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# Mosco convergence results for common fixed point problems and generalized equilibrium problems in Banach spaces

Muhammad Aqeel Ahmad Khan[1*], Hafiz Fukhar-ud-din[1,2] and Abdul Rahim Khan[2]

Dedicated to Professor Wataru Takahashi on the occasion of his 70th birthday

*Correspondence:
itsakb@hotmail.com
[1]Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur, 63100, Pakistan
Full list of author information is available at the end of the article

**Abstract**

In this paper, we propose and analyze an explicit type algorithm for finding a common element of the set of solutions of a finite family of generalized equilibrium problems and the set of common fixed points of two countable families of total quasi-$\varphi$-asymptotically nonexpansive mappings in a Banach space $E$. As an application of our result, we suggest a framework for finding a common solution of a finite family of generalized equilibrium problems and common zeros of two finite families of maximal monotone operators on $E$.

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**Keywords:** total quasi-$\varphi$-asymptotically nonexpansive mapping; asymptotic fixed point; equilibrium problem; variational inequality problem; maximal monotone operator; Mosco convergence

## 1 Introduction and preliminaries

Let $E$ be a real Banach space with the norm $\|\cdot\|$ and $E^*$ be its dual. Let $C$ be a nonempty subset of $E$ and $T : C \to C$ be a mapping. We denote by $F(T) = \{x \in C : x = Tx\}$ the set of fixed points of $T$. We symbolize weak convergence and strong convergence of a sequence $\{x_n\}$ in $E$ as $x_n \rightharpoonup x$ and $x_n \to x$, respectively. Let $f : C \times C \to \mathbb{R}$ (the set of reals) be a bifunction and $A : C \to E^*$ be a nonlinear mapping. A generalized equilibrium problem is to find the set

$$GEP(f) = \big\{x \in C : f(x,y) + \langle Ax, y - x \rangle \geq 0 \text{ for all } y \in C\big\}, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ stands for the duality product.

Note that:

(i) if $A \equiv 0$, then problem (1.1) reduces to the following equilibrium problem $EP(f)$:

$$\text{find } x \in C \text{ such that } f(x,y) \geq 0 \text{ for all } y \in C;$$

(ii) if $f \equiv 0$, then problem (1.1) reduces to the classical variational inequality problem $VI(C, A)$:

$$\text{find } x \in C \text{ such that } \langle Ax, y - x \rangle \geq 0 \text{ for all } y \in C.$$

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The equilibrium problem provides a unified approach to finding a solution of a large number of problems arising in physics, optimization, economics and fixed point problems [1]. Moreover, the generalized equilibrium problem addresses monotone inclusion problems, variational inequality problems, minimization problems and vector equilibrium problem [2–4]. Since an algorithmic construction plays a key role in solving nonlinear equations in various fields of investigation, numerous implicit and explicit algorithms have been developed for the approximate solution of nonlinear equations as well as for the approximation of fixed points of various mappings [5–7].

Recall that a Banach space $E$ is said to be:

(i) *strictly convex* if for all $x, y \in S_E := \{z \in E : \|z\| = 1\}$ with $x \neq y$, we have $\|x + y\| < 2$;

(ii) *uniformly convex* if for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that

$$x, y \in S_E, \quad \|x - y\| \geq \epsilon \quad \Longrightarrow \quad \left\| \frac{x + y}{2} \right\| \leq 1 - \delta.$$

A Banach space $E$ is said to have the Kadec-Klee property if for any sequence $\{x_n\}$ in $E$ with $x_n \rightharpoonup x_0 \in E$ and $\lim_{n \to \infty} \|x_n\| = \|x_0\|$, we have $x_n \to x_0$. Note that every uniformly convex Banach space $E$ is strictly convex and enjoys the Kadec-Klee property but the converse is not true.

Furthermore, define $h : S_E \times S_E \times \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by

$$h(x, y, t) = \frac{\|x + ty\| - \|x\|}{t}$$

for $x, y \in S_E$ and $t \in \mathbb{R} \setminus \{0\}$. The norm of $E$ is said to be *Gâteaux differentiable* if $\lim_{t \to 0} h(x, y, t)$ exists for each $x, y \in S_E$ and in this case $E$ is *smooth*.

The normalized duality mapping $J : E \to E^*$ is defined by

$$Jx = \left\{ x^* \in E^* : \|x\|^2 = \langle x, x^* \rangle = \left\| x^* \right\|^2 \right\} \quad (x \in E).$$

It is remarked that the set-valued mapping $J$ is nonempty, closed and convex in a real Banach space whereas $J$ is single-valued in a reflexive, strictly convex and smooth Banach space. Furthermore, $J^{-1} : E^* \to E$, the inverse of the normalized duality mapping $J$, is also a duality mapping in a uniformly convex and smooth Banach space. Both $J$ and $J^{-1}$ are uniformly norm-to-norm continuous on each bounded subset of $E$ and $E^*$, respectively. If $E$ is reflexive and strictly convex, then $J^{-1}$ is norm-to-weak continuous. For more details, see [8, 9].

The Lyapunov functional $\varphi : E \times E \to \mathbb{R}$ is defined by

$$\varphi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \quad \text{for all } x, y \in E.$$

It is obvious from the definition of $\varphi$ that $\varphi(x, y) \geq 0$ for all $x, y \in E$. In a real Hilbert space, $\varphi(x, y) = \|x - y\|^2$ for all $x, y \in E$. For details, see [5, 10].

Let $E$ be a reflexive, strictly convex and smooth Banach space, and let $C$ be a nonempty, closed and convex subset of $E$. Then, for arbitrarily fixed $x \in E$, there exists a unique point $y_x \in C$ such that $\varphi(x, y_x) = \min_{y \in C} \varphi(x, y)$. Following the notation of [5], we let $\Pi_C(x) = y_x$ and call $\Pi_C$ a generalized projection onto $C$. Note that the generalized projection operator coincides with the metric projection in a Hilbert space.

A point $x \in C$ is said to be an *asymptotic fixed point* [11] of $T : C \to C$ if there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup x$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of all asymptotic fixed points of $T$ is denoted by $\widehat{F}(T)$.

