\omega}^n f(z))']^2} = p(z) + zp'(z)$ ,  $z \in U$ , and (2.9) becomes

$$p(z) + zp'(z) < h(z), \quad z \in U.$$

Using Lemma 1.1, we have

$$p(z) < q(z), \quad z \in U, \quad \text{i.e.,} \quad \frac{L_{\omega}^n f(z)}{z(L_{\omega}^n f(z))'} < q(z) = \frac{1}{z} \int_0^z h(t) dt, \quad z \in U,$$

and  $q$  is the best dominant. □

**Corollary 2.10** Let  $h(z) = \frac{1+(2\beta-1)z}{1+z}$  be a convex function in  $U$ , where  $0 \leq \beta < 1$ .

If  $\alpha \geq 0$ ,  $n \in \mathbb{N}$ ,  $f \in \mathcal{A}$  and satisfies the differential subordination

$$1 - \frac{L_{\omega}^n f(z) \cdot (L_{\omega}^n f(z))'}{[(L_{\omega}^n f(z))']^2} < h(z), \quad z \in U, \tag{2.10}$$

then

$$\frac{L_{\omega}^n f(z)}{z(L_{\omega}^n f(z))'} < q(z), \quad z \in U,$$

where  $q$  is given by  $q(z) = (2\beta - 1) + 2(1 - \beta) \frac{\ln(1+z)}{z}$ ,  $z \in U$ . The function  $q$  is convex and it is the best dominant.

*Proof* Following the same steps as in the proof of Theorem 2.9 and considering  $p(z) = \frac{L_{\omega}^n f(z)}{z(L_{\omega}^n f(z))'}$ , the differential subordination (2.10) becomes

$$p(z) + zp'(z) < h(z) = \frac{1 + (2\beta - 1)z}{1 + z}, \quad z \in U.$$

By using Lemma 1.1 for  $\gamma = 1$ , we have  $p(z) < q(z)$ , i.e.,

$$\begin{aligned} \frac{L_{\omega}^n f(z)}{z(L_{\omega}^n f(z))'} < q(z) &= \frac{1}{z} \int_0^z h(t) dt = \frac{1}{z} \int_0^z \frac{1 + (2\beta - 1)t}{1 + t} dt \\ &= \frac{1}{z} \int_0^z \left[ (2\beta - 1) + \frac{2(1 - \beta)}{1 + t} \right] dt = (2\beta - 1) + 2(1 - \beta) \frac{\ln(1 + z)}{z}, \quad z \in U. \quad \square \end{aligned}$$

**Example 2.1** Let  $h(z) = \frac{1-z}{1+z}$  be a convex function in  $U$  with  $h(0) = 1$  and  $\text{Re}(\frac{zh''(z)}{h'(z)} + 1) > -\frac{1}{2}$ .

Let  $f(z) = z + z^2$ ,  $z \in U$ . For  $n = 1$ ,  $\alpha = 2$ , we obtain  $L_2^1 f(z) = -R^1 f(z) + 2S^1 f(z) = -zf'(z) + 2zf''(z) = zf'(z) = z + 2z^2$ .

Then  $(L_2^1 f(z))' = 1 + 4z$ ,

$$\begin{aligned} \frac{L_2^1 f(z)}{z(L_2^1 f(z))'} &= \frac{z + 2z^2}{z(1 + 4z)} = \frac{1 + 2z}{1 + 4z}, \\ 1 - \frac{L_2^1 f(z) \cdot (L_2^1 f(z))''}{[(L_2^1 f(z))']^2} &= 1 - \frac{(z + 2z^2) \cdot 4}{(1 + 4z)^2} = \frac{8z^2 + 4z + 1}{(1 + 4z)^2}. \end{aligned}$$

We have  $q(z) = \frac{1}{z} \int_0^z \frac{1-t}{1+t} dt = -1 + \frac{2\ln(1+z)}{z}$ .



