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A general iterative method for two maximal monotone operators and 2-generalized hybrid mappings in Hilbert spaces

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Abstract

Let C be a closed and convex subset of a real Hilbert space H. Let T be a 2-generalized hybrid mapping of C into itself, let A be an α-inverse strongly-monotone mapping of C into H, and let B and F be maximal monotone operators on $D(B) \subset C$ and $D(F) \subset C$ respectively. The purpose of this paper is to introduce a general iterative scheme for finding a point of $F(T) \cap (A+B)^{-1}0 \cap F^{-1}0$ which is a unique solution of a hierarchical variational inequality, where $F(T)$ is the set of fixed points of T, $(A+B)^{-1}0$ and $F^{-1}0$ are the sets of zero points of $A+B$ and F, respectively. A strong convergence theorem is established under appropriate conditions imposed on the parameters. Further, we consider the problem for finding a common element of the set of solutions of a mathematical model related to mixed equilibrium problems and the set of fixed points of a 2-generalized hybrid mapping in a real Hilbert space.

Keywords: 2-generalized hybrid mapping; inverse strongly monotone mapping; maximal monotone mapping; hierarchical variational inequality

1 Introduction

Let H be a Hilbert space, and let C be a nonempty closed convex subset of H. Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of all positive integers and real numbers, respectively. Let $\varphi : C \to \mathbb{R}$ be a real-valued function, and let $f : C \times C \to \mathbb{R}$ be an equilibrium bifunction, that is, $f(u,u) = 0$ for each $u \in C$. The mixed equilibrium problem is to find $x \in C$ such that

$$f(x,y) + \varphi(y) - \varphi(x) \geq 0 \quad \text{for all } y \in C. \tag{1.1}$$

Denote the set of solutions of (1.1) by $MEP(f,\varphi)$. In particular, if $\varphi = 0$, this problem reduces to the equilibrium problem, which is to find $x \in C$ such that

$$f(x,y) \geq 0 \quad \text{for all } y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $EP(f)$. The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, min-max problems, the Nash equilibrium problems in noncooperative games and others; see, for example, Blum-Oettli [1] and Moudafi [2]. Numerous problems in physics, optimization and economics reduce to finding a solution of the problem (1.2).

Let T be a mapping of C into C. We denote by $F(T) := \{x \in C : Tx = x\}$ the set of fixed points of T. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The mapping $T : C \to C$ is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle \quad \text{for all } x, y \in C; \tag{1.3}$$

see, for instance, Browder [3] and Goebel and Kirk [4]. The mapping $T : C \to C$ is said to be firmly nonspreading [5] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2 \tag{1.4}$$

for all $x, y \in C$. Iemoto and Takahashi [6] proved that $T : C \to C$ is nonspreading if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle \tag{1.5}$$

for all $x, y \in C$. It is not hard to know that a nonspreading mapping is deduced from a firmly nonexpansive mapping; see [7, 8] and a firmly nonexpansive mapping is a nonexpansive mapping.

In 2010, Kocourek *et al.* [9] introduced a class of nonlinear mappings, say generalized hybrid mappings. A mapping $T : C \to C$ is said to be generalized hybrid if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{1.6}$$

for all $x, y \in C$. We call such a mapping an (α, β)-generalized hybrid mapping. We observe that the mappings above generalize several well-known mappings. For example, an (α, β)-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, nonspreading for $\alpha = 2$ and $\beta = 1$, and hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$.

Recently, Maruyama *et al.* [10] defined a more general class of nonlinear mappings than the class of generalized hybrid mappings. Such a mapping is a 2-generalized hybrid mapping. A mapping T is called 2-generalized hybrid if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}
&\alpha_1\left\|T^2x - Ty\right\|^2 + \alpha_2\|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \\
&\quad\leq \beta_1\left\|T^2x - y\right\|^2 + \beta_2\|Tx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2
\end{aligned} \tag{1.7}$$

for all $x, y \in C$; see [10] for more details. We call such a mapping an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid mapping. We can also show that if T is a 2-generalized hybrid mapping and $x = Tx$, then for any $y \in C$,

$$\begin{aligned}
&\alpha_1\|x - Ty\|^2 + \alpha_2\|x - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \\
&\quad\leq \beta_1\|x - y\|^2 + \beta_2\|x - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2,
\end{aligned}$$

and hence $\|x - Ty\| \leq \|x - y\|$. This means that a 2-generalized hybrid mapping with a fixed point is quasi-nonexpansive. We observe that the 2-generalized hybrid mappings above generalize several well-known mappings. For example, a $(0, \alpha_2, 0, \beta_2)$-generalized hybrid mapping is an (α_2, β_2)-generalized hybrid mapping in the sense of Kocourek *et al.* [9].

Recall that a linear bounded operator B is strongly positive if there is a constant $\bar{\gamma} > 0$ with the property

$$\langle Vx, x\rangle \geq \bar{\gamma}\|x\|^2 \quad \text{for all } x \in H. \tag{1.8}$$

In general, a nonlinear operator $V : H \to H$ is called strongly monotone if there exists $\bar{\gamma} > 0$ such that

$$\langle x - y, Vx - Vy\rangle \geq \bar{\gamma}\|x - y\|^2 \quad \text{for all } x, y \in H. \tag{1.9}$$

Such V is called $\bar{\gamma}$-strongly monotone. A nonlinear operator $V : H \to H$ is called Lipschitzian continuous if there exists $L > 0$ such that

$$\|Vx - Vy\| \leq L\|x - y\| \quad \text{for all } x, y \in H. \tag{1.10}$$

Such V is called L-Lipschitzian continuous. A mapping $A : C \to H$ is said to be α-inverse-strongly monotone if $\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2$ for all $x, y \in C$. It is known that $\|Ax - Ay\| \leq (\frac{1}{\alpha})\|x - y\|$ for all $x, y \in C$ if A is α-inverse-strongly monotone; see, for example, [11–13].

Many studies have been done for structuring the fixed point of a nonexpansive mapping T. In 1953, Mann [14] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \tag{1.11}$$

where the initial guess $x_1 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0, 1]$. It is known that under appropriate settings the sequence $\{x_n\}$ converges weakly to a fixed point of T. However, even in a Hilbert space, Mann iteration may fail to converge strongly; for example, see [15]. Some attempts to construct an iteration method guaranteeing the strong convergence have been made. For example, Halpern [16] proposed the so-called Halpern iteration

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \tag{1.12}$$

where $u, x_1 \in C$ are arbitrary and $\{\alpha_n\}$ is a real sequence in $[0, 1]$ which satisfies $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$. Then $\{x_n\}$ converges strongly to a fixed point of T; see [16, 17].

In 1975, Baillon [18] first introduced the nonlinear ergodic theorem in a Hilbert space as follows:

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x \tag{1.13}$$

converges weakly to a fixed point of T for some $x \in C$. Recently Hojo *et al.* [19] proved the strong convergence theorem of Halpern type [20] for 2-generalized hybrid mappings in a Hilbert space as follows.

Theorem 1.1 *Let C be a nonempty, closed and convex subset of a Hilbert space H. Let $T : C \to C$ be a 2-generalized hybrid mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated by $x_1 = x \in C$, $u \in C$ and*

$$x_{n+1} = \gamma_n u + (1-\gamma_n)\frac{1}{n}\sum_{k=0}^{n-1} T^k x_n, \quad \forall n \in \mathbb{N}, \tag{1.14}$$

where $0 \leq \gamma_n \leq 1$, $\lim_{n\to\infty} \gamma_n = 0$ and $\sum_{n=1}^{\infty} \gamma_n = \infty$. Then $\{x_n\}$ converges strongly to $P_{F(T)}u$.