Recall that a mapping $T : C \to C$ is:

(i) *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$;

(ii) *relatively nonexpansive* if $\widehat{F}(T) = F(T) \ne \emptyset$ and $\varphi(p, Tx) \le \varphi(p, x)$ for all $p \in F(T)$ and $x \in C$;

(iii) *quasi-$\varphi$-nonexpansive* if $F(T) \ne \emptyset$ and $\varphi(p, Tx) \le \varphi(p, x)$ for all $p \in F(T)$ and $x \in C$;

(iv) *quasi-$\varphi$-asymptotically nonexpansive* if there exists a real sequence $\{k_n\}$ with $k_n \ge 1$; $k_n \overset{n\to\infty}{\to} 1$ and $F(T) \ne \emptyset$ such that $\varphi(p, T^n x) \le k_n \varphi(p, x)$ for all $n \ge 1$, $p \in F(T)$ and $x \in C$;

(v) *total quasi-$\varphi$-asymptotically nonexpansive* if there exist nonnegative real sequences $\{k_n\}$ and $\{\eta_n\}$ with $k_n \overset{n\to\infty}{\to} 0$, $\eta_n \overset{n\to\infty}{\to} 0$ and $F(T) \ne \emptyset$ such that

$$\varphi(p, T^n x) \le \varphi(p, x) + k_n \xi(\varphi(p, x)) + \eta_n \quad \text{for all } n \ge 1, p \in F(T) \text{ and } x \in C,$$

where $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing continuous function with $\xi(0) = 0$.

It is worth mentioning that the class of total quasi-$\varphi$-asymptotically nonexpansive mappings properly contains the mappings defined in (i)-(iv), but the converse is not true.

A mapping $T : C \to C$ is said to be uniformly $L$-Lipschitzian if

$$\|T^n x - T^n y\| \le L\|x - y\| \quad \text{for some } L > 0 \text{ and } x, y \in C.$$

Recently, numerous attempts have been made to guarantee strong convergence through explicit and implicit algorithms for finding a common solution of the set of fixed points of (relatively nonexpansive, quasi-$\varphi$-nonexpansive, quasi-$\varphi$-asymptotically nonexpansive, total quasi-$\varphi$-asymptotically nonexpansive) mappings and the set of solutions of equilibrium problems; see [12–21] and the references cited therein.

In 2008, Takahashi and Zembayashi [21] introduced an explicit algorithm based on the shrinking projection method for finding a common solution of the set of fixed points of a relatively nonexpansive mapping $T$ and the set of solutions of an equilibrium problem. Their algorithm reads as follows:

$$
\begin{cases}
x_0 = x \in C = C_0, \\
y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JTx_n), \\
u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \ge 0 \quad \text{for all } y \in C, \\
C_{n+1} = \{z \in C_n : \varphi(z, u_n) \le \varphi(z, x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \ge 0,
\end{cases}
\tag{1.2}
$$

where $J$ is the duality mapping on $E$ and $\Pi_C$ is the generalized projection from $E$ onto $C$. They proved that the sequence $\{x_n\}$ generated by (1.2) converges strongly to $\Pi_{F(T) \cap EP(f)} x_0$ under some appropriate conditions.

In 2010, Chang *et al.* [12] proved a strong convergence theorem for finding a common element of the set of solutions for generalized equilibrium problem (1.1) and the set of common fixed points for a pair of relatively nonexpansive mappings in Banach spaces.

Their algorithm reads as follows:

$$
\begin{cases}
x_0 \in C \text{ chosen arbitrarily,} \\
z_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JTx_n), \\
y_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n)JSz_n), \\
u_n \in C \text{ such that } f(u_n, y) + \langle Au_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0 \\
\quad \text{for all } y \in C, \\
H_n = \{z \in C : \varphi(z, u_n) \leq \beta_n \varphi(z, x_n) + (1 - \beta_n)\varphi(z, z_n) \leq \varphi(z, x_n)\}, \\
W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\
x_{n+1} = \Pi_{H_n \cap W_n} x_0, \quad n \geq 0,
\end{cases}
\tag{1.3}
$$

where $J$, $T$, $S$ and $\Pi_C$ are as in (1.2). The authors showed that the sequence $\{x_n\}$ generated by (1.3) converges strongly to $\Pi_{F(S) \cap F(T) \cap GEP(f)} x_0$ under some appropriate conditions.

In 2009, Wattanawitton and Kumam [22] approximated a common solution for a pair of relatively quasi-nonexpansive mappings and an equilibrium problem. Recently, Qin *et al.* [18] established strong convergence results for a pair of asymptotically quasi-$\varphi$-nonexpansive mappings in a Banach space. It is worth mentioning that the hybrid algorithms proposed in [12, 18, 22] are computationally complex. Therefore, it is natural to have improved and computationally simpler counterparts.

Quite recently, Zuo *et al.* [16] proposed a hybrid algorithm for total quasi-$\varphi$-asymptotically nonexpansive mappings and established strong convergence results in a Banach space. Moreover, they characterized such strong convergence results by using the notion of Mosco convergence; see also [23]. Inspired and motivated by the work of Takahashi and Zembayashi [21], Chang *et al.* [12] and Zuo *et al.* [16], we aim to introduce and analyze a general algorithm based on the shrinking projection method for finding a common element of the set of common solutions of a finite family of generalized equilibrium problems and the set of common fixed points of two countable families of total quasi-$\varphi$-asymptotically nonexpansive mappings. We also characterize the set of common solutions for families of total quasi-$\varphi$-asymptotically nonexpansive mappings and equilibrium problems in terms of Mosco convergence.

We now introduce the notion of Mosco convergence.

Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of a reflexive Banach space $E$. We denote the set of all strong limit points of $\{C_n\}$ by $s - Li_n C_n$, that is, $x \in s - Li_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to $x$ and that $x_n \in C_n$ for all $n$. Similarly, we define the set of all weak subsequential limit points by $w - Ls_n C_n$; $y \in w - Ls_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to $y$ and $y_i \in C_{n_i}$ for all $i$. If $C_0$ satisfies $C_0 = s - Li_n C_n = w - Ls_n C_n$, then we say that $\{C_n\}$ converges to $C_0$ in the sense of Mosco and we write $C_0 = M\text{-}\lim_n C_n$. By definition, it always holds that $s - Li_n C_n \subset w - Ls_n C_n$. Therefore, to prove $C_0 = M\text{-}\lim_n C_n$, it suffices to show that $w - Ls_n C_n \subset C_0 \subset s - Li_n C_n$. One of the simplest examples of Mosco convergence is a decreasing sequence $\{C_n\}$ with respect to inclusion. The Mosco limit of such a sequence is $\bigcap_{n=1}^{\infty} C_n$. For more details, we refer to [23, 24].

For a relation between a sequence of closed convex sets and the corresponding generalized projections, we state the following lemma which plays a key role in our main result.

**Lemma 1.1** ([23], Theorem 2.2) *Let E be a smooth, reflexive and strictly convex Banach space having the Kadec-Klee property. Let $\{C_n\}$ be a sequence of nonempty closed subset of E. If $C_0 = M\text{-}\lim_n C_n$ exists and is nonempty, then $\{\Pi_{C_n} x\}$ converges strongly to $\Pi_{C_0} x$ for each $x \in C$.*

The following two results can be found as Remark 7.3 in [5].