Using Theorem 2.9, we obtain

$$\frac{8z^2 + 4z + 1}{(1 + 4z)^2} < \frac{1 - z}{1 + z}, \quad z \in U,$$

induce

$$\frac{1 + 2z}{1 + 4z} < -1 + \frac{2 \ln(1 + z)}{z}, \quad z \in U.$$

**Theorem 2.11** *Let  $g$  be a convex function such that  $g(0) = 0$  and let  $h$  be the function  $h(z) = g(z) + zg'(z)$ ,  $z \in U$ .*

*If  $\alpha \geq 0$ ,  $n \in \mathbb{N}$ ,  $f \in \mathcal{A}$  and the differential subordination*

$$\left[ (L_\alpha^n f(z))' \right]^2 + L_\alpha^n f(z) \cdot (L_\alpha^n f(z))'' < h(z), \quad z \in U, \tag{2.11}$$

*holds, then*

$$\frac{L_\alpha^n f(z) \cdot (L_\alpha^n f(z))'}{z} < g(z), \quad z \in U.$$

*This result is sharp.*

*Proof* Let  $p(z) = \frac{L_\alpha^n f(z) \cdot (L_\alpha^n f(z))'}{z}$ . We deduce that  $p \in \mathcal{H}[0, 1]$ .

Differentiating, we obtain  $\left[ (L_\alpha^n f(z))' \right]^2 + L_\alpha^n f(z) \cdot (L_\alpha^n f(z))'' = p(z) + zp'(z)$ ,  $z \in U$ .

Using the notation in (2.11), the differential subordination becomes

$$p(z) + zp'(z) < h(z) = g(z) + zg'(z).$$

By using Lemma 1.2, we have

$$p(z) < g(z), \quad z \in U, \quad \text{i.e.,} \quad \frac{L_\alpha^n f(z) \cdot (L_\alpha^n f(z))'}{z} < g(z), \quad z \in U,$$

and this result is sharp. □

**Theorem 2.12** *Let  $h$  be a holomorphic function which satisfies the inequality  $\operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$ ,  $z \in U$ , and  $h(0) = 0$ .*

*If  $\alpha \geq 0$ ,  $n \in \mathbb{N}$ ,  $f \in \mathcal{A}$  and satisfies the differential subordination*

$$\left[ (L_\alpha^n f(z))' \right]^2 + L_\alpha^n f(z) \cdot (L_\alpha^n f(z))'' < h(z), \quad z \in U, \tag{2.12}$$

*then*

$$\frac{L_\alpha^n f(z) \cdot (L_\alpha^n f(z))'}{z} < q(z), \quad z \in U,$$

where  $q(z) = \frac{1}{z} \int_0^z h(t) dt$ . The function  $q$  is convex and it is the best dominant.

*Proof* Let  $p(z) = \frac{L_\alpha^n f(z) \cdot (L_\alpha^n f(z))'}{z}$ ,  $z \in U$ ,  $p \in \mathcal{H}[0, 1]$ .

Differentiating, we obtain  $[(L_\alpha^n f(z))']^2 + L_\alpha^n f(z) \cdot (L_\alpha^n f(z))'' = p(z) + zp'(z)$ ,  $z \in U$ , and (2.12) becomes

$$p(z) + zp'(z) < h(z), \quad z \in U.$$

Using Lemma 1.1, we have

$$p(z) < q(z), \quad z \in U, \quad \text{i.e.,} \quad \frac{L_\alpha^n f(z) \cdot (L_\alpha^n f(z))'}{z} < q(z) = \frac{1}{z} \int_0^z h(t) dt, \quad z \in U,$$

and  $q$  is the best dominant. □

**Corollary 2.13** Let  $h(z) = \frac{1+(2\beta-1)z}{1+z}$  be a convex function in  $U$ , where  $0 \leq \beta < 1$ .

If  $\alpha \geq 0$ ,  $n \in \mathbb{N}$ ,  $f \in \mathcal{A}$  and satisfies the differential subordination

$$[(L_\alpha^n f(z))']^2 + L_\alpha^n f(z) \cdot (L_\alpha^n f(z))'' < h(z), \quad z \in U, \tag{2.13}$$

then

$$\frac{L_\alpha^n f(z) \cdot (L_\alpha^n f(z))'}{z} < q(z), \quad z \in U,$$

where  $q$  is given by  $q(z) = (2\beta - 1) + 2(1 - \beta) \frac{\ln(1+z)}{z}$ ,  $z \in U$ . The function  $q$  is convex and it is the best dominant.

