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A new iterative scheme with nonexpansive mappings for equilibrium problems

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Abstract

In this paper, we suggest a new iteration scheme for finding a common of the solution set of monotone, Lipschitz-type continuous equilibrium problems and the set of fixed points of a nonexpansive mapping. The scheme is based on both hybrid method and extragradient-type method. We obtain a strong convergence theorem for the sequences generated by these processes in a real Hilbert space. Based on this result, we also get some new and interesting results. The results in this paper generalize, extend, and improve some well-known results in the literature.

AMS 2010 Mathematics subject classification: 65 K10, 65 K15, 90 C25, 90 C33.

Keywords: Equilibrium problems, nonexpansive mappings, monotone, Lipschitz-type continuous, fixed point

1 Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $||\cdot||$. Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. A mapping $S : C \to C$ is a *contraction* with a constant $\delta \in (0, 1)$, if

$$||S(x) - S(y)|| \le \delta||x - y||, \ \forall x, \ y \in C.$$

If $\delta = 1$, then S is called *nonexpansive* on C. $Fix(S)$ is denoted by the set of fixed points of S. Let $f : C \times C \to \mathcal{R}$ be a bifunction such that $f(x, x) = 0$ for all $x \in C$. We consider the equilibrium problem in the sense of Blum and Oettli (see [1]) which is presented as follows:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \ge 0 \text{ for all } y \in C. \qquad EP(f, C)$$

The set of solutions of $EP(f, C)$ is denoted by $Sol(f, C)$. The bifunction f is called *strongly monotone* on C with $\beta > 0$, if

$$f(x, y) + f(y, x) \le -\beta||x - y||^2, \ \forall x, \ y \in C;$$

monotone on C, if

$$f(x, y) + f(y, x) \le 0, \ \forall x, \ y \in C;$$

pseudomonotone on C, if

$$f(x, y) \ge 0 \text{ implies } f(y, x) \le 0, \ \forall x, y \in C;$$

Lipschitz-type continuous on C with constants $c_1 > 0$ and $c_2 > 0$ (see [2]), if

$$f(x,\ y) + f(y,\ z) \geq f(x,\ z) - c_1||x - y||^2 - c_2||y - z||^2,\ \forall x,\ y,\ z \in C.$$

It is well-known that Problem *EP*(f, C) includes, as particular cases, the optimization problem, the variational inequality problem, the Nash equilibrium problem in noncooperative games, the fixed point problem, the nonlinear complementarity problem and the vector minimization problem (see [2-6]).

In recent years, the problem to find a common point of the solution set of problem (*EP*) and the set of fixed points of a nonexpansive mapping becomes an attractive field for many researchers (see [7-15]). An important special case of equilibrium problems is the variational inequalities (shortly (*VIP*)), where $F : C \to \mathcal{H}$ and $f(x, y) = \langle F(x), y - x\rangle$. Various methods have been developed for finding a common point of the solution set of problem (*VIP*) and the set of fixed points of a nonexpansive mapping when F is monotone (see [16-18]).

Motivated by fixed point techniques of Takahashi and Takahashi in [19] and an improvement set of extragradient-type iteration methods in [20], we introduce a new iteration algorithm for finding a common of the solution set of equilibrium problems with a *monotone* and *Lipschitz-type continuous* bifunction and the set of fixed points of a nonexpansive mapping. We show that all of the iterative sequences generated by this algorithm convergence strongly to the common element in a real Hilbert space.

2 Preliminaries

Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. We write $x^n \rightharpoonup x$ to indicate that the sequence $\{x^n\}$ converges weakly to x as $n \to \infty$, $x^n \to x$ implies that $\{x^n\}$ converges strongly to x. For any $x \in \mathcal{H}$, there exists a nearest point in C, denoted by $Pr_C(x)$, such that

$$||x - Pr_C(x)|| \ \leq \ ||x - y||,\ \forall y \in C.$$

Pr_C is called the metric projection of $\mathcal{H}$ to C. It is well known that Pr_C satisfies the following properties:

$$\langle x - y, Pr_C(x) - Pr_C(y)\rangle \geq \ ||Pr_C(x) - Pr_C(y)||^2,\ \forall x, y \in \mathcal{H}, \tag{2.1}$$

$$\langle x - Pr_C(x), Pr_C(x) - y >\rangle \geq 0,\ \forall x \in \mathcal{H}, y \in C, \tag{2.2}$$

$$||x - y||^2 \geq \ ||x - Pr_C(x)||^2 + ||y - Pr_C(x)||^2,\ \forall x \in \mathcal{H}, y \in C. \tag{2.3}$$