**Lemma 1.2** *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, let $x_0 \in E$ and let $x \in C$. Then $\Pi_C x_0 = x$ if and only if*

$$\langle x - y, Jx_0 - Jx \rangle \geq 0 \quad \text{for all } y \in C.$$

**Lemma 1.3** *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\varphi(y, \Pi_C x) + \varphi(\Pi_C x, x) \leq \varphi(y, x) \quad \text{for all } y \in C.$$

The following well-known results are also needed in the sequel for the development of our main result.

**Lemma 1.4** ([25], Proposition 2) *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$, $\{y_n\}$ be two sequences of E. If $\varphi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

**Lemma 1.5** ([6], Lemma 1.4) *Let E be a uniformly convex Banach space and let $B_r[0]$ be a closed ball in E. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\alpha x + \beta y + \gamma z\|^2 \leq \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha \beta g(\|x - y\|)$$

*for all $x, y \in B_r[0]$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$.*

For solving the equilibrium problem, let us assume that the bifunction $f$ satisfies the following conditions (*cf.* [2, 26]):

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) $f$ is monotone, *i.e.*, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) $\limsup_{t \downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y)$ for all $x, y, z \in C$;

(A4) $f(x, \cdot)$ is convex and lower semicontinuous for all $x \in C$.

The following result is stated as Lemma 2.7 in [21] (see also [2]).

**Lemma 1.6** *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4), let $r > 0$ and $x \in E$. Then there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0 \quad \text{for all } y \in C.$$

**Lemma 1.7** ([21], Lemma 2.8) *Let $C$ be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space $E$. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *For $r > 0$ and $x \in E$, define a mapping $V_r : E \to C$ by*

$$V_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0 \text{ for all } y \in C \right\}$$

*for all $x \in C$. Then the following hold:*

(1) *$EP(f)$ is closed and convex;*

(2) *$V_r$ is single-valued;*

(3) *$V_r$ is firmly nonexpansive-type mapping, i.e.,*

$$\langle V_r x - V_r y, JV_r x - JV_r y \rangle \leq \langle V_r x - V_r y, Jx - Jy \rangle \quad \text{for all } x, y \in E,$$

(4) *$F(V_r) = EP(f)$.*

## 2 Main results

Let $C$ be a nonempty closed convex subset of a strictly convex, reflexive and smooth Banach space $E$ having the Kadec-Klee property. Let $T_i, S_i : C \to C$ be two countable families of uniformly $L$-Lipschitzian and uniformly total quasi-$\varphi$-asymptotically nonexpansive mappings with sequences $\{k_n^T\}$, $\{\eta_n^T\}$ and $\{k_n^S\}$, $\{\eta_n^S\}$, respectively. Let $f_n : C \times C \to \mathbb{R}$, $n = 1, 2, 3, \ldots, N$, be a finite family of bifunctions such that $f_n \equiv f_n (\mathrm{mod}\, N)$ (here the $\mathrm{mod}\, N$ function takes values in $\{1, 2, 3, \ldots, N\}$).

**Algorithm**

$$\begin{cases} x_0 \in C_0 = C, \\ y_{n,i} = J^{-1}(\alpha_{n,i} Jx_n + \beta_{n,i} JT_i^n x_n + \gamma_{n,i} JS_i^n x_n), \\ f_n(u_{n,i}, y) + \langle A_n u_{n,i}, y - u_{n,i} \rangle + \frac{1}{r_n} \langle y - u_{n,i}, Ju_{n,i} - Jy_{n,i} \rangle \geq 0 \quad \text{for all } y \in C, \\ C_{n+1} = \{z \in C_n : \sup_{i \geq 1} \varphi(z, u_{n,i}) \leq \varphi(z, x_n) + \Theta_n \}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \geq 0, \end{cases} \qquad (2.1)$$

where $\alpha_{n,i} + \beta_{n,i} + \gamma_{n,i} = 1$ and $\Theta_n := (1 - \alpha_{n,i})\{k_n \sup_{u \in F} \xi(\varphi(u, x_n)) + \eta_n\}$.

**Theorem 2.1** *Let $C$ be a nonempty closed convex subset of a strictly convex, reflexive and smooth Banach space $E$ having the Kadec-Klee property and let $T_i, S_i : C \to C$ be two countable families of uniformly $L$-Lipschitzian and uniformly total quasi-$\varphi$-asymptotically nonexpansive mappings with sequences $\{k_n\}$ and $\{\eta_n\}$, where $0 < L := \max\{L_i^T, L_i^S\}$ for each $i \geq 1$, $\{k_n\} := \max_{n \geq 1}\{\{k_n^T\}, \{k_n^S\}\}$ and $\{\eta_n\} := \max_{n \geq 1}\{\{\eta_n^T\}, \{\eta_n^S\}\}$. Let $f_n : C \times C \to \mathbb{R}$, $n = 1, 2, 3, \ldots, N$, be a finite family of bifunctions satisfying* (A1)-(A4) *such that $f_n = f_n(\mathrm{mod}\, N)$. Let $a \leq \alpha_{n,i}, \beta_{n,i}, \gamma_{n,i} \leq b$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfying:*

(C1) $\liminf_{n \to \infty} r_n > 0$.

*Assume that $F := [\bigcap_{i=1}^{\infty} F(T_i)] \cap [\bigcap_{i=1}^{\infty} F(S_i)] \cap [\bigcap_{n=1}^{N} GEP(f_n)] \neq \emptyset$. Then the sequence $\{x_n\}$ generated by (2.1) converges strongly to $x = \Pi_F x_0$, where $\Pi_F$ is the generalized projection of $E$ onto $F$.*

*Proof* For each $n = 1, 2, 3, \ldots, N$, define $G_n : C \times C \to \mathbb{R}$ by $G_n(x, y) = f_n(x, y) + \langle A_n x, y - x \rangle$. Then $G_n$ coincides with the classical equilibrium problem and satisfies (A1)-(A4). Now,

we show that algorithm (2.1) is well defined. By Lemma 1.7(1) we have that $F$ is closed and convex. Next we show that $C_{n+1}$ is closed and convex. Clearly, $C_0 = C$ is closed and convex. Suppose that $C_k$ is closed and convex for $k \in \mathbb{N}$ (set of naturals). For each $z \in C_k$, we observe that

$$
\begin{aligned}
C_{k+1} &= \left\{ z \in C_k : \sup_{i \geq 1} \varphi(z, u_{k,i}) \leq \varphi(z, x_k) + \Theta_k \right\} \\
&= \bigcap_{i \geq 1} \left\{ z \in C_k : \varphi(z, u_{k,i}) \leq \varphi(z, x_k) + \Theta_k \right\} \\
&= \bigcap_{i \geq 1} \left\{ z \in C_k : 2\langle z, Jx_k - Ju_{k,i}\rangle + \|u_{k,i}\|^2 - \|x_k\|^2 - \Theta_k \leq 0 \right\} \cap C_k.
\end{aligned}
$$