*Proof* Following the same steps as in the proof of Theorem 2.12 and considering  $p(z) = \frac{L_\alpha^n f(z) \cdot (L_\alpha^n f(z))'}{z}$ , the differential subordination (2.13) becomes

$$p(z) + zp'(z) < h(z) = \frac{1 + (2\beta - 1)z}{1 + z}, \quad z \in U.$$

By using Lemma 1.1 for  $\gamma = 1$ , we have  $p(z) < q(z)$ , i.e.,

$$\begin{aligned} \frac{L_\alpha^n f(z) \cdot (L_\alpha^n f(z))'}{z} < q(z) &= \frac{1}{z} \int_0^z h(t) dt = \frac{1}{z} \int_0^z \frac{1 + (2\beta - 1)t}{1 + t} dt \\ &= \frac{1}{z} \int_0^z \left[ (2\beta - 1) + \frac{2(1 - \beta)}{1 + t} \right] dt \\ &= (2\beta - 1) + 2(1 - \beta) \frac{\ln(1 + z)}{z}, \quad z \in U. \end{aligned} \quad \square$$

**Example 2.2** Let  $h(z) = \frac{1-z}{1+z}$  be a convex function in  $U$  with  $h(0) = 1$  and  $\text{Re}(\frac{zh''(z)}{h'(z)} + 1) > -\frac{1}{2}$ .

Let  $f(z) = z + z^2$ ,  $z \in U$ . For  $n = 1$ ,  $\alpha = 2$ , we obtain  $L_2^1 f(z) = -R^1 f(z) + 2S^1 f(z) = -zf'(z) + 2zf''(z) = zf'(z) = z + 2z^2$ ,  $z \in U$ .

Then  $(L_2^1 f(z))' = 1 + 4z$ ,

$$\frac{L_2^1 f(z) \cdot (L_2^1 f(z))'}{z} = \frac{(z + 2z^2)(1 + 4z)}{z} = 8z^2 + 6z + 1,$$

$$[(L_2^1 f(z))']^2 + L_2^1 f(z) \cdot (L_2^1 f(z))'' = (1 + 4z)^2 + (z + 2z^2) \cdot 4 = 24z^2 + 12z + 1.$$

We have  $q(z) = \frac{1}{z} \int_0^z \frac{1-t}{1+t} dt = -1 + \frac{2 \ln(1+z)}{z}$ .

Using Theorem 2.12, we obtain

$$24z^2 + 12z + 1 < \frac{1-z}{1+z}, \quad z \in U,$$

induce

$$8z^2 + 6z + 1 < -1 + \frac{2 \ln(1+z)}{z}, \quad z \in U.$$

**Theorem 2.14** *Let  $g$  be a convex function such that  $g(0) = 0$  and let  $h$  be the function  $h(z) = g(z) + \frac{z}{1-\delta} g'(z)$ ,  $z \in U$ .*

*If  $\alpha \geq 0$ ,  $\delta \in (0, 1)$ ,  $n \in \mathbb{N}$ ,  $f \in \mathcal{A}$  and the differential subordination*

$$\left( \frac{z}{L_\alpha^n f(z)} \right)^\delta \frac{L_\alpha^{n+1} f(z)}{1-\delta} \left( \frac{(L_\alpha^{n+1} f(z))'}{L_\alpha^{n+1} f(z)} - \delta \frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} \right) < h(z), \quad z \in U, \tag{2.14}$$

*holds, then*

$$\frac{L_\alpha^{n+1} f(z)}{z} \cdot \left( \frac{z}{L_\alpha^n f(z)} \right)^\delta < g(z), \quad z \in U.$$

*This result is sharp.*

*Proof* Let  $p(z) = \frac{L_\alpha^{n+1} f(z)}{z} \cdot \left( \frac{z}{L_\alpha^n f(z)} \right)^\delta$ . We deduce that  $p \in \mathcal{H}[1, 1]$ .