Let B be a mapping of H into 2^H. The effective domain of B is denoted by $D(B)$, that is, $D(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping B on H is said to be monotone if $\langle x-y, u-v\rangle \geq 0$ for all $x, y \in D(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1} : H \to D(B)$, which is called the resolvent of B for r. We denote by $A_r = \frac{1}{r}(I - J_r)$ the Yosida approximation of B for $r > 0$. We know [21] that

$$A_r x \in BJ_r x, \quad \forall x \in H, r > 0. \tag{1.15}$$

Let B be a maximal monotone operator on H, and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$. It is known that the resolvent J_r is firmly nonexpansive and $B^{-1}0 = F(J_r)$ for all $r > 0$, *i.e.*,

$$\|J_r x - J_r y\| \leq \langle x-y, J_r x - J_r y\rangle, \quad \forall x, y \in H. \tag{1.16}$$

Recently, in the case when $T : C \to C$ is a nonexpansive mapping, $A : C \to H$ is an α-inverse strongly monotone mapping and $B \in H \times H$ is a maximal monotone operator, Takahashi *et al.* [22] proved a strong convergence theorem for finding a point of $F(T) \cap (A+B)^{-1}0$, where $F(T)$ is the set of fixed points of T and $(A+B)^{-1}0$ is the set of zero points of $A+B$. In 2011, for finding a point of the set of fixed points of T and the set of zero points of $A+B$ in a Hilbert space, Manaka and Takahashi [23] introduced an iterative scheme as follows:

$$x_{n+1} = \beta_n x_n + (1-\beta_n)T\big(J_{\lambda_n}(I - \lambda_n A)x_n\big), \tag{1.17}$$

where T is a nonspreading mapping, A is an α-inverse strongly monotone mapping and B is a maximal monotone operator such that $J_\lambda = (I - \lambda B)^{-1}$; $\{\beta_n\}$ and $\{\lambda_n\}$ are sequences which satisfy $0 < c \leq \beta_n \leq d < 1$ and $0 < a \leq \lambda_n \leq b < 2\alpha$. Then they proved that $\{x_n\}$ converges weakly to a point $p = \lim_{n\to\infty} P_{F(T)\cap(A+B)^{-1}(0)} x_n$.

Very recently, Liu *et al.* [24] generalized the iterative algorithm (1.17) for finding a common element of the set of fixed points of a nonspreading mapping T and the set of zero points of a monotone operator $A+B$ (A is an α-inverse strongly monotone mapping and B is a maximal monotone operator). More precisely, they introduced the following iterative scheme:

$$\begin{cases} x_1 = x \in H \quad \text{arbitrarily}, \\ z_n = J_{\lambda_n}(I - \lambda_n A)x_n, \\ y_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k z_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n \quad \text{for all } n \in \mathbb{N}, \end{cases} \tag{1.18}$$

where $\{\alpha_n\}$ is an appropriate sequence in $[0,1]$. They obtained strong convergence theorems about a common element of the set of fixed points of a nonspreading mapping and the set of zero points of an α-inverse strongly monotone mapping and a maximal monotone operator in a Hilbert space.

On the other hand, iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, *e.g.*, [25–28] and the references therein. Convex minimization problems have a great impact and influence on the development of almost all branches of pure and applied sciences. A typical problem is to minimize a quadratic function over the set of fixed points a nonexpansive mapping on a real Hilbert space:

$$\theta(x) = \min_{x\in C} \frac{1}{2}\langle Vx, x\rangle - \langle x, b\rangle, \tag{1.19}$$

where V is a linear bounded operator, C is the fixed point set of a nonexpansive mapping T and b is a given point in H. Let H be a real Hilbert space. In [29], Marino and Xu introduced the following general iterative scheme based on the viscosity approximation method introduced by Moudafi [30]:

$$x_{n+1} = (I - \alpha_n V)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{1.20}$$

where V is a strongly positive bounded linear operator on H. They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.20) converges strongly to the unique solution of the variational inequality

$$\langle (V - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in C,$$

which is the optimality condition for the minimization problem

$$\min_{x\in C} \frac{1}{2}\langle Vx, x\rangle - h(x), \tag{1.21}$$

where h is a potential function for γf (*i.e.*, $h'(x) = \gamma f(x)$ for $x \in H$).

Recently, Tian [31] introduced the following general iterative scheme based on the viscosity approximation method induced by a $\bar{\gamma}$-strongly monotone and a L-Lipschitzian continuous operator V on H

$$x_{n+1} = \alpha_n \gamma g(x_n) + (I - \mu\alpha_n V)Tx_n,$$

for all $n \in \mathbb{N}$, where $\mu, \gamma \in \mathbb{R}$ satisfying $0 < \mu < \frac{2\bar{\gamma}}{L^2}$, $0 < \gamma < \mu(\bar{\gamma} - \frac{L^2\mu}{2})/k$, g is a k-contraction of H into itself and T is a nonexpansive mapping on H. It is proved, under some restrictions on the parameters, in [31] that $\{x_n\}$ converges strongly to a point $p_0 \in F(T)$ which is a unique solution of the variational inequality

$$\big\langle (V - \gamma g)p_0, q - p_0 \big\rangle \geq 0, \quad \forall q \in F(T).$$

Very recently, Lin and Takahashi [32] obtained the strong convergence theorem for finding a point $p_0 \in (A + B)^{-1}0 \cap F^{-1}0$ which is a unique solution of a hierarchical variational inequality, where A is an α-inverse strongly-monotone mapping of C into H, and B and F are maximal monotone operators on $D(B) \subset C$ and $D(F) \subset C$, respectively. More precisely, they introduced the following iterative scheme: Let $x_1 = x \in H$ and let $\{x_n\} \subset H$ be a sequence generated

$$x_{n+1} = \alpha_n \gamma g(x_n) + (I - \alpha_n V)J_{\lambda_n}(I - \lambda_n A)T_{r_n}x_n \quad \text{for all } n \in \mathbb{N}, \tag{1.22}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0,\infty)$ and $\{r_n\} \subset (0,\infty)$ satisfy certain appropriate conditions, $J_\lambda = (I + \lambda B)^{-1}$ and $T_r = (I + rF)^{-1}$ are the resolvents of B for $\lambda > 0$ and F for $r > 0$, respectively.

In this paper, motivated by the mentioned results, let C be a closed and convex subset of a real Hilbert space H. Let T be a 2-generalized hybrid mapping of C into itself, let A be an α-inverse strongly-monotone mapping of C into H, and let B and F be maximal monotone operators on $D(B) \subset C$ and $D(F) \subset C$ respectively. We introduce a new general iterative scheme for finding a common element of $F(T) \cap (A + B)^{-1}0 \cap F^{-1}0$ which is a unique solution of a hierarchical variational inequality, where $F(T)$ is the set of fixed points of T, $(A + B)^{-1}0$ and $F^{-1}0$ are the sets of zero points of $A + B$ and F, respectively. Then, we prove a strong convergence theorem. Further, we consider the problem for finding a common element of the set of solutions of a mathematical model related to mixed equilibrium problems and the set of fixed points of a 2-generalized hybrid mapping in a real Hilbert space.

2 Preliminaries

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, *i.e.*,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle \tag{2.1}$$

for all $x, y \in H$. Furthermore, $\langle P_C x - P_C y, x - y \rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [33]. Let $\alpha > 0$ be a given constant.

We also know the following lemma from [22].

Lemma 2.1 *Let H be a real Hilbert space, and let B be a maximal monotone operator on H. For $r > 0$ and $x \in H$, define the resolvent $J_r x$. Then the following holds*:

$$\frac{s - t}{s} \langle J_s x - J_t x, J_s x - x \rangle \geq \|J_s x - J_t x\|^2 \tag{2.2}$$

for all $s, t > 0$ *and* $x \in H$.

From Lemma 2.1, we have that

$$\|J_\lambda x - J_\mu x\| \leq \bigl(|\lambda - \mu|/\lambda\bigr)\|x - J_\lambda x\| \tag{2.3}$$

for all $\lambda, \mu > 0$ and $x \in H$; see also [33, 34]. To prove our main result, we need the following lemmas.

Remark 2.2 It is not hard to know that if A is an α-inverse strongly monotone mapping, then it is $\frac{1}{\alpha}$ -Lipschitzian and hence uniformly continuous. Clearly, the class of monotone mappings includes the class of α-inverse strongly monotone mappings.

Remark 2.3 It is well known that if $T : C \to C$ is a nonexpansive mapping, then $I - T$ is $\frac{1}{2}$-inverse strongly monotone, where I is the identity mapping on H; see, for instance, [21]. It is known that the resolvent J_r is firmly nonexpansive and $B^{-1}0 = F(J_r)$ for all $r > 0$.

Lemma 2.4 [23] *Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let $\alpha > 0$. Let A be an α-inverse strongly monotone mapping of C into H, and let B be a maximal monotone operator on H such that the domain of B is included in C. Let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for any $\lambda > 0$. Then the following hold*:

(i) *if $u, v \in (A + B)^{-1}(0)$, then $Au = Av$;*
(ii) *for any $\lambda > 0$, $u \in (A + B)^{-1}(0)$ if and only if $u = J_\lambda(I - \lambda A)u$.*

Lemma 2.5 [26, 35] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \leq (1 - t_n)a_n + b_n + t_n c_n,$$

where $\{t_n\}$, $\{b_n\}$ and $\{c_n\}$ satisfy the restrictions:

(i) $\sum_{n=0}^{\infty} t_n = \infty$;
(ii) $\sum_{n=0}^{\infty} b_n < \infty$;
(iii) $\limsup_{n\to\infty} c_n \leq 0$.

Then $\{a_n\}$ converges to zero as $n \to \infty$.