This implies that $C_{k+1}$ is closed and convex. This implies, inductively, that $C_{n+1}$ is closed and convex for all $n \geq 0$. Now we prove that $F \subset C_{n+1}$ for each $n \geq 0$. Obviously, $F \subset C_0 = C$. Suppose that $F \subset C_k$ for some $k \in \mathbb{N}$. It follows from Lemma 1.6 that $u_{k,i} = V_{r_k} y_{k,i}$ and $V_{r_k}$ is relatively nonexpansive. Hence, for any $p \in F$, we have

$$
\begin{aligned}
\varphi(p, u_{k,i}) = \varphi(p, V_{r_k} y_{k,i}) &\leq \varphi(p, y_{k,i}) \\
&= \varphi\big(p, J^{-1}\big(\alpha_{k,i} Jx_k + \beta_{k,i} JT_i^n x_k + \gamma_{k,i} JS_i^n x_k\big)\big) \\
&= \|p\|^2 - 2\big\langle p, \alpha_{k,i} Jx_k + \beta_{k,i} JT_i^n x_k + \gamma_{k,i} JS_i^n x_k \big\rangle \\
&\quad + \big\| \alpha_{k,i} Jx_k + \beta_{k,i} JT_i^n x_k + \gamma_{k,i} JS_i^n x_k \big\|^2 \\
&\leq \|p\|^2 - 2\alpha_{k,i}\langle p, Jx_k\rangle - 2\beta_{k,i}\big\langle p, JT_i^n x_k\big\rangle - 2\gamma_{k,i}\big\langle p, JS_i^n x_k\big\rangle \\
&\quad + \alpha_{k,i}\|x_k\|^2 + \beta_{k,i}\big\| T_i^n x_k \big\|^2 + \gamma_{k,i}\big\| S_i^n x_k \big\|^2 \\
&= \alpha_{k,i}\big(\|p\|^2 - 2\langle p, Jx_k\rangle + \|x_k\|^2\big) \\
&\quad + \beta_{k,i}\big(\|p\|^2 - 2\big\langle p, JT_i^n x_k\big\rangle + \big\| T_i^n x_k \big\|^2\big) \\
&\quad + \gamma_{k,i}\big(\|p\|^2 - 2\big\langle p, JS_i^n x_k\big\rangle + \big\| S_i^n x_k \big\|^2\big) \\
&= \alpha_{k,i}\varphi(p, x_k) + \beta_{k,i}\varphi\big(p, T_i^n x_k\big) + \gamma_{k,i}\varphi\big(p, S_i^n x_k\big) \\
&\leq \alpha_{k,i}\varphi(p, x_k) + \beta_{k,i}\big\{\varphi(p, x_k) + k_k \xi\big(\varphi(p, x_k)\big) + \eta_k\big\} \\
&\quad + \gamma_{k,i}\big\{\varphi(p, x_k) + k_k \xi\big(\varphi(p, x_k)\big) + \eta_k\big\}.
\end{aligned}
$$

That is,

$$
\varphi(p, u_{k,i}) \leq \varphi(p, x_k) + (1 - \alpha_{k,i})\big\{ k_k \xi\big(\varphi(p, x_k)\big) + \eta_k\big\} = \varphi(p, x_k) + \Theta_k, \tag{2.2}
$$

where $\Theta_k := (1 - \alpha_{k,i})\{k_k \sup_{u \in F} \xi(\varphi(u, x_k)) + \eta_k\}$. This shows that $p \in C_{k+1}$; consequently, $F \subset C_{k+1}$. By induction, we also get that $F \subset C_{n+1}$ for all $n \geq 0$ with $\Theta_n := (1 - \alpha_{n,i}) \times \{k_n \sup_{u \in F} \xi(\varphi(u, x_n)) + \eta_n\}$. Since $F \neq \emptyset$ and $C_{n+1}$ is a nonempty closed convex subset of $E$, hence both $\Pi_{C_{n+1}} x$ and $\{x_n\}$ are well defined.

Note that $\varphi(x_n, x^*)$ is nondecreasing. In fact, from the definition of $C_{n+1}$, we conclude that $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$; hence

$$
\varphi(x_n, x^*) \leq \varphi(x_{n+1}, x^*).
$$

Moreover, from Lemma 1.3 we get that

$$\varphi(x_n, x^*) = \varphi(\Pi_{C_n} x_0, x^*) \leq \varphi(p, x^*) - \varphi(p, \Pi_{C_n} x_0) \leq \varphi(p, x^*)$$

for each $p \in F$.

So $\varphi(x_n, x^*)$ is nondecreasing and bounded. This implies that $\lim_{n \to \infty} \varphi(x_n, x^*)$ exists.
Let $m \in \mathbb{Z}^+$ (a set of positive integers). Then Lemma 1.3 implies

$$\begin{aligned} \varphi(x_{n+m}, x_n) = \varphi(x_{n+m}, \Pi_{C_n} x_0) &\leq \varphi(x_{n+m}, x^*) - \varphi(\Pi_{C_n} x_0, x^*) \\ &= \varphi(x_{n+m}, x^*) - \varphi(x_n, x^*). \end{aligned}$$