Differentiating, we obtain  $\left( \frac{z}{L_\alpha^n f(z)} \right)^\delta \frac{L_\alpha^{n+1} f(z)}{1-\delta} \left( \frac{(L_\alpha^{n+1} f(z))'}{L_\alpha^{n+1} f(z)} - \delta \frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} \right) = p(z) + \frac{1}{1-\delta} z p'(z)$ ,  $z \in U$ .

Using the notation in (2.14), the differential subordination becomes

$$p(z) + \frac{1}{1-\delta} z p'(z) < h(z) = g(z) + \frac{z}{1-\delta} g'(z).$$

By using Lemma 1.2, we have

$$p(z) < g(z), \quad z \in U, \quad \text{i.e.,} \quad \frac{L_\alpha^{n+1} f(z)}{z} \cdot \left( \frac{z}{L_\alpha^n f(z)} \right)^\delta < g(z), \quad z \in U,$$

and this result is sharp. □

**Theorem 2.15** *Let  $h$  be a holomorphic function which satisfies the inequality  $\operatorname{Re}(1 + \frac{zh''(z)}{h'(z)}) > -\frac{1}{2}$ ,  $z \in U$ , and  $h(0) = 1$ .*

*If  $\alpha \geq 0$ ,  $\delta \in (0, 1)$ ,  $n \in \mathbb{N}$ ,  $f \in \mathcal{A}$  and satisfies the differential subordination*

$$\left( \frac{z}{L_\alpha^n f(z)} \right)^\delta \frac{L_\alpha^{n+1} f(z)}{1-\delta} \left( \frac{(L_\alpha^{n+1} f(z))'}{L_\alpha^{n+1} f(z)} - \delta \frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} \right) < h(z), \quad z \in U, \tag{2.15}$$

*then*

$$\frac{L_\alpha^{n+1} f(z)}{z} \cdot \left( \frac{z}{L_\alpha^n f(z)} \right)^\delta < q(z), \quad z \in U,$$

where  $q(z) = \frac{1-\delta}{z^{1-\delta}} \int_0^z h(t) t^{-\delta} dt$ . The function  $q$  is convex and it is the best dominant.

*Proof* Let  $p(z) = \frac{L_\alpha^{n+1}f(z)}{z} \cdot \left(\frac{z}{L_\alpha^n f(z)}\right)^\delta$ ,  $z \in U$ ,  $p \in \mathcal{H}[0,1]$ .

Differentiating, we obtain  $\left(\frac{z}{L_\alpha^n f(z)}\right)^\delta \frac{L_\alpha^{n+1}f(z)}{1-\delta} \left(\frac{L_\alpha^{n+1}f(z)}{L_\alpha^n f(z)}\right)' - \delta \frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} = p(z) + \frac{1}{1-\delta} zp'(z)$ ,  $z \in U$ , and (2.15) becomes

$$p(z) + \frac{1}{1-\delta} zp'(z) < h(z), \quad z \in U.$$

Using Lemma 1.1, we have

$$p(z) < q(z), \quad z \in U, \quad \text{i.e.,}$$

$$\frac{L_\alpha^{n+1}f(z)}{z} \cdot \left(\frac{z}{L_\alpha^n f(z)}\right)^\delta < q(z) = \frac{1-\delta}{z^{1-\delta}} \int_0^z h(t)t^{-\delta} dt, \quad z \in U,$$

and  $q$  is the best dominant. □

#### Competing interests

The author declares that she has no competing interests.

#### Authors' contributions

The author drafted the manuscript, read and approved the final manuscript.

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