Lemma 2.6 [32] *Let H be a Hilbert space, and let $g : H \to H$ be a k-contraction with $0 < k < 1$. Let V be a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous operator on H with $\bar{\gamma} > 0$ and $L > 0$. Let a real number γ satisfy $0 < \gamma < \frac{\bar{\gamma}}{k}$. Then $V - \gamma g : H \to H$ is a $(\bar{\gamma} - \gamma k)$-strongly monotone and $(L + \gamma k)$-Lipschitzian continuous mapping. Furthermore, let C be a nonempty closed convex subset of H. Then $P_C(I - V + \gamma g)$ has a unique fixed point z_0 in C. This point $z_0 \in C$ is also a unique solution of the variational inequality*

$$\bigl\langle (V - \gamma f)z_0, q - z_0 \bigr\rangle \geq 0, \quad \forall q \in C.$$

3 Main results

In this section, we are in a position to propose a new general iterative sequence for 2-generalized hybrid mappings and establish a strong convergence theorem for the proposed sequence.

Theorem 3.1 *Let H be a real Hilbert space, and let C be a nonempty, closed and convex subset of H. Let $\alpha > 0$ and A be an α-inverse-strongly monotone mapping of C into H. Let the set-valued maps $B : D(B) \subset C \to 2^H$ and $F : D(F) \subset C \to 2^H$ be maximal monotone. Let $J_\lambda = (I + \lambda B)^{-1}$ and $T_r = (I + rF)^{-1}$ be the resolvents of B for $\lambda > 0$ and F for $r > 0$, respectively. Let $0 < k < 1$ and let g be a k-contraction of H into itself. Let V be a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous operator with $\bar{\gamma} > 0$ and $L > 0$. Let $T : C \to C$ be a 2-generalized hybrid mapping such that $\Omega := F(T) \cap (A + B)^{-1}0 \cap F^{-1}0 \neq \emptyset$. Take $\mu, \gamma \in \mathbb{R}$ as follows:*

$$0 < \mu < \frac{2\bar{\gamma}}{L^2}, \qquad 0 < \gamma < \frac{\bar{\gamma} - \frac{L^2\mu}{2}}{k}.$$

Let the sequence $\{x_n\} \subset H$ be generated by

$$\begin{cases} x_1 = x \in H \quad \text{arbitrarily}, \\ z_n = J_{\lambda_n}(I - \lambda_n A)T_{r_n}x_n, \\ y_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k z_n, \\ x_{n+1} = \alpha_n \gamma g(x_n) + (I - \alpha_n V)y_n, \quad \forall n = 1, 2, \ldots, \end{cases} \tag{3.1}$$

where the sequences $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{r_n\}$ satisfy the following restrictions:

(i) $\{\alpha_n\} \subset [0,1]$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) *there exist constants a and b such that $0 < a \le \lambda_n \le b < 2\alpha$ for all $n \in \mathbb{N}$;*
(iii) $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to a point p_0 of Ω, where p_0 is a unique fixed point of $P_\Omega(I - V + \gamma g)$. This point $p_0 \in \Omega$ is also a unique solution of the hierarchical variational inequality

$$\langle (V - \gamma g)p_0, q - p_0 \rangle \ge 0, \quad \forall q \in \Omega. \tag{3.2}$$

Proof First we prove that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \Omega$. Let $p \in \Omega$, we have that $p = J_{\lambda_n}(I - \lambda_n A)p$ and $p = T_{r_n}p$. Putting $u_n = T_{r_n}x_n$, we have that

$$\begin{aligned} \|z_n - p\|^2 &= \|J_{\lambda_n}(I - \lambda_n A)T_{r_n}x_n - J_{\lambda_n}(I - \lambda_n A)p\|^2 \\ &\le \|(T_{r_n}x_n - T_{r_n}p) - \lambda_n(AT_{r_n}x_n - AT_{r_n}p)\|^2 \\ &= \|T_{r_n}x_n - T_{r_n}p\|^2 - 2\lambda_n\langle u_n - p, Au_n - Ap\rangle + \lambda_n^2\|Au_n - Ap\|^2 \\ &\le \|u_n - p\|^2 - 2\lambda_n\alpha\|Au_n - Ap\|^2 + \lambda_n^2\|Au_n - Ap\|^2 \\ &\le \|x_n - p\|^2 - \lambda_n(2\alpha - \lambda_n)\|Au_n - Ap\|^2 \\ &\le \|x_n - p\|^2. \end{aligned} \tag{3.3}$$

This together with quasi-nonexpansiveness of T implies that

$$\begin{aligned} \|y_n - p\| &= \left\| \frac{1}{n}\sum_{k=0}^{n-1} T^k z_n - p \right\| \\ &\le \frac{1}{n}\sum_{k=0}^{n-1} \|T^k z_n - p\| \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{n}\sum_{k=0}^{n-1}\|z_n - p\| \\
&= \|z_n - p\| \leq \|x_n - p\|. \qquad (3.4)
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\|x_{n+1} - p\| &= \left\|\alpha_n\bigl(\gamma g(x_n) - Vp\bigr) + (I - \alpha_n V)y_n - (I - \alpha_n V)p\right\| \\
&\leq \alpha_n\left\|\gamma g(x_n) - Vp\right\| + \left\|(I - \alpha_n V)y_n - (I - \alpha_n V)p\right\| \\
&\leq \alpha_n\gamma k\|x_n - p\| + \alpha_n\left\|\gamma g(p) - Vp\right\| + \left\|(I - \alpha_n V)y_n - (I - \alpha_n V)p\right\|. \qquad (3.5)
\end{aligned}
$$

Putting $\tau = \bar{\gamma} - \frac{L^2\mu}{2}$, we can calculate the following:

$$
\begin{aligned}
\left\|(I - \alpha_n V)y_n - (I - \alpha_n V)p\right\|^2 &= \left\|(y_n - p) - \alpha_n(Vy_n - Vp)\right\|^2 \\
&= \|y_n - p\|^2 - 2\alpha_n\langle y_n - p, Vy_n - Vp\rangle + \alpha_n^2\|Vy_n - Vp\|^2 \\
&\leq \|y_n - p\|^2 - 2\alpha_n\bar{\gamma}\|y_n - p\|^2 + \alpha_n^2 L^2\|y_n - p\|^2 \\
&= \bigl(1 - 2\alpha_n\bar{\gamma} + \alpha_n^2 L^2\bigr)\|y_n - p\|^2 \\
&= \bigl(1 - 2\alpha_n\tau - \alpha_n L^2\mu + \alpha_n^2 L^2\bigr)\|y_n - p\|^2 \\
&\leq \bigl(1 - 2\alpha_n\tau - \alpha_n\bigl(L^2\mu - \alpha_n L^2\bigr) + \alpha_n^2\tau^2\bigr)\|y_n - p\|^2 \\
&\leq \bigl(1 - 2\alpha_n\tau + \alpha_n^2\tau^2\bigr)\|y_n - p\|^2 \\
&= (1 - \alpha_n\tau)^2\|y_n - p\|^2. \qquad (3.6)
\end{aligned}
$$

Since $1 - \alpha_n\tau > 0$, we obtain that

$$
\left\|(I - \alpha_n V)y_n - (I - \alpha_n V)p\right\| \leq (1 - \alpha_n\tau)\|y_n - p\|.
$$

Therefore, by (3.5), we have

$$
\begin{aligned}
\|x_{n+1} - p\| &\leq \alpha_n\gamma k\|x_n - p\| + \alpha_n\left\|\gamma g(p) - Vp\right\| + (1 - \alpha_n\tau)\|y_n - p\| \\
&\leq \alpha_n\gamma k\|x_n - p\| + \alpha_n\left\|\gamma g(p) - Vp\right\| + (1 - \alpha_n\tau)\|x_n - p\| \\
&= \bigl(1 - \alpha_n(\tau - \gamma k)\bigr)\|x_n - p\| + \alpha_n\left\|\gamma g(p) - Vp\right\| \\
&= \bigl(1 - \alpha_n(\tau - \gamma k)\bigr)\|x_n - p\| + \alpha_n(\tau - \gamma k)\frac{\|\gamma g(p) - Vp\|}{\tau - \gamma k} \\
&\leq \max\left\{\|x_n - p\|, \frac{\|\gamma g(p) - Vp\|}{\tau - \gamma k}\right\} \quad \text{for all } n \in \mathbb{N},
\end{aligned}
$$

which yields that the sequence $\{\|x_n - p\|\}$ is bounded, so are $\{x_n\}$, $\{y_n\}$, $\{Vy_n\}$, $\{g(x_n)\}$ and $\{T^n z_n\}$. Using Lemma 2.6, we can take a unique $p_0 \in \Omega$ of the hierarchical variational inequality

$$
\bigl\langle(V - \gamma g)p_0, q - p_0\bigr\rangle \geq 0, \quad \forall q \in \Omega. \qquad (3.7)
$$

We show that $\limsup_{n\to\infty}\langle (V-\gamma g)p_0, x_n - p_0\rangle \geq 0$. We may assume, without loss of generality, that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to $w \in C$, as $k \to \infty$, such that

$$\limsup_{n\to\infty}\langle (V-\gamma g)p_0, x_n - p_0\rangle = \lim_{k\to\infty}\langle (V-\gamma g)p_0, x_{n_k} - p_0\rangle.$$

Since $\{\|x_{n_k} - p\|\}$ is bounded, there exists a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that $\lim_{i\to\infty}\|x_{n_{k_i}} - p\|$ exists. Now we shall prove that $w \in \Omega$.