Letting $n \to \infty$, we have $\varphi(x_{n+m}, x_n) \to 0$. By Lemma 1.4, we have

$$\|x_{n+m} - x_n\| \overset{n \to \infty}{\to} 0. \tag{2.3}$$

Hence $\{x_n\}$ is Cauchy. Since $C$ is a closed subset of the Banach space $E$, we can assume
that there exists a point $x \in C$ such that

$$x_n \to x \quad \text{as } n \to \infty. \tag{2.4}$$

Note that $\{C_n\}$ is a decreasing sequence of closed convex subsets of $E$ with $C_0 = \bigcap_{n=1}^{\infty} C_n$
is nonempty. That is,

$$M\text{-}\lim_n C_n = C_0 = \bigcap_{n=1}^{\infty} C_n \neq \emptyset.$$

Hence Lemma 1.1 asserts that $\{\Pi_{C_n} x_0\}$ converges to $\Pi_{C_0} x_0$.
In what follows, we show that:
Step 1. $x \in [\bigcap_{i=1}^{\infty} F(T_i)] \cap [\bigcap_{i=1}^{\infty} F(S_i)]$;
Step 2. $x \in [\bigcap_{n=1}^{N} EP(G_n)]$;
Step 3. $x = \Pi_F x_0$.
Proof of Step 1. As $x_{n+1} \in C_{n+1}$, so $\sup_{i \geq 1} \varphi(x_{n+1}, u_{n,i}) \leq \varphi(x_{n+1}, x_n) + \Theta_n$. It follows from
(2.3) and the fact that $\Theta_n \overset{n \to \infty}{\to} 0$, we get $\lim_{n \to \infty} \varphi(x_{n+1}, u_{n,i}) = 0$ for all $i \geq 1$. Again by
Lemma 1.4, we have

$$\lim_{n \to \infty} \|x_{n+1} - u_{n,i}\| = 0, \quad i \geq 1. \tag{2.5}$$

Observe that the following implication of the triangle inequality

$$\|x_n - u_{n,i}\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - u_{n,i}\|$$

yields that

$$\lim_{n \to \infty} \|x_n - u_{n,i}\| = 0. \tag{2.6}$$

As $J$ is uniformly norm-to-norm continuous on bounded sets, so we have

$$\lim_{n\to\infty} \|Jx_n - Ju_{n,i}\| = 0. \tag{2.7}$$

Moreover, it follows from (2.4) and (2.6) that

$$\lim_{n\to\infty} \|u_{n,i} - x\| = 0 \quad \text{for all } i \geq 1. \tag{2.8}$$

From (2.2), we know that $\varphi(p, y_{n,i}) \leq \varphi(p, x_n) + \Theta_n$ for all $i \geq 1$ and for all $p \in F$. So, by Lemma 1.3, we have

$$
\begin{aligned}
\varphi(u_{n,i}, y_{n,i}) &= \varphi(V_{r_n} y_{n,i}, y_{n,i}) \leq \varphi(p, y_{n,i}) - \varphi(p, V_{r_n} y_{n,i}) \\
&\leq \varphi(p, x_n) + \Theta_n - \varphi(p, V_{r_n} y_{n,i}) \\
&= \varphi(p, x_n) - \varphi(p, u_{n,i}) + \Theta_n \\
&= \|x_n\|^2 - \|u_{n,i}\|^2 - 2\langle p, Jx_n - Ju_{n,i}\rangle + \Theta_n \\
&\leq \|x_n - u_{n,i}\| (\|x_n\| + \|u_{n,i}\|) - 2\langle p, Jx_n - Ju_{n,i}\rangle + \Theta_n.
\end{aligned}
$$

Letting $n \to \infty$ in the above estimate and using (2.7), we have $\lim_{n\to\infty} \varphi(u_{n,i}, y_{n,i}) = 0$ for all $i \geq 1$; consequently, Lemma 1.4 asserts that

$$\lim_{n\to\infty} \|u_{n,i} - y_{n,i}\| = 0, \quad i \geq 1. \tag{2.9}$$

From (2.6) and (2.9), we have

$$\lim_{n\to\infty} \|x_n - y_{n,i}\| = 0, \quad i \geq 1.$$

Hence, we conclude that

$$y_{n,i} \overset{n\to\infty}{\to} x \quad \text{for all } i \geq 1. \tag{2.10}$$

On the other hand, from Lemma 1.5, we have

$$
\begin{aligned}
\varphi(p, u_{n,i}) &= \varphi(p, V_{r_n} y_{n,i}) \leq \varphi(p, y_{n,i}) \\
&= \varphi\big(p, J^{-1}\big(\alpha_{n,i} Jx_n + \beta_{n,i} JT_i^n x_n + \gamma_{n,i} JS_i^n x_n\big)\big) \\
&= \|p\|^2 - 2\langle p, \alpha_{n,i} Jx_n + \beta_{n,i} JT_i^n x_n + \gamma_{n,i} JS_i^n x_n\rangle \\
&\quad + \big\|\alpha_{n,i} Jx_n + \beta_{n,i} JT_i^n x_n + \gamma_{n,i} JS_i^n x_n\big\|^2 \\
&\leq \|p\|^2 - 2\alpha_{n,i}\langle p, Jx_n\rangle - 2\beta_{n,i}\langle p, JT_i^n x_n\rangle - 2\gamma_{n,i}\langle p, JS_i^n x_n\rangle \\
&\quad + \alpha_{n,i}\|x_n\|^2 + \beta_{n,i}\big\|T_i^n x_n\big\|^2 + \gamma_{n,i}\big\|S_i^n x_n\big\|^2 - \alpha_{n,i}\beta_{n,i} g\big(\big\|Jx_n - JT_i^n x_n\big\|\big) \\
&= \alpha_{n,i}\varphi(p, x_n) + \beta_{n,i}\varphi\big(p, T_i^n x_n\big) + \gamma_{n,i}\varphi\big(p, S_i^n x_n\big) \\
&\quad - \alpha_{n,i}\beta_{n,i} g\big(\big\|Jx_n - JT_i^n x_n\big\|\big) \\
&\leq \alpha_{n,i}\varphi(p, x_n) + (1 - \alpha_{n,i})\varphi(p, x_n) + (1 - \alpha_{n,i})\Theta_n - \alpha_{n,i}\beta_{n,i} g\big(\big\|Jx_n - JT_i^n x_n\big\|\big).
\end{aligned}
$$

Since $a \leq \alpha_{n,i}, \beta_{n,i}$, re-arranging the terms of the above estimate and simplifying, we get

$$a^2 g\big(\|Jx_n - JT_i^n x_n\|\big) \leq \varphi(p, x_n) - \varphi(p, u_{n,i}) + (1 - \alpha_{n,i})\Theta_n.$$

Reasoning as above and then utilizing (2.6), we get that

$$\lim_{n \to \infty} g\big(\|Jx_n - JT_i^n x_n\|\big) = 0, \quad i \geq 1.$$

Note that $g$ is a continuous function and $g(0) = 0$, so we have

$$\lim_{n \to \infty} \|Jx_n - JT_i^n x_n\| = 0, \quad i \geq 1. \tag{2.11}$$

In view of estimate (2.4), we conclude that $Jx_n \overset{n \to \infty}{\to} Jx$. Hence, from (2.11) we obtain

$$JT_i^n x_n \overset{n \to \infty}{\to} Jx, \quad i \geq 1. \tag{2.12}$$

Since $J^{-1}$ is uniformly norm-to-norm continuous, (2.11) implies that

$$\lim_{n \to \infty} \|x_n - T_i^n x_n\| = 0, \quad i \geq 1. \tag{2.13}$$

Observe that

$$\big\| T_i^n x_n - x \big\| \leq \big\| T_i^n x_n - x_n \big\| + \|x_n - x\|.$$