Let us assume that a bifunction $f : C \times C \to \mathcal{R}$ and a nonexpansive mapping $S : C \to C$ satisfy the following conditions:

A_1. f is Lipschitz-type continuous on C;

A_2. f is monotone on C;

A_3. for each $x \in C$, $f(x, \cdot)$ is subdifferentiable and convex on C;

A_4. $Fix(S) \cap Sol(f, C) \neq \emptyset$.

Recently, Takahashi and Takahashi in [19] first introduced an iterative scheme by the viscosity approximation method. The sequence $\{x^k\}$ is defined by:

$$\begin{cases} x^0 \in \mathcal{H}, \\ \text{Find } u^k \in C \text{ such that } f(u^k, y) + \frac{1}{r_k}\langle y - u^k, u^k - x^k\rangle \ge 0, \quad \forall y \in C, \\ x^{k+1} = \alpha_k g(x^k) + (1-\alpha_k)S(u^k), \ \forall k \ge 0, \end{cases}$$

where C is a nonempty closed convex subset of $\mathcal{H}$ and g is a contractive mapping of $\mathcal{H}$ into itself. The authors showed that under certain conditions over $\{\alpha_k\}$ and $\{r_k\}$, sequences $\{x_k\}$ and $\{u_k\}$ converge strongly to $z = Pr_{Sol(f,C)\cap Fix(S)}(g(x^0))$. Recently, iterative methods for finding a common element of the set of solutions of equilibrium problems and the set of fixed points of a nonexpansive mapping have further developed by many authors. These methods require to solve approximation auxilary equilibrium problems.

In this paper, we introduce a new iteration method for finding a common point of the set of fixed points of a nonexpansive mapping S and the set of solutions of problem $EP(f, C)$. At each our iteration, the main steps are to solve two strongly convex problems

$$\begin{cases} y^k = \operatorname{argmin}\{\lambda_k f(x^k, y) + \frac{1}{2}||y - x^k||^2 : y \in C\}, \\ t^k = \operatorname{argmin}\{\lambda_k f(y^k, y) + \frac{1}{2}||y - x^k||^2 : y \in C\}, \end{cases} \tag{2.4}$$

and compute the next iteration point by Mann-type fixed points

$$x^{k+1} = \alpha_k g(x^k) + (1-\alpha_k)S(t^k), \tag{2.5}$$

where $g : C \to C$ is a δ-contraction with $0 < \delta < \frac{1}{2}$.

To investigate the convergence of this scheme, we recall the following technical lemmas which will be used in the sequel.

Lemma 2.1 *(see* [21]*) Let* $\{a_n\}$ *be a sequence of nonnegative real numbers such that:*

$$a_{n+1} \le (1-\alpha_n)a_n + \beta_n, n \ge 0,$$

where $\{\alpha_n\}$, *and* $\{\beta_n\}$ *satisfy the conditions:*

(*i*) $\alpha_n \subset (0, 1)$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(*ii*) $\limsup_{n\to\infty} \frac{\beta_n}{\alpha_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\beta_n| < \infty$.

Then

$$\lim_{n\to\infty} a_n = 0.$$

Lemma 2.2 *(*[22]*) Assume that S is a nonexpansive self-mapping of a nonempty closed convex subset C of a real Hilbert space* $\mathcal{H}$*. If* $Fix(S) \ne \emptyset$*, then I - S is demiclosed; that is, whenever* $\{x^k\}$ *is a sequence in C weakly converging to some* $\bar{x} \in C$ *and the sequence* $\{(I - S)(x^k)\}$ *strongly converges to some* $\bar{y}$*, it follows that* $(I-S)(\bar{x}) = \bar{y}$*. Here I is the identity operator of* $\mathcal{H}$*.*