(a) We first prove $w \in F(T)$. We notice that

$$\|x_{n+1} - y_n\| = \|\alpha_n\gamma g(x_n) + (I - \alpha_n V)y_n - y_n\| = \alpha_n\|\gamma g(x_n) - Vy_n\|.$$

In particular, replacing n by n_{k_i} and taking $i \to \infty$ in the last equality, we have

$$\lim_{i\to\infty}\|x_{n_{k_i}+1} - y_{n_{k_i}}\| = 0,$$

so we have $y_{n_{k_i}} \rightharpoonup w$. Since T is 2-generalized hybrid, there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}
&\alpha_1\|T^2x - Ty\|^2 + \alpha_2\|Tx - Ty\|^2 + (1-\alpha_1-\alpha_2)\|x - Ty\|^2 \\
&\quad\leq \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1-\beta_1-\beta_2)\|x - y\|^2
\end{aligned}$$

for all $x, y \in C$. For any $n \in \mathbb{N}$ and $k = 0, 1, 2, \ldots, n-1$, we compute the following:

$$\begin{aligned}
0 &\leq \beta_1\|T^2T^kz_n - y\|^2 + \beta_2\|TT^kz_n - y\|^2 + (1-\beta_1-\beta_2)\|T^kz_n - y\|^2 \\
&\quad - \alpha_1\|T^2T^kz_n - Ty\|^2 - \alpha_2\|TT^kz_n - Ty\|^2 - (1-\alpha_1-\alpha_2)\|T^kz_n - Ty\|^2 \\
&= \beta_1\|T^{k+2}z_n - y\|^2 + \beta_2\|T^{k+1}z_n - y\|^2 + (1-\beta_1-\beta_2)\|T^kz_n - y\|^2 \\
&\quad - \alpha_1\|T^{k+2}z_n - Ty\|^2 - \alpha_2\|T^{k+1}z_n - Ty\|^2 - (1-\alpha_1-\alpha_2)\|T^kz_n - Ty\|^2 \\
&\leq \beta_1\{\|T^{k+2}z_n - Ty\|^2 + \|Ty - y\|^2\} + \beta_2\{\|T^{k+1}z_n - Ty\|^2 + \|Ty - y\|^2\} \\
&\quad + (1-\beta_1-\beta_2)\{\|T^kz_n - Ty\|^2 + \|Ty - y\|^2\} - \alpha_1\|T^{k+2}z_n - Ty\|^2 \\
&\quad - \alpha_2\|T^{k+1}z_n - Ty\|^2 - (1-\alpha_1-\alpha_2)\|T^kz_n - Ty\|^2 \\
&= \beta_1\{\|T^{k+2}z_n - Ty\|^2 + \|Ty - y\|^2 + 2\langle T^{k+2}z_n - Ty, Ty - y\rangle\} \\
&\quad + \beta_2\{\|T^{k+1}z_n - Ty\|^2 + \|Ty - y\|^2 + 2\langle T^{k+1}z_n - Ty, Ty - y\rangle\} \\
&\quad + (1-\beta_1-\beta_2)\{\|T^kz_n - Ty\|^2 + \|Ty - y\|^2 + 2\langle T^kz_n - Ty, Ty - y\rangle\} \\
&\quad - \alpha_1\|T^{k+2}z_n - Ty\|^2 - \alpha_2\|T^{k+1}z_n - Ty\|^2 - (1-\alpha_1-\alpha_2)\|T^kz_n - Ty\|^2 \\
&= (\beta_1-\alpha_1)\|T^{k+2}z_n - Ty\|^2 + (\beta_2-\alpha_2)\|T^{k+1}z_n - Ty\|^2 \\
&\quad + (\alpha_1+\alpha_2-\beta_1-\beta_2)\|T^kz_n - Ty\|^2 \\
&\quad \times (\beta_1+\beta_2+1-\beta_1-\beta_2)\|Ty - y\|^2 + 2\langle \beta_1T^{k+2}z_n - \beta_1Ty + \beta_2T^{k+1}z_n - \beta_2Ty \\
&\quad + (1-\beta_1-\beta_2)T^kz_n - (1-\beta_1-\beta_2)Ty, Ty - y\rangle \\
&= (\beta_1-\alpha_1)\|T^{k+2}z_n - Ty\|^2 + (\beta_2-\alpha_2)\|T^{k+1}z_n - Ty\|^2
\end{aligned}$$

$$
\begin{aligned}
&-\big((\beta_1-\alpha_1)+(\alpha_2-\beta_2)\big)\big\|T^kz_n-Ty\big\|^2+\|Ty-y\|^2\\
&+2\big\langle\beta_1T^{k+2}z_n+\beta_2T^{k+1}z_n+(1-\beta_1-\beta_2)T^kz_n-Ty,Ty-y\big\rangle\\
={}&(\beta_1-\alpha_1)\big(\big\|T^{k+2}z_n-Ty\big\|^2-\big\|T^kz_n-Ty\big\|^2\big)\\
&+(\beta_2-\alpha_2)\big(\big\|T^{k+1}z_n-Ty\big\|^2-\big\|T^kz_n-Ty\big\|^2\big)\\
&+\|Ty-y\|^2+2\big\langle\beta_1T^{k+2}z_n+\beta_2T^{k+1}z_n+(1-\beta_1-\beta_2)T^kz_n-Ty,Ty-y\big\rangle\\
={}&\|Ty-y\|^2+2\big\langle T^kz_n-Ty,Ty-y\big\rangle\\
&+2\big\langle\beta_1\big(T^{k+2}z_n-T^kx_n\big)+\beta_2\big(T^{k+1}z_n-T^kz_n\big),Ty-y\big\rangle\\
&+(\beta_1-\alpha_1)\big(\big\|T^{k+2}z_n-Ty\big\|^2-\big\|T^kz_n-Ty\big\|^2\big)\\
&+(\beta_2-\alpha_2)\big(\big\|T^{k+1}z_n-Ty\big\|^2-\big\|T^kz_n-Ty\big\|^2\big).
\end{aligned}
$$

Summing up these inequalities from $k=0$ to $n-1$, we get

$$
\begin{aligned}
0\le{}&\sum_{k=0}^{n-1}\|Ty-y\|^2+2\Big\langle\sum_{k=0}^{n-1}\big(T^kz_n-Ty\big),Ty-y\Big\rangle\\
&+2\Big\langle\beta_1\sum_{k=0}^{n-1}\big(T^{k+2}z_n-T^kz_n\big)+\beta_2\sum_{k=0}^{n-1}\big(T^{k+1}z_n-T^kz_n\big),Ty-y\Big\rangle\\
&+(\beta_1-\alpha_1)\sum_{k=0}^{n-1}\big(\big\|T^{k+2}z_n-Ty\big\|^2-\big\|T^kz_n-Ty\big\|^2\big)\\
&+(\beta_2-\alpha_2)\sum_{k=0}^{n-1}\big(\big\|T^{k+1}z_n-Ty\big\|^2-\big\|T^kz_n-Ty\big\|^2\big)\\
={}&n\|Ty-y\|^2+2\Big\langle\sum_{k=0}^{n-1}T^kz_n-nTy,Ty-y\Big\rangle\\
&+2\big\langle\beta_1\big(T^{n+1}z_n-T^nz_n-z_n-Tz_n\big)+\beta_2\big(T^nz_n-z_n\big),Ty-y\big\rangle\\
&+(\beta_1-\alpha_1)\big(\big\|T^{n+1}z_n-Ty\big\|^2+\big\|T^nz_n-Ty\big\|^2-\|z_n-Ty\|^2-\|Tz_n-Ty\|^2\big)\\
&+(\beta_2-\alpha_2)\big(\big\|T^nz_n-Ty\big\|^2-\|z_n-Ty\|^2\big).
\end{aligned}
$$