So, we conclude from (2.4) and (2.13) that

$$\lim_{n \to \infty} \big\| T_i^n x_n - x \big\| = 0, \quad i \geq 1.$$

Since each $T_i$ is uniformly $L$-Lipschitzian, we have

$$\begin{aligned}
\big\| T_i^{n+1} x_n - T_i^n x_n \big\| &\leq \big\| T_i^{n+1} x_n - T_i^{n+1} x_{n+1} \big\| + \big\| T_i^{n+1} x_{n+1} - x_{n+1} \big\| \\
&\quad + \|x_{n+1} - x_n\| + \big\| x_n - T_i^n x_n \big\| \\
&\leq (L+1)\|x_{n+1} - x_n\| + \big\| T_i^{n+1} x_{n+1} - x_{n+1} \big\| + \big\| x_n - T_i^n x_n \big\|.
\end{aligned}$$

Hence, from (2.3) and (2.13), we obtain that

$$\lim_{n \to \infty} \big\| T_i^{n+1} x_n - T_i^n x_n \big\| = 0, \quad i \geq 1.$$

Moreover, it yields that $\lim_{n \to \infty} \|T_i^{n+1} x_n - x\| = 0$, $i \geq 1$; consequently, we have $T_i(T_i^n x_n) \overset{n \to \infty}{\to} x$, $i \geq 1$. So we infer that $x \in \bigcap_{i=1}^{\infty} F(T_i)$.

Reasoning as above, one can also show that $x \in \bigcap_{i=1}^{\infty} F(S_i)$.

Proof of Step 2. We first show that $x \in EP(G_1)$, where $G_1 = G_{n_j}$ for some $j \geq 1$. In view of estimate (2.9) and (C1), *i.e.*, $\liminf_{n \to \infty} r_n > 0$, we observe that

$$\lim_{j \to \infty} \frac{\|Ju_{n_j,i} - Jy_{n_j,i}\|}{r_{n_j}} = 0, \quad i \geq 1.$$

From $u_{n_j,i} = V_{r_{n_j}} y_{n_j,i}$, for all $n \geq 0$ and $i \geq 1$, we have

$$G_1(u_{n_j,i}, y) + \frac{1}{r_{n_j}} \langle y - u_{n_j,i}, Ju_{n_j,i} - Jy_{n_j,i} \rangle \geq 0 \quad \text{for all } y \in C.$$

Using (A2), the above estimate yields that

$$\frac{1}{r_{n_j}} \langle y - u_{n_j,i}, Ju_{n_j,i} - Jy_{n,i} \rangle \geq -G_1(u_{n_j,i}, y) \geq G_1(y, u_{n_j,i}) \quad \text{for all } y \in C.$$

From $u_{n_j,i} \to x$ and (A4), we obtain $G_1(y, x) \leq 0$ for all $y \in C$. Let $y_t = ty + (1 - t)x_0$ for $0 < t < 1$ and $y \in C$. Then $y_t \in C$ and hence $G_1(y_t, x) \leq 0$. From (A1) and (A4), we have $0 = G_1(y_t, y_t) \leq tG_1(y_t, y) + (1 - t)G_1(y_t, x) \leq tG_1(y_t, y)$. Thus, $G_1(y_t, y) \geq 0$. From (A3), we have $G_1(x, y) \geq 0$ for all $y \in C$. Hence $x \in EP(G_1)$. In a similar fashion, we have some $k \geq 1$ such that $G_2 = G_{n_k}$ and $x \in EP(G_2)$. Therefore, $x \in \bigcap_{n=1}^N EP(G_n)$ and hence $x \in F$.

Proof of Step 3. Lemma 1.1 asserts that the sequence $\{x_n\} = \{\Pi_{C_n} x_0\}$ converges to $\Pi_{C_0} x_0$. Let $p = \Pi_{C_0} x_0 \in F$ and $F$ is a nonempty closed convex subset of $C_0 = \bigcap_{n=1}^\infty C_n$. Therefore $p = \Pi_F x_0$. It suffices to show that $x = p$. For this, we reason as follows:

Since $x_n = \Pi_{C_n} x_0$ and $p \in F \subset C_n$, we have

$$\varphi(x_n, x_0) \leq \varphi(p, x_0).$$

Since the norm is weakly lower semicontinuous, we have

$$\begin{aligned}
\varphi(x, x_0) &= \|x\|^2 - 2\langle x, Jx_0 \rangle + \|x_0\|^2 \\
&\leq \liminf_{k \to \infty} \left( \|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx_0 \rangle + \|x_0\|^2 \right) \\
&\leq \liminf_{k \to \infty} \varphi(x_{n_k}, x_0) \\
&\leq \limsup_{k \to \infty} \varphi(x_{n_k}, x_0) \leq \varphi(p, x_0).
\end{aligned}$$

From the definition of $\Pi_F$, we have $x = p$. Hence $\lim_{k \to \infty} \varphi(x_{n_k}, x_0) = \varphi(p, x_0)$. Therefore, we have

$$\begin{aligned}
0 &= \lim_{k \to \infty} \left( \varphi(x_{n_k}, x_0) - \varphi(x, x_0) \right) \\
&= \lim_{k \to \infty} \left( \|x_{n_k}\|^2 - \|x\|^2 + 2\langle x_{n_k} - x, Jx_0 \rangle \right) \\
&= \lim_{k \to \infty} \left( \|x_{n_k}\|^2 - \|x\|^2 \right).
\end{aligned}$$

Since $E$ has the Kadec-Klee property, we have that $\lim_{k \to \infty} x_{n_k} = x = \Pi_F x_0$.

The arbitrariness of $\{x_{n_k}\}$ implies that $\{x_n\}$ converges strongly to $x = \Pi_F x_0$. □

**Remark 2.2** If $\{T_i\}_{i=1}^N$ and $\{S_i\}_{i=1}^N$ are finite families in Theorem 2.1, then its conclusion can be strengthened as follows:

The sequence $\{x_n\}$ generated by (2.1) converges strongly to some $x \in F$ if and only if $\lim_{n \to \infty} d(x_n, F) = 0$, where $x = \Pi_F x_0$ and $d(q, F) = \inf\{d(q, y) : y \in F\}$.