Lemma 2.3 *(see* [20]*, Lemma 3.1) Let C be a nonempty closed convex subset of a real Hilbert space* $\mathcal{H}$*. Let* $f : C \times C \to \mathcal{R}$ *be a pseudomonotone, Lipschitz-type continuous bifunction with constants* $c_1 > 0$ *and* $c_2 > 0$*. For each* $x \in C$*, let* $f(x, \cdot)$ *be convex and subdifferentiable on C. Suppose that the sequences* $\{x^k\}$, $\{y^k\}$, $\{t^k\}$ *generated by Scheme* (2.4) *and* $x^* \in Sol(f, C)$*. Then*

$$||t^k - x^*||^2 \le ||x^k - x^*||^2 - (1-2\lambda_k c_1)||x^k - y^k||^2 - (1-2\lambda_k c_2)||y^k - t^k||^2, \ \forall k \ge 0.$$

3 Main results

Now, we prove the main convergence theorem.

Theorem 3.1 *Suppose that Assumptions A_1-A_4 are satisfied, $x^0 \in C$ and two positive sequences $\{\lambda_k\}$, $\{a_k\}$ satisfy the following restrictions:*

$$\begin{cases} \sum_{k=0}^{\infty} |\alpha_{k+1} - \alpha_k| < \infty, \\ \lim_{k\to\infty} \alpha_k = 0, \\ \sum_{k=0}^{\infty} \alpha_k = \infty, \\ \sum_{k=0}^{\infty} \sqrt{|\lambda_{k+1} - \lambda_k|} < \infty, \\ \{\lambda_k\} \subset [a,\ b] \text{ for some } a,\ b \in (0,\ \frac{1}{L}),\ \text{where } L = \max\{2c_1, 2c_2\}. \end{cases}$$

Then the sequences $\{x^k\}$, $\{y^k\}$ and $\{t^k\}$ generated by (2.4) *and* (2.5) *converge strongly to the same point x^*, where*

$$x^* = Pr_{Fix(S)\cap Sol(f,C)} g(x^*).$$

The proof of this theorem is divided into several steps.

Step 1. Claim that

$$\lim_{k\to\infty} ||x^k - t^k|| = 0.$$

Proof of Step 1. For each $x^* \in Fix(S) \cap Sol(f, C)$, it follows from $x^{k+1} = a_k g(x^k) + (1 - a_k)S(t^k)$, Lemma 2.3 and $\delta \in (0, \frac{1}{2})$ that

$$\begin{aligned} ||x^{k+1} - x^*||^2 &= ||\alpha_k(g(x^k) - x^*) + (1-\alpha_k)(S(t^k) - S(x^*))||^2 \\ &\le \alpha_k||g(x^k) - x^*||^2 + (1-\alpha_k)||S(t^k) - S(x^*)||^2 \\ &= \alpha_k||(g(x^k) - g(x^*)) + (g(x^*) - x^*)||^2 + (1-\alpha_k)||S(t^k) - S(x^*)||^2 \\ &\le 2\delta^2\alpha_k||x^k - x^*||^2 + 2\alpha_k||g(x^*) - x^*||^2 + (1-\alpha_k)||t^k - x^*||^2 \\ &\le 2\delta^2\alpha_k||x^k - x^*||^2 + 2\alpha_k||g(x^*) - x^*||^2 + (1-\alpha_k)||x^k - x^*||^2 \\ &\quad - (1-\alpha_k)(1-2\lambda_k c_1)||x^k - y^k||^2 - (1-\alpha_k)(1-2\lambda_k c_2)||y^k - t^k||^2 \\ &\le ||x^k - x^*||^2 + 2\alpha_k||g(x^*) - x^*||^2 - (1-\alpha_k)(1-2\lambda_k c_1)||x^k - y^k||^2 \\ &\quad - (1-\alpha_k)(1-2\lambda_k c_2)||y^k - t^k||^2. \end{aligned}$$