Dividing this inequality by n, we get

$$
\begin{aligned}
0\le{}&\|Ty-y\|^2+2\langle y_n-Ty,Ty-y\rangle\\
&+2\Big\langle\frac{1}{n}\beta_1\big(T^{n+1}z_n-T^nz_n-z_n-Tz_n\big)+\frac{1}{n}\beta_2\big(T^nz_n-z_n\big),Ty-y\Big\rangle\\
&+\frac{1}{n}(\beta_1-\alpha_1)\big(\big\|T^{n+1}z_n-Ty\big\|^2+\big\|T^nz_n-Ty\big\|^2-\|z_n-Ty\|^2-\|Tz_n-Ty\|^2\big)\\
&+\frac{1}{n}(\beta_2-\alpha_2)\big(\big\|T^nz_n-Ty\big\|^2-\|z_n-Ty\|^2\big).
\end{aligned}
$$

Replacing n by n_{k_i} and letting $i\to\infty$ in the last inequality, we have

$$0\le\|Ty-y\|^2+2\langle w-Ty,Ty-y\rangle\quad\text{for all } y\in C. \tag{3.8}$$

In particular, replacing y by w in (3.8), we obtain that

$$0 \leq \|Tw - w\|^2 + 2\langle w - Tw, Tw - w\rangle = -\|Tw - w\|^2,$$

which ensures that $w \in F(T)$.

(b) We prove that $w \in (A+B)^{-1}0$. From (3.3), (3.4) and (3.6), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \big\|(I - \alpha_n V)y_n - (I - \alpha_n V)p\big\|^2 + 2\alpha_n\big\langle \gamma g(x_n) - Vp, x_{n+1} - p\big\rangle \\
&\leq (1 - \alpha_n \tau)^2\|y_n - p\|^2 + 2\alpha_n\big\langle \gamma g(x_n) - Vp, x_{n+1} - p\big\rangle \\
&\leq (1 - \alpha_n \tau)^2\|z_n - p\|^2 + 2\alpha_n\big\langle \gamma g(x_n) - Vp, x_{n+1} - p\big\rangle \\
&\leq (1 - \alpha_n \tau)^2\big\{\|x_n - p\|^2 - \lambda_n(2\alpha - \lambda_n)\|Au_n - Ap\|^2\big\} \\
&\quad + 2\alpha_n\big\langle \gamma g(x_n) - Vp, x_{n+1} - p\big\rangle \\
&= \big(1 - 2\alpha_n\tau + \alpha_n^2\tau^2\big)\|x_n - p\|^2 - (1 - \alpha_n\tau)^2\lambda_n(2\alpha - \lambda_n)\|Au_n - Ap\|^2 \\
&\quad + 2\alpha_n\big\langle \gamma g(x_n) - Vp, x_{n+1} - p\big\rangle \\
&\leq \|x_n - p\|^2 + \alpha_n^2\tau^2\|x_n - p\|^2 - (1 - \alpha_n\tau)^2\lambda_n(2\alpha - \lambda_n)\|Au_n - Ap\|^2 \\
&\quad + 2\alpha_n\big\langle \gamma g(x_n) - Vp, x_{n+1} - p\big\rangle,
\end{aligned} \tag{3.9}$$

and hence

$$\begin{aligned}
(1 - \alpha_n\tau)^2\lambda_n(2\alpha - \lambda_n)\|Au_n - Ap\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n^2\tau^2\|x_n - p\|^2 \\
&\quad + 2\alpha_n\big\langle \gamma g(x_n) - Vp, x_{n+1} - p\big\rangle.
\end{aligned} \tag{3.10}$$

Replacing n by n_{k_i} in (3.10), we have

$$\begin{aligned}
&(1 - \alpha_{n_{k_i}}\tau)^2\lambda_{n_{k_i}}(2\alpha - \lambda_{n_{k_i}})\|Au_{n_{k_i}} - Ap\|^2 \\
&\quad \leq \|x_{n_{k_i}} - p\|^2 - \|x_{n_{k_i}+1} - p\|^2 + \alpha_{n_{k_i}}^2\tau^2\|x_{n_{k_i}} - p\|^2 \\
&\qquad + 2\alpha_{n_{k_i}}\big\langle \gamma g(x_n) - Vp, x_{n_{k_i}+1} - p\big\rangle.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $0 < a \leq \lambda_n \leq b < 2\alpha$ and the existence of $\lim_{i\to\infty}\|x_{n_{k_i}} - p\|$, we have

$$\lim_{i\to\infty}\|Au_{n_{k_i}} - Ap\| = 0. \tag{3.11}$$

We also have from (1.16) that

$$\begin{aligned}
2\|u_n - p\|^2 &= 2\|T_{r_n}x_n - T_{r_n}p\|^2 \\
&\leq 2\langle x_n - p, u_n - p\rangle \\
&= \|x_n - p\|^2 + \|u_n - p\|^2 - \|u_n - x_n\|^2,
\end{aligned}$$

and hence

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|u_n - x_n\|^2. \tag{3.12}$$

From (3.3), (3.4), (3.6) and (3.12), we obtain the following:

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \bigl\|(I-\alpha_n V)y_n-(I-\alpha_n V)p\bigr\|^2+2\alpha_n\bigl\langle\gamma g(x_n)-Vp,x_{n+1}-p\bigr\rangle\\
&\le (1-\alpha_n\tau)^2\|y_n-p\|^2+2\alpha_n\bigl\langle\gamma g(x_n)-Vp,x_{n+1}-p\bigr\rangle\\
&\le (1-\alpha_n\tau)^2\|z_n-p\|^2+2\alpha_n\bigl\langle\gamma g(x_n)-Vp,x_{n+1}-p\bigr\rangle\\
&\le (1-\alpha_n\tau)^2\bigl\{\|u_n-p\|^2-\lambda_n(2\alpha-\lambda_n)\|Au_n-Ap\|^2\bigr\}\\
&\quad+2\alpha_n\bigl\langle\gamma g(x_n)-Vp,x_{n+1}-p\bigr\rangle\\
&\le (1-\alpha_n\tau)^2\bigl\{\|x_n-p\|^2-\|u_n-x_n\|^2\bigr\}\\
&\quad-(1-\alpha_n\tau)^2\lambda_n(2\alpha-\lambda_n)\|Au_n-Ap\|^2\\
&\quad+2\alpha_n\bigl\langle\gamma g(x_n)-Vp,x_{n+1}-p\bigr\rangle\\
&\le \bigl(1-2\alpha_n\tau+\alpha_n^2\tau^2\bigr)\|x_n-p\|^2-(1-\alpha_n\tau)^2\|u_n-x_n\|^2\\
&\quad-(1-\alpha_n\tau)^2\lambda_n(2\alpha-\lambda_n)\|Au_n-Ap\|^2\\
&\quad+2\alpha_n\bigl\langle\gamma g(x_n)-Vp,x_{n+1}-p\bigr\rangle\\
&\le \|x_n-p\|^2+\alpha_n^2\tau^2\|x_n-p\|^2-(1-\alpha_n\tau)^2\|u_n-x_n\|^2\\
&\quad-(1-\alpha_n\tau)^2\lambda_n(2\alpha-\lambda_n)\|Au_n-Ap\|^2\\
&\quad+2\alpha_n\bigl\langle\gamma g(x_n)-Vp,x_{n+1}-p\bigr\rangle,
\end{aligned}$$

and hence

$$\begin{aligned}
(1-\alpha_n\tau)^2\|u_n-x_n\|^2 &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n^2\tau^2\|x_n-p\|^2\\
&\quad-(1-\alpha_n\tau)^2\lambda_n(2\alpha-\lambda_n)\|Au_n-Ap\|^2\\
&\quad+2\alpha_n\bigl\langle\gamma g(x_n)-Vp,x_{n+1}-p\bigr\rangle.
\end{aligned}\tag{3.13}$$

Replacing n by n_{k_i} in (3.13), we have

$$\begin{aligned}
(1-\alpha_{n_{k_i}}\tau)^2\|u_{n_{k_i}}-x_{n_{k_i}}\|^2 &\le \|x_{n_{k_i}}-p\|^2-\|x_{n_{k_i}+1}-p\|^2+\alpha_{n_{k_i}}^2\tau^2\|x_{n_{k_i}}-p\|^2\\
&\quad-(1-\alpha_{n_{k_i}}\tau)^2\lambda_{n_{k_i}}(2\alpha-\lambda_{n_{k_i}})\|Au_{n_{k_i}}-Ap\|^2\\
&\quad+2\alpha_{n_{k_i}}\bigl\langle\gamma g(x_{n_{k_i}})-Vp,x_{n_{k_i}+1}-p\bigr\rangle.
\end{aligned}$$