*Proof* The necessity is obvious. Conversely, suppose that $\lim_{n\to\infty} d(x_n, F) = 0$. Since $\varphi(x_n, x^*)$ is bounded and $\lim_{n\to\infty} \varphi(x_n, x^*)$ exists (established above). Moreover, Lemma 1.3 and the following estimate

$$\varphi(x_{n+m}, x_n) = \varphi(x_{n+m}, \Pi_{C_n} x) \leq \varphi(x_m, x^*) - \varphi(\Pi_{C_n} x, x^*)$$
$$= \varphi(x_{n+m}, x^*) - \varphi(x_n, x^*)$$

imply $\varphi(x_{n+m}, x_n) \to 0$ when $m, n \to \infty$. Hence, Lemma 1.4 asserts that $\|x_{n+m} - x_n\| \to 0$. Thus, $\{x_n\}$ is a Cauchy sequence. Therefore, there exists a point $x \in C$ such that $x_n \to x$ as $n \to \infty$. Thus,

$$d(x, F) = \lim_{n\to\infty} d(x_n, F) = 0.$$

Since $F$ is closed, this implies $x \in F$. Now, as in the proof of Step 3 in Theorem 2.1, we have $x = \Pi_F x_0$. □

To proceed further, we need the following concept.

Let $f$ be a nondecreasing self-mapping on $[0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$. Let $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of total quasi-$\varphi$-asymptotically nonexpansive mappings on $C$ with $F \neq \emptyset$. Then the two families are said to satisfy condition (I) on $C$ if

$$\|x - Tx\| \geq f(d(x, F)) \quad \text{or} \quad \|x - Sx\| \geq f(d(x, F)) \quad \text{for all } x \in C.$$

As an application of Theorem 2.1 to the case of finite families of mappings $\{T_i\}_{i=1}^N$ and $\{S_i\}_{i=1}^N$, we obtain the following new strong convergence result.

**Theorem 2.3** *Let $C$ be a nonempty closed convex subset of a strictly convex, reflexive and smooth Banach space $E$ having the Kadec-Klee property and let $T_i, S_i : C \to C$ be two finite families of uniformly L-Lipschitzian and uniformly total quasi-$\varphi$-asymptotically nonexpansive mappings with sequences $\{k_n\}$ and $\{\eta_n\}$, where $0 < L := \max\{L_i^T, L_i^S\}$ for each $i \geq 1$, $\{k_n\} := \max_{n\geq 1}\{\{k_n^T\}, \{k_n^S\}\}$ and $\{\eta_n\} := \max_{n\geq 1}\{\{\eta_n^T\}, \{\eta_n^S\}\}$. Let $f_n : C \times C \to \mathbb{R}$, $n = 1, 2, 3, \ldots, N$, be a finite family of bifunctions satisfying (A1)-(A4) such that $f_n = f_n \pmod{N}$. Let $a \leq \alpha_{n,i}, \beta_{n,i}, \gamma_{n,i} \leq b$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfying*

(C1) $\liminf_{n\to\infty} r_n > 0$.

*Assume that $T_i$ and $S_i$ satisfy condition (I) and $F \neq \emptyset$. Then the sequence $\{x_n\}$ generated by (2.1) converges strongly to some $x \in F$, where $x = \Pi_F x_0$.*

*Proof* It follows from Theorem 2.1 that

$$\lim_{n\to\infty} \|x_n - T_i^n x_n\| = 0 = \lim_{n\to\infty} \|x_n - S_i^n x_n\|.$$

Since $\{T_i^n : i \in I\}$ and $\{S_i^n : i \in I\}$ satisfy condition (I), so we have either

$$\lim_{n\to\infty} f(d(u_{n,i}, F)) \leq \lim_{n\to\infty} \|x_n - T_i^n x_n\| = 0,$$

or

$$\lim_{n\to\infty} f\big(d(x_n, F)\big) \le \lim_{n\to\infty} \big\| x_n - S_i^n x_n \big\| = 0.$$

This implies that $\lim_{n\to\infty} f(d(u_{n,i}, F)) = 0$. Since $f$ is nondecreasing and $f(0) = 0$, we have $\lim_{n\to\infty} d(u_{n,i}, F) = 0$. The rest of the proof follows from Remark 2.2 and is, therefore, omitted. □

As another application of Theorem 2.1, we establish a result for finding a common element in the set of solutions of a finite family of generalized equilibrium problems and the set of common zeros of two finite families of maximal monotone operators on a Banach space.

First we recall some preliminary concepts as follows.

A multi-valued operator $T : E \to E^*$ is said to be monotone if for any $x, y \in E$ and $x^*, y^* \in E^*$ with $x^* \in Tx$ and $y^* \in Ty$ it holds that

$$\langle x^* - y^*, x - y \rangle \ge 0.$$

A point $u \in E$ satisfying $0 \in Tu$ is called a zero of $T$ and the set of all such points is denoted by $T^{-1}0$. $T$ is said to be maximal monotone if $T$ has no monotone extension. Equivalently, $T$ is maximal monotone if the graph of $T$, *i.e.*, $G(T) = \{(x, y) : y \in Tx\}$, is not properly contained in the graph of any other monotone operator (*cf.* [27], p.264).

Let $E$ be a strictly convex, reflexive and smooth Banach space and let $T : E \to E^*$ be a maximal monotone operator. For a positive real number $r$, we can define a single-valued mapping $J_r : E \to D(T)$ by $J_r(x) = (J + rT)^{-1}Jx$ for each $x \in E$. This mapping is called resolvent of $T$ for $r > 0$. It is known that if $T$ is maximal monotone, then $T^{-1}0 = F(J_r)$ for each $r > 0$. Moreover, $T^{-1}0$ is a closed convex subset of $E$. For each $r > 0$, we can define *Yosida approximation* of $T$ by $Y_r x = \frac{1}{r}(Jx - JJ_r x)$ for all $x \in E$. We know that $Y_r x \in T(J_r x)$; for more details, see [9].

**Theorem 2.4** *Let $C$ be a nonempty closed convex subset of a strictly convex, reflexive and smooth Banach space $E$ having the Kadec-Klee property. Let $T_i, S_i : E \to E^*$, $i = 1, 2, 3, \ldots, N$, be two finite families of maximal monotone operators and let $J_{r,i}^{T_i}$ and $J_{r,i}^{S_i}$ be the corresponding finite families of resolvents of $T_i$ and $S_i$, respectively, where $r > 0$. Let $f_n : C \times C \to \mathbb{R}$, $n = 1, 2, 3, \ldots, N$, be a finite family of bifunctions satisfying (A1)-(A4). Let $\{\alpha_{n,i}\}$, $\{\beta_{n,i}\}$ and $\{\gamma_{n,i}\}$ be three sequences in $[0, 1]$ such that $\alpha_{n,i} + \beta_{n,i} + \gamma_{n,i} = 1$ and $\{r_n\} \subset (0, \infty)$ satisfying*

(C1) $\liminf_{n\to\infty} r_n > 0$.