Then, we have

$$\begin{aligned} (1-\alpha_k)(1-2bc_1)||x^k - y^k||^2 &\le (1-\alpha_k)(1-2\lambda_k c_1)||x^k - y^k||^2 \\ &\le ||x^k - x^*||^2 - ||x^{k+1} - x^*||^2 + 2\alpha_k||g(x^*) - x^*||^2 \\ &\to 0 \text{ as } k \to \infty, \end{aligned}$$

and

$$\lim_{k\to\infty} ||x^k - y^k|| = 0. \tag{3.1}$$

By the similar way, also

$$\lim_{k\to\infty} ||y^k - t^k|| = 0.$$

Combining this, (3.1) and the inequality $||x^k - t^k|| = ||x^k - y^k|| + ||y^k - t^k||$, we have

$$\lim_{k\to\infty} ||x^k - t^k|| = 0. \tag{3.2}$$

Step 2. Claim that

$$\lim_{k\to\infty} ||x^{k+1} - x^k|| = 0.$$

Proof of Step 2. It is easy to see that $t^k = \operatorname{argmin} \{\frac{1}{2}||t - x^k||^2 + \lambda_k f(y^k, t) : t \in C\}$ if and only if

$$0 \in \partial_2(\lambda_k f(y^k, y) + \tfrac{1}{2}||y - x^k||^2)(t^k) + N_C(t^k),$$

where $N_C(x)$ is the (outward) normal cone of C at $x \in C$. This means that $0 = \lambda_k w + t^k - x^k + \bar{w}$, where $w \in \partial_2 f(y^k, t^k)$ and $\bar{w} \in N_C(t^k)$. By the definition of the normal cone N_C we have, from this relation that

$$\langle t^k - x^k, t - t^k\rangle \geq \lambda_k\langle w, t^k - t\rangle \forall t \in C.$$

Substituting $t = t^{k+1}$ into this inequality, we get

$$\langle t^k - x^k, t^{k+1} - t^k\rangle \geq \lambda_k\langle w, t^k - t^{k+1}\rangle. \tag{3.3}$$

Since $f(x, \cdot)$ is convex on C for all $x \in C$, we have

$$f(y^k, t) - f(y^k, t^k) \geq \langle w, t - t^k\rangle \forall t \in C, w \in \partial_2 f(y^k, t^k).$$

Using this and (3.3), we have

$$\begin{aligned} \langle t^k - x^k, t^{k+1} - t^k\rangle &\geq \lambda_k\langle w, t^k - t^{k+1}\rangle \\ &\geq \lambda_k(f(y^k, t^k) - f(y^k, t^{k+1})). \end{aligned} \tag{3.4}$$

By the similar way, we also have

$$\langle t^{k+1} - x^{k+1}, t^k - t^{k+1}\rangle \geq \lambda_{k+1}(f(y^{k+1}, t^{k+1}) - f(y^{k+1}, t^k)). \tag{3.5}$$

Using (3.4), (3.5) and f is Lipschitz-type continuous and monotone, we get

$$\begin{aligned} &\frac{1}{2}||x^{k+1} - x^k||^2 - \frac{1}{2}||t^{k+1} - t^k||^2 \\ &\quad\geq \langle t^{k+1} - t^k, t^k - x^k - t^{k+1} + x^{k+1}\rangle \\ &\quad\geq \lambda_k(f(y^k, t^k) - f(y^k, t^{k+1})) \\ &\quad\quad + \lambda_{k+1}(f(y^{k+1}, t^{k+1}) - f(y^{k+1}, t^k)) \\ &\quad\geq \lambda_k(-f(t^k, t^{k+1}) - c_1||y^k - t^k||^2 - c_2||t^k - t^{k+1}||^2) \\ &\quad\quad + \lambda_{k+1}(-f(t^{k+1}, t^k) - c_1||y^{k+1} - t^{k+1}||^2 - c_2||t^k - t^{k+1}||^2) \\ &\quad\geq (\lambda_{k+1} - \lambda_k)f(t^k, t^{k+1}) \\ &\quad\geq -|\lambda_{k+1} - \lambda_k||f(t^k, t^{k+1})|. \end{aligned}$$

Hence

$$\begin{aligned} ||t^{k+1} - t^k|| &\leq \sqrt{||x^{k+1} - x^k||^2 + 2|\lambda_{k+1} - \lambda_k||f(t^k, t^{k+1})|} \\ &\leq ||x^{k+1} - x^k|| + \sqrt{2|\lambda_{k+1} - \lambda_k||f(t^k, t^{k+1})|} \end{aligned} \tag{3.6}$$