From (3.11), $\lim_{n\to\infty}\alpha_n=0$ and the existence of $\lim_{i\to\infty}\|x_{n_{k_i}}-p\|$, we have

$$\lim_{i\to\infty}\|u_{n_{k_i}}-x_{n_{k_i}}\|=0. \tag{3.14}$$

On the other hand, since J_{λ_n} is firmly nonexpansive and $u_n=T_{r_n}x_n$, we have that

$$\begin{aligned}
\|z_n-p\|^2 &= \bigl\|J_{\lambda_n}(I-\lambda_nA)u_n-J_{\lambda_n}(I-\lambda_nA)p\bigr\|^2\\
&\le \bigl\langle z_n-p,(I-\lambda_nA)u_n-(I-\lambda_nA)p\bigr\rangle\\
&= \frac{1}{2}\bigl(\|z_n-p\|^2+\bigl\|(I-\lambda_nA)u_n-(I-\lambda_nA)p\bigr\|^2
\end{aligned}$$

$$
\begin{aligned}
&- \|z_n - p - (I - \lambda_n A)u_n + (I - \lambda_n A)p\|^2\big) \\
&\le \frac{1}{2}\big\{\|z_n - p\|^2 + \|u_n - p\|^2 - \|z_n - p - (I - \lambda_n A)u_n + (I - \lambda_n A)p\|^2\big\} \\
&\le \frac{1}{2}\big(\|z_n - p\|^2 + \|x_n - p\|^2 - \|z_n - u_n\|^2 - 2\lambda_n\langle z_n - u_n, Au_n - Ap\rangle \\
&\quad - \lambda_n^2\|Au_n - Ap\|^2\big),
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\|z_n - p\|^2 \\
&\quad \le \|x_n - p\|^2 - \|z_n - u_n\|^2 - 2\lambda_n\langle z_n - u_n, Au_n - Ap\rangle - \lambda_n^2\|Au_n - Ap\|^2. \qquad (3.15)
\end{aligned}
$$

From (3.3), (3.4), (3.6) and (3.15), we have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\le \|(I - \alpha_n V)y_n - (I - \alpha_n V)p\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vp, x_{n+1} - p\rangle \\
&\le (1 - \alpha_n\tau)^2\|y_n - p\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vp, x_{n+1} - p\rangle \\
&\le (1 - \alpha_n\tau)^2\|z_n - p\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vp, x_{n+1} - p\rangle \\
&\le (1 - \alpha_n\tau)^2\big(\|x_n - z\|^2 - \|z_n - u_n\|^2 - 2\lambda_n\langle z_n - u_n, Au_n - Ap\rangle \\
&\quad - \lambda_n^2\|Au_n - Ap\|^2\big) + 2\alpha_n\langle \gamma g(x_n) - Vp, x_{n+1} - p\rangle \\
&\le \|x_n - p\|^2 + \alpha_n^2\tau^2\|x_n - p\|^2 - (1 - \alpha_n\tau)^2\|z_n - u_n\| \\
&\quad - 2(1 - \alpha_n\tau)^2\lambda_n(\lambda_n - 2\alpha)\|z_n - u_n\|\|Au_n - Ap\| \\
&\quad - (1 - \alpha_n\tau)^2\lambda_n^2\|Au_n - Ap\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vp, x_{n+1} - p\rangle,
\end{aligned}
$$

and hence

$$
\begin{aligned}
&(1 - \alpha_n\tau)^2\|z_n - u_n\| \\
&\quad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n^2\tau^2\|x_n - p\|^2 \\
&\quad\quad - 2(1 - \alpha_n\tau)^2\lambda_n(\lambda_n - 2\alpha)\|z_n - u_n\|\|Au_n - Ap\| \\
&\quad\quad - (1 - \alpha_n\tau)^2\lambda_n^2\|Au_n - Ap\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vp, x_{n+1} - p\rangle. \qquad (3.16)
\end{aligned}
$$

Replacing n by n_{k_i} in (3.16), we have

$$
\begin{aligned}
&(1 - \alpha_{n_{k_i}}\tau)^2\|z_{n_{k_i}} - u_{n_{k_i}}\|^2 \\
&\quad \le \|x_{n_{k_i}} - p\|^2 - \|x_{n_{k_i}+1} - p\|^2 + \alpha_{n_{k_i}}^2\tau^2\|x_{n_{k_i}} - p\|^2 \\
&\quad\quad - 2(1 - \alpha_{n_{k_i}}\tau)^2\lambda_{n_{k_i}}(\lambda_{n_{k_i}} - 2\alpha)\|z_{n_{k_i}} - u_{n_{k_i}}\|\|Au_{n_{k_i}} - Ap\| \\
&\quad\quad - (1 - \alpha_{n_{k_i}}\tau)^2\lambda_{n_{k_i}}^2\|Au_{n_{k_i}} - Ap\|^2 + 2\alpha_{n_{k_i}}\langle \gamma g(x_{n_{k_i}}) - Vp, x_{n_{k_i}+1} - p\rangle.
\end{aligned}
$$

From (3.11), $\lim_{n\to\infty}\alpha_n = 0$ and the existence of $\lim_{i\to\infty}\|x_{n_{k_i}} - p\|$, we obtain that

$$
\lim_{i\to\infty}\|z_{n_{k_i}} - u_{n_{k_i}}\| = 0. \qquad (3.17)
$$

Since $\|z_{n_{k_i}} - x_{n_{k_i}}\| \leq \|z_{n_{k_i}} - u_{n_{k_i}}\| + \|u_{n_{k_i}} - x_{n_{k_i}}\|$, by (3.14) and (3.17), we obtain that

$$\lim_{i\to\infty} \|z_{n_{k_i}} - x_{n_{k_i}}\| = 0. \tag{3.18}$$

Since A is Lipschitz continuous, we also obtain

$$\lim_{i\to\infty} \|Az_{n_{k_i}} - Ax_{n_{k_i}}\| = 0. \tag{3.19}$$

Since $z_n = J_\lambda(I - \lambda A)u_n$, we have that

$$\begin{aligned} z_n &= (I + \lambda_n B)^{-1}(I - \lambda_n A)u_n \\ &\Leftrightarrow \quad (I - \lambda_n A)u_n \in (I + \lambda_n B)z_n = z_n + \lambda_n Bz_n \\ &\Leftrightarrow \quad u_n - z_n - \lambda_n Au_n \in \lambda_n Bz_n \\ &\Leftrightarrow \quad \frac{1}{\lambda_n}(u_n - z_n - \lambda_n Au_n) \in Bz_n. \end{aligned}$$

Since B is monotone, we have that for $(u, v) \in B$,

$$\left\langle z_n - u, \frac{1}{\lambda_n}(u_n - z_n - \lambda_n Au_n) - v \right\rangle \geq 0,$$

and hence

$$\langle z_n - u, u_n - z_n - \lambda_n(Au_n + v)\rangle \geq 0. \tag{3.20}$$

Replacing n by n_{k_i} in (3.20), we have that

$$\langle z_{n_{k_i}} - u, u_{n_{k_i}} - z_{n_{k_i}} - \lambda_{n_{k_i}}(Au_{n_{k_i}} + v)\rangle \geq 0. \tag{3.21}$$

Since $x_{n_{k_i}} \rightharpoonup w$ and $x_{n_{k_i}} - u_{n_{k_i}} \to 0$, so $u_{n_{k_i}} \rightharpoonup w$. From (3.17), we get that $z_{n_{k_i}} \rightharpoonup w$, together with (3.21), we have that

$$\langle w - u, -Aw - v\rangle \geq 0.$$

Since B is maximal monotone, $(-Aw) \in Bw$, that is, $w \in (A + B)^{-1}0$.