*Assume that $F := [\bigcap_{i=1}^{N} T_i^{-1}] \cap [\bigcap_{i=1}^{N} S_i^{-1}] \cap [\bigcap_{n=1}^{N} GEP(f_n)] \ne \emptyset$. Then the sequence $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C_0 = C, \\ y_{n,i} = J^{-1}(\alpha_{n,i} Jx_n + \beta_{n,i} JJ_{r_n,i}^{T_i} x_n + \gamma_{n,i} JJ_{r_n,i}^{S_i} x_n), \\ f_n(u_{n,i}, y) + \langle A_n u_{n,i}, y - u_{n,i} \rangle + \frac{1}{r_n} \langle y - u_{n,i}, Ju_{n,i} - Jy_{n,i} \rangle \ge 0 \quad \text{for all } y \in C, \\ C_{n+1} = \{z \in C_n : \sup_{i \ge 1} \varphi(z, u_{n,i}) \le \varphi(z, x_n) + \Theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \ge 0, \end{cases} \qquad (2.14)$$

*converges strongly to $\Pi_F x_0$, where $\Pi_F$ is the generalized projection of $E$ onto $F$ and $\Theta_n := (1 - \alpha_{n,i})\{k_n \sup_{u \in F} \xi(\varphi(u, x_n)) + \eta_n\}$.*

*Proof* The first part of this proof - $J_{r_{n,i}}^{T_i}$ (resp. $J_{r_{n,i}}^{S_i}$) are relatively nonexpansive mappings for each $i \geq 1$ - is essentially due to Matsushita and Takahashi (*cf.* [27, p.265]) which we include for completeness. Note that $\widehat{F}(J_{r_{n,i}}^{T_i}) \subset T_i^{-1}0$ for each $i = 1, 2, 3, \ldots, N$. Let $p \in \widehat{F}(J_{r_{n,i}}^{T_i})$. Then there exists $\{z_n\} \subset E$ such that $z_n \rightharpoonup p$ and $\lim_{n\to\infty} \|z_n - J_{r_{n,i}}^{T_i} z_n\| = 0$. Since $\liminf_{n\to\infty} r_n > 0$, we have

$$\frac{1}{r_{n,i}} \left\| Jz_n - JJ_{r_{n,i}}^{T_i} z_n \right\| \to 0.$$

It follows from $\frac{1}{r_{n,i}}(Jz_n - JJ_{r_{n,i}}^{T_i} z_n) \in T_i(J_{r_{n,i}}^{T_i} z_n)$ and the monotonicity of $T_i$ that

$$\left\langle w - J_{r_{n,i}}^{T_i} z_n, w^* - \frac{1}{r_n}\left(Jz_n - JJ_{r_{n,i}}^{T_i} z_n\right)\right\rangle \geq 0$$

for all $w \in D(T_i)$ and $w^* \in T_i w$. Letting $n \to \infty$, we have

$$\left\langle w - p, w^* \right\rangle \geq 0$$

for all $w \in D(T_i)$ and $w^* \in T_i w$. Therefore, from the maximality of $T_i$, we obtain that $p \in T_i^{-1}0$. On the other hand, $T_i^{-1}0 = F(J_{r,i}^{T_i})$ and $F(J_{r,i}^{T_i}) \subset \widehat{F}(J_{r,i}^{T_i})$, so we have $T_i^{-1}0 = F(J_{r,i}^{T_i}) = \widehat{F}(J_{r,i}^{T_i})$ for each $i = 1, 2, 3, \ldots, N$. Similarly, we can show that $p \in S_i^{-1}0$ and $S_i^{-1}0 = F(J_{r,i}^{S_i}) = \widehat{F}(J_{r,i}^{S_i})$ for each $i = 1, 2, 3, \ldots, N$. Moreover, the resolvent $J_{r,i}^{T_i}$ of $T_i$ (resp. $J_{r,i}^{S_i}$ of $S_i$) with $r > 0$ satisfies $\varphi(u, J_{r,i}^{T_i} x) \leq \varphi(u, x)$ for all $u \in F(J_{r,i}^{T_i})$ and $x \in E$ (resp. $\varphi(u, J_{r,i}^{S_i} x) \leq \varphi(u, x)$ for all $u \in F(J_{r,i}^{S_i})$ and $x \in E$). Hence $J_{r,i}^{T_i}$ and $J_{r,i}^{S_i}$ are relatively nonexpansive mappings. Since the class of relatively nonexpansive mappings is properly contained in the class of total quasi-$\varphi$-asymptotically nonexpansive mappings, for a finitely many mappings case, one can derive the desired result from Theorem 2.1. □

**Remark 2.5** It is worth to mention that Theorem 2.1 and Theorem 2.4 improve and generalize various results available in the current literature. In particular, we highlight some significant features of both these theorems as follows.

(i) Algorithm (2.1) is comparatively more general and computationally simpler than the algorithms which appeared in [12, 18] and [22], respectively, in the context of (two) countable families of total quasi-$\varphi$-asymptotically nonexpansive mappings.

(ii) Theorem 2.1 improves and extends [12, Theorem 3.1], [16], [21, Theorem 3.1] and [22, Theorem 3.1] for two countable families of total quasi-$\varphi$-asymptotically nonexpansive mappings and a finite family of generalized equilibrium problems.

(iii) Theorem 2.4 improves and extends [12, Theorem 4.1] for two finite families of maximal monotone operators. Moreover, our algorithm (2.14) is computationally simpler in comparison with algorithm (4.3) in [12, Theorem 3.1].

(iv) Theorem 2.4 improves and extends [28, Theorem 4.1] for two finite families of maximal monotone operators in the more general domain of a strictly convex, reflexive and smooth Banach space $E$.

(v) In the context of an approximate common solution for an equilibrium problem, a maximal monotone operator and a countable family of relatively nonexpansive mappings, algorithms (2.1) and (2.14) are more general and computationally simpler in comparison with algorithms (3.1) and (4.2), respectively, in [28].

(vi) Algorithm (2.14) is computationally simpler in comparison with algorithms (3.6) in [29] and (3.1) in [30]. Moreover, Theorem 2.4 improves [29, Theorem 3.2] and [30, Theorem 3.1] by removing the $H_n$ and $W_n$ conditions in the corresponding algorithms.

(vii) Theorem 2.4 is an analogue of [17, Theorem 3.1] for the approximation of a common zero for a finite family of maximal monotone operators.

## Author details
[1]Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur, 63100, Pakistan. [2]Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran, 3126, Saudi Arabia.

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