Since (3.6), $a_{k+1} - a_k \to 0$ as $k \to \infty$, g is contractive on C, Lemma 2.3, Step 2 and the definition of x^{k+1} that $x^{k+1} = a_k g(x^k) + a_k S(t^k)$, we have

$$\begin{aligned}
||x^{k+1} - x^k|| &= ||\alpha_k g(x^k) + \alpha_k S(t^k) - \alpha_{k-1} g(x^{k-1}) - \alpha_{k-1} S(t^{k-1})|| \\
&= ||(\alpha_k - \alpha_{k-1})(g(x^{k-1}) - S(t^{k-1})) + (1-\alpha_k)(S(t^k) - S(t^{k-1})) \\
&\quad + \alpha_k (g(x^k) - g(x^{k-1}))|| \\
&\le |\alpha_k - \alpha_{k-1}|||g(x^{k-1}) - S(t^{k-1})|| + (1-\alpha_k)||t^k - t^{k-1}|| + \alpha_k \delta ||x^k - x^{k-1}|| \\
&\le |\alpha_k - \alpha_{k-1}|||g(x^{k-1}) - S(t^{k-1})|| + (1-\alpha_k)(||x^k - x^{k-1}|| \\
&\quad + \sqrt{2|\lambda_k - \lambda_{k-1}||f(t^{k-1}, t^k)|}) + \alpha_k \delta ||x^k - x^{k-1}|| \\
&= (1 - (1-\delta)\alpha_k)||x^k - x^{k-1}|| + |\alpha_k - \alpha_{k-1}|||g(x^{k-1}) - S(t^{k-1})|| \\
&\quad + (1-\alpha_k)\sqrt{2|\lambda_k - \lambda_{k-1}||f(t^{k-1}, t^k)|} \\
&\le (1 - (1-\delta)\alpha_k)||x^k - x^{k-1}|| + M|\alpha_k - \alpha_{k-1}| + K(1-\alpha_k)\sqrt{2|\lambda_k - \lambda_{k-1}|},
\end{aligned}$$

where δ is contractive constant of the mapping g, $M = \sup\{||g(x^{k-1}) - S(t^{k-1})||: k = 0, 1, \ldots\}$ and $K = \sup\left\{\sqrt{|f(t^{k-1}, t^k)|} \ : \ k = 0, 1, \cdots\right\}$, since $\sum_{k=0}^{\infty} |\alpha_k - \alpha_{k-1}| < \infty$ and $\sum_{k=0}^{\infty} \sqrt{|\lambda_k - \lambda_{k-1}|} < \infty$, in view of Lemma 2.1, we have $\lim_{k\to\infty} ||x^{k+1} - x^k|| = 0$.

Step 3. Claim that

$$\lim_{k\to\infty} ||t^k - S(t^k)|| = 0.$$

Proof of Step 3. From $x^{k+1} = a_k g(x^k) + (1 - a_k)S(t^k)$, we have

$$\begin{aligned}
x^{k+1} - x^k &= \alpha_k g(x^k) + (1-\alpha_k)S(t^k) - x^k \\
&= \alpha_k (g(x^k) - x^k) + (1-\alpha_k)(t^k - x^k) + (1-\alpha_k)(S(t^k) - t^k)
\end{aligned}$$

and hence

$$(1-\alpha_k)||S(t^k) - t^k|| \le ||x^{k+1} - x^k|| + \alpha_k ||g(x^k) - x^k|| + (1-\alpha_k)||t^k - x^k||.$$

Using this, $\lim_{k\to\infty} \alpha_k = 0$, Step 1 and Step 2, we have

$$\lim_{k\to\infty} ||t^k - S(t^k)|| = 0.$$

Step 4. Claim that

$$\limsup_{k\to\infty} \langle x^* - g(x^*), S(t^k) - x^* \rangle \ge 0.$$

Proof of Step 4. By Step 1, $\{t^k\}$ is bounded, there exists a subsequence $\{t^{k_i}\}$ of $\{t^k\}$ such that

$$\limsup_{k\to\infty} \langle x^* - g(x^*), t^k - x^* \rangle = \lim_{i\to\infty} \langle x^* - g(x^*), t^{k_i} - x^* \rangle.$$