(c) Next, we show that $w \in F^{-1}0$. Since F is a maximal monotone operator, we have from (1.15) that $A_{r_{n_{k_i}}} x_{n_{k_i}} \in FT_{r_{n_{k_i}}} x_{n_{k_i}}$, where A_r is the Yosida approximation of F for $r > 0$. Furthermore, we have that for any $(u, v) \in F$,

$$\left\langle u - u_{n_{k_i}}, v - \frac{x_{n_{k_i}} - u_{n_{k_i}}}{r_{n_{k_i}}} \right\rangle \geq 0.$$

Since $\liminf_{n\to\infty} r_n > 0$, $u_{n_{k_i}} \rightharpoonup w$ and $x_{n_{k_i}} - u_{n_{k_i}} \to 0$, we have

$$\langle u - w, v\rangle \geq 0.$$

Since F is a maximal monotone operator, we have $0 \in Fw$, that is, $w \in F^{-1}0$. By (a), (b) and (c), we conclude that

$$w \in F(T) \cap (A+B)^{-1}0 \cap F^{-1}0.$$

Using (3.7), we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle (V-\gamma g)p_0, x_n - p_0\rangle &= \lim_{k\to\infty}\langle (V-\gamma g)p_0, x_{n_k} - p_0)\rangle \\ &= \langle (V-\gamma g)p_0, w - p_0)\rangle \geq 0.\end{aligned}$$

Finally, we prove that $x_n \to p_0$. Notice that

$$x_{n+1} - p_0 = \alpha_n\bigl(\gamma g(x_n) - p_0\bigr) + (I - \alpha_n V)y_n - (I - \alpha_n V)p_0,$$

we have

$$\begin{aligned}\|x_{n+1} - p_0\|^2 &\leq (1-\alpha_n\tau)^2\|y_n - p_0\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vp_0, x_{n+1} - p_0\rangle \\ &\leq (1-\alpha_n\tau)^2\|x_n - p_0\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vp_0, x_{n+1} - p_0\rangle \\ &\leq (1-\alpha_n\tau)^2\|x_n - p_0\|^2 + 2\alpha_n\gamma k\|x_n - p_0\|\|x_{n+1} - p_0\| \\ &\quad + 2\alpha_n\langle \gamma g(p_0) - Vp_0, x_{n+1} - p_0\rangle \\ &\leq (1-\alpha_n\tau)^2\|x_n - p_0\|^2 + \alpha_n\gamma k\bigl(\|x_n - p_0\|^2 + \|x_{n+1} - p_0\|^2\bigr) \\ &\quad + 2\alpha_n\langle \gamma g(p_0) - Vp_0, x_{n+1} - p_0\rangle \\ &\leq \bigl\{(1-\alpha_n\tau)^2 + \alpha_n\gamma k\bigr\}\|x_n - p_0\|^2 + \alpha_n\gamma k\|x_{n+1} - p_0\|^2 \\ &\quad + 2\alpha_n\langle \gamma g(p_0) - Vp_0, x_{n+1} - p_0\rangle,\end{aligned}$$

and hence

$$\begin{aligned}\|x_{n+1} - p_0\|^2 &\leq \frac{1 - 2\alpha_n\tau + (\alpha_n\tau)^2 + \alpha_n\gamma k}{1-\alpha_n\gamma k}\|x_n - p_0\|^2 \\ &\quad + \frac{2\alpha_n}{1-\alpha_n\gamma k}\langle \gamma g(p_0) - Vp_0, x_{n+1} - p_0\rangle \\ &= \left\{1 - \frac{2(\tau-\gamma k)\alpha_n}{1-\alpha_n\gamma k}\right\}\|x_n - p_0\|^2 + \frac{(\alpha_n\tau)^2}{1-\alpha_n\gamma k}\|x_n - p_0\|^2 \\ &\quad + \frac{2\alpha_n}{1-\alpha_n\gamma k}\langle \gamma g(p_0) - Vp_0, x_{n+1} - p_0\rangle \\ &= \left\{1 - \frac{2(\tau-\gamma k)\alpha_n}{1-\alpha_n\gamma k}\right\}\|x_n - p_0\|^2 + \frac{\alpha_n\cdot\alpha_n\tau^2}{1-\alpha_n\gamma k}\|x_n - p_0\|^2 \\ &\quad + \frac{2\alpha_n}{1-\alpha_n\gamma k}\langle \gamma g(p_0) - Vp_0, x_{n+1} - p_0\rangle \\ &= (1-\beta_n)\|x_n - p_0\|^2 \\ &\quad + \beta_n\left\{\frac{\alpha_n\tau^2\|x_n - p_0\|^2}{2(\tau-\gamma k)} + \frac{1}{\tau-\gamma k}\langle \gamma g(p_0) - Vp_0, x_{n+1} - p_0\rangle\right\},\end{aligned} \tag{3.22}$$

where $\beta_n = \frac{2(\tau-\gamma k)\alpha_n}{1-\alpha_n\gamma k}$. Since $\sum_{n=1}^{\infty}\beta_n = \infty$, we have from Lemma 2.5 and (3.22) that $x_n \to p_0$. This completes the proof. □

4 Applications

Let H be a Hilbert space, and let f be a proper lower semicontinuous convex function of H into $(-\infty,\infty]$. Then the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{z \in H : f(x) + \langle z, y-x\rangle \le f(y), y \in H\}$$

for all $x \in H$; see, for instance, [36]. From Rockafellar [37], we know that ∂f is maximal monotone. Let C be a nonempty closed convex subset of H, and let i_C be the indicator function of C, *i.e.*,

$$i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

Then i_C is a proper lower semicontinuous convex function of H into $(-\infty,\infty]$, and then the subdifferential ∂_{i_C} of i_C is a maximal monotone operator. So, we can define the resolvent J_λ of ∂_{i_C} for $\lambda > 0$, *i.e.*,

$$J_\lambda x = (I + \lambda\partial_{i_C})^{-1}x$$

for all $x \in H$. We have that for any $x \in H$ and $u \in C$,

$$\begin{aligned} u = J_\lambda x \quad &\Leftrightarrow \quad x \in u + \lambda\partial_{i_C}u \\ &\Leftrightarrow \quad x \in u + \lambda N_C u \\ &\Leftrightarrow \quad x - u \in \lambda N_C u \\ &\Leftrightarrow \quad \frac{1}{\lambda}\langle x-u, v-u\rangle \le 0, \quad \forall v \in C \\ &\Leftrightarrow \quad \langle x-u, v-u\rangle \le 0, \quad \forall v \in C \\ &\Leftrightarrow \quad u = P_C x, \end{aligned}$$

where $N_C u$ is the normal cone to C at u, *i.e.*,

$$N_C u = \{x \in H : \langle z, v-u\rangle \le 0, \forall v \in C\}.$$

Let C be a nonempty, closed and convex subset of H, and let $f : C \times C \to \mathbb{R}$ be a bifunction. For solving the equilibrium problem, let us assume that the bifunction $f : C \times C \to \mathbb{R}$ satisfies the following conditions.

For solving the mixed equilibrium problem, let us give the following assumptions for the bifunction F, φ and the set C:

(A1) $f(x,x) = 0$ for all $x \in C$;

(A2) f is monotone, *i.e.*, $f(x,y) + f(y,x) \le 0$ for any $x, y \in C$;

(A3) for all $x, y, z \in C$,

$$\limsup_{t\downarrow 0} f\big(tz + (1-t)x, y\big) \le f(x,y);$$

(A4) for all $x \in C, f(x,\cdot)$ is convex and lower semicontinuous;

(B1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$f(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z);$$

(B2) C is a bounded set.

We know the following lemma which appears implicitly in Blum and Oettli [1].

Lemma 4.1 [1] *Let C be a nonempty closed convex subset of H, and let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A5). *Let $r > 0$ and $x \in H$. Then there exists a unique $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C.$$

By a similar argument as that in [38, Lemma 2.3], we have the following result.

Lemma 4.2 [38] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions* (A1)-(A4), *and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either* (B1) *or* (B2) *holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows*:

$$T_r(x) = \left\{ z \in C : f(z,y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \ge \varphi(z), \forall y \in C \right\}$$

for all $x \in H$. Then following conclusions hold:

(1) *For each $x \in H$, $T_r(x) \neq \emptyset$*;

(2) *T_r is single-valued*;

(3) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$*,

$$\big\| T_r(x) - T_r(y)\big\|^2 \le \big\langle T_r(x) - T_r(y), x - y\big\rangle;$$

(4) $\mathrm{Fix}(T_r) = MEP(f,\varphi)$;

(5) *$MEP(f,\varphi)$ is closed and convex.*

We call such T_r the resolvent of f for $r > 0$. Using Lemmas 4.1 and 4.2, Takahashi *et al.* [22] obtained the following lemma. See [39] for a more general result.

Lemma 4.3 [22] *Let H be a Hilbert space, and let C be a nonempty closed convex subset of H. Let $f : C \times C \to \mathbb{R}$ satisfy* (A1)-(A5). *Let A_f be a set-valued mapping of H into itself defined by*

$$A_f x = \begin{cases} \{z \in H : f(x,y) \ge \langle y - x, z\rangle, \forall y \in C\}, & \forall x \in C, \\ \emptyset, & \forall x \notin C. \end{cases}$$

Then $MEP(f) = A_f^{-1}0$ and A_f is a maximal monotone operator with $\operatorname{dom} A_f \subset C$. *Furthermore, for any $x \in H$ and $r > 0$, the resolvent T_r of f coincides with the resolvent of A_f, i.e.,*

$$T_r x = (I + rA_f)^{-1}x.$$

Applying the idea of the proof in Lemma 4.3, we have the following results.

Lemma 4.4 *Let H be a Hilbert space, and let C be a nonempty closed convex subset of H. Let $f : C \times C \to \mathbb{R}$ satisfy* (A1)-(A4), *and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either* (B1) *or* (B2) *hold. Let $A_{(f,\varphi)}$ be a set-valued mapping of H into itself defined by*

$$A_{(f,\varphi)}x = \begin{cases} \{z \in H : f(x,y) + \varphi(y) - \varphi(x) \geq \langle y - x, z\rangle, \forall y \in C\}, & \forall x \in C, \\ \emptyset, & \forall x \notin C. \end{cases} \tag{4.1}$$

Then $MEP(f,\varphi) = A_{(f,\varphi)}^{-1}0$ and $A_{(f,\varphi)}$ is a maximal monotone operator with $\operatorname{dom} A_{(f,\varphi)} \subset C$. *Furthermore, for any $x \in H$ and $r > 0$, the resolvent T_r of f coincides with the resolvent of $A_{(f,\varphi)}$, i.e.,*

$$T_r x = (I + rA_{(f,\varphi)})^{-1}x.$$

Proof It is obvious that $MEP(f,\varphi) = A_{(f,\varphi)}^{-1}0$. In fact, we have that

$$\begin{aligned} z \in MEP(f,\varphi) \quad &\Leftrightarrow \quad f(z,y) + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C \\ &\Leftrightarrow \quad f(z,y) + \varphi(y) - \varphi(z) \geq \langle y - z, 0\rangle, \quad \forall y \in C \\ &\Leftrightarrow \quad 0 \in A_{(f,\varphi)}z \\ &\Leftrightarrow \quad z \in A_{(f,\varphi)}^{-1}0. \end{aligned}$$

We show that $A_{(f,\varphi)}$ is monotone. Let $(x_1,z_1), (x_2,z_2) \in A_{(f,\varphi)}$ be given. Then we have, for all $y \in C$,

$$f(x_1,y) + \varphi(y) - \varphi(x_1) \geq \langle y - x_1, z_1\rangle \quad \text{and} \quad f(x_2,y) + \varphi(y) - \varphi(x_2) \geq \langle y - x_2, z_2\rangle,$$

and hence

$$f(x_1,x_2) + \varphi(x_2) - \varphi(x_1) \geq \langle x_2 - x_1, z_1\rangle \quad \text{and} \quad f(x_2,x_1) + \varphi(x_1) - \varphi(x_2) \geq \langle x_1 - x_2, z_2\rangle.$$

It follows from (A2) that

$$0 \geq f(x_1,x_2) + f(x_2,x_1) \geq \langle x_2 - x_1, z_1\rangle + \langle x_1 - x_2, z_2\rangle = -\langle x_1 - x_2, z_1 - z_2\rangle.$$

This implies that $A_{(f,\varphi)}$ is monotone. We next prove that $A_{(f,\varphi)}$ is maximal monotone. To show that $A_{(f,\varphi)}$ is maximal monotone, it is sufficient to show from [33] that $R(I + rA_{(f,\varphi)}) =$

H for all $r > 0$, where $R(I + rA_{(f,\varphi)})$ is the range of $I + rA_{(f,\varphi)}$. Let $x \in H$ and $r > 0$. Then, from Lemma 4.2, there exists $z \in C$ such that

$$f(z,y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

So, we have that

$$f(z,y) + \varphi(y) - \varphi(z) \geq \left\langle y - z, \frac{1}{r}(x - z)\right\rangle, \quad \forall y \in C.$$

By the definition of $A_{(f,\varphi)}$, we get

$$A_{(f,\varphi)}z \ni \frac{1}{r}(x - z),$$

and hence $x \in z + rA_{(f,\varphi)}z$. Therefore, $H \subset R(I + rA_{(f,\varphi)})$ and $R(I + rA_{(f,\varphi)}) = H$. Also, $x \in z + rA_{(f,\varphi)}z$ implies that $T_r x = (I + rA_{(f,\varphi)})^{-1}x$ for all $x \in H$ and $r > 0$. □

Using Theorem 3.1, we obtain the following results for an inverse-strongly monotone mapping.

Theorem 4.5 *Let H be a real Hilbert space, and let C be a nonempty, closed and convex subset of H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H. Let $0 < k < 1$ and let g be a k-contraction of H into itself. Let V be a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous operator with $\bar{\gamma} > 0$ and $L > 0$. Let $T : C \to C$ be a 2-generalized hybrid mapping such that $\Gamma := F(T) \cap VI(C,A) \neq \emptyset$. Take $\mu, \gamma \in \mathbb{R}$ as follows*:

$$0 < \mu < \frac{2\bar{\gamma}}{L^2}, \qquad 0 < \gamma < \frac{\bar{\gamma} - \frac{L^2\mu}{2}}{k}.$$

Let $\{x_n\} \subset H$ be a sequence generated by

$$\begin{cases} x_1 = x \in H \quad \textit{arbitrarily}, \\ z_n = P_C(I - \lambda_n A)P_C x_n, \\ y_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k z_n, \quad \forall n = 1, 2, \ldots, \\ x_{n+1} = \alpha_n \gamma g(x_n) + (I - \alpha_n V)y_n \quad \textit{for all } n \in \mathbb{N}, \end{cases} \tag{4.2}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \textit{and} \quad \liminf_{n\to\infty} r_n > 0.$$

Then $\{x_n\}$ converges strongly to a point p_0 of Γ, where p_0 is a unique fixed point of $P_\Gamma(I - V + \gamma g)$. This point $p_0 \in \Gamma$ is also a unique solution of the hierarchical variational inequality

$$\langle (V - \gamma g)p_0, q - p_0\rangle \geq 0, \quad \forall q \in VI(C,A). \tag{4.3}$$

Proof Put $B = F = \partial i_C$ in Theorem 3.1. Then, for $\lambda_n > 0$ and $r_n > 0$, we have that

$$J_{\lambda_n} = T_{r_n} = P_C.$$

Furthermore we have, from the proof of [32, Theorem 12], that

$$(\partial i_C)^{-1}0 = C \quad \text{and} \quad (A + \partial i_C)^{-1} = VI(C, A).$$

Thus we obtained the desired results by Theorem 3.1. □

Using Theorem 3.1, we finally prove a strong convergence theorem for inverse-strongly monotone operators and equilibrium problems in a Hilbert space.

Theorem 4.6 *Let H be a real Hilbert space, and let C be a nonempty, closed and convex subset of H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H. Let $B : D(B) \subset C \to 2^H$ be maximal monotone. Let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $0 < k < 1$ and let g be a k-contraction of H into itself. Let V be a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous operator with $\bar{\gamma} > 0$ and $L > 0$. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions* (A1)-(A4), *and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either* (B1) *or* (B2) *holds. Let $T : C \to C$ be a 2-generalized hybrid mapping with $\Theta := F(T) \cap (A + B)^{-1}0 \cap MEP(f, \varphi) \neq \emptyset$. Take $\mu, \gamma \in \mathbb{R}$ as follows*:

$$0 < \mu < \frac{2\bar{\gamma}}{L^2}, \qquad 0 < \gamma < \frac{\bar{\gamma} - \frac{L^2\mu}{2}}{k}.$$

Let $\{x_n\} \subset H$ be a sequence generated by

$$\begin{cases} x_1 = x \in H \quad \textit{arbitrarily}, \\ f(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ z_n = J_{\lambda_n}(I - \lambda_n A)u_n, \\ y_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k z_n, \quad \forall n = 1, 2, \ldots, \\ x_{n+1} = \alpha_n \gamma g(x_n) + (I - \alpha_n V)y_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{4.4}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \textit{and} \quad \liminf_{n\to\infty} r_n > 0.$$

Then $\{x_n\}$ converges strongly to a point p_0 of Θ, where p_0 is a unique fixed point of $P_\Theta(I - V + \gamma g)$. This point $p_0 \in \Theta$ is also a unique solution of the hierarchical variational inequality

$$\big\langle (V - \gamma g)p_0, q - p_0\big\rangle \geq 0, \quad \forall q \in \Theta. \tag{4.5}$$

Proof Since f is a bifunction of $C \times C$ into $\mathbb{R}$ satisfying conditions (A1)-(A4) and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous and convex function, we have that the mapping

A_f^φ defined by (4.1) is a maximal monotone operator with $\operatorname{dom} A_f^\varphi \subset C$. Put $F = A_f^\varphi$ in Theorem 3.1. Then we obtain that $u_n = T_{r_n} x_n$. Therefore, we arrive at the desired results. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors read and approved the final manuscript.

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