Since the sequence $\{t^{k_i}\}$ is bounded, there exists a subsequence $\{t^{k_{i_j}}\}$ of $\{t^{k_i}\}$ which converges weakly to $\bar{t}$. Without loss of generality we suppose that the sequence $\{t^{k_i}\}$ converges weakly to $\bar{t}$ such that

$$\limsup_{k\to\infty}\langle x^* - g(x^*), t^k - x^*\rangle = \lim_{i\to\infty}\langle x^* - g(x^*), t^{k_i} - x^*\rangle. \tag{3.7}$$

Since Lemma 2.2 and Step 3, we have

$$S(\bar{t}) = \bar{t} \Leftrightarrow \bar{t} \in Fix(S). \tag{3.8}$$

Now we show that $\bar{t} \in Sol(f, C)$. By Step 1, we also have

$$x^{k_i} \rightharpoonup \bar{t},\ y^{k_i} \rightharpoonup \bar{t}.$$

Since y^k is the unique solution of the strongly convex problem

$$\min\{\tfrac{1}{2}||y - x^k||^2 + f(x^k,\ y) : y \in C\},$$

we have

$$0 \in \partial_2(\lambda_k f(x^k,\ y) + \frac{1}{2}||y - x^k||^2)(y^k) + N_C(y^k).$$

This follows that

$$0 = \lambda_k w + y^k - x^k + w^k,$$

where $w \in \partial_2 f(x^k, y^k)$ and $w^k \in N_C(y^k)$. By the definition of the normal cone N_C, we have

$$\langle y^k - x^k, y - y^k\rangle \geq \lambda_k\langle w, y^k - y\rangle,\ \forall y \in C. \tag{3.9}$$

On the other hand, since $f(x^k, \cdot)$ is subdifferentiable on C, by the well-known Moreau-Rockafellar theorem, there exists $w \in \partial_2 f(x^k, y^k)$ such that

$$f(x^k,\ y) - f(x^k,\ y^k) \geq \langle w,\ y - y^k\rangle,\ \forall y \in C.$$

Combining this with (3.9), we have

$$\lambda_k(f(x^k,\ y) - f(x^k,\ y^k)) \geq \langle y^k - x^k,\ y^k - y\rangle,\ \forall y \in C.$$

Hence

$$\lambda_{k_j}(f(x^{k_j},\ y) - f(x^{k_j},\ y^{k_j})) \geq \langle y^{k_j} - x^{k_j},\ y^{k_j} - y\rangle, \quad \forall y \in C.$$

Then, using $\{\lambda_k\} \subset [a,\ b] \subset (0, \frac{1}{L})$ and the continuity of f, we have

$$f(\bar{t},\ y) \geq 0,\ \forall y \in C.$$

Combining this and (3.8), we obtain

$$t^{k_i} \rightharpoonup \bar{t} \in Fix(S) \cap Sol(f,\ C).$$

By (3.7) and the definition of x^*, we have

$$\limsup_{k\to\infty}\langle x^* - g(x^*), t^k - x^*\rangle = \langle x^* - g(x^*), \bar{t} - x^*\rangle \geq 0.$$

Using this and Step 3, we get

$$\limsup_{k\to\infty}\langle x^* - g(x^*), S(t^k) - x^*\rangle = \langle x^* - g(x^*), \bar{t} - x^*\rangle \geq 0.$$

Step 5. Claim that the sequences $\{x^k\}$, $\{y^k\}$ and $\{t^k\}$ converge strongly to x^*.

Proof of Step 5. Using $x^{k+1} = \alpha_k g(x^k) + (1 - a_k)S(t^k)$ and Lemma 2.3, we have

$$\begin{aligned}||x^{k+1} - x^*||^2 &= ||\alpha_k(g(x^k) - x^*) + (1-\alpha_k)(S(t^k) - x^*)||^2 \\ &= \alpha_k^2||g(x^k) - x^*||^2 + (1-\alpha_k)^2||S(t^k) - x^*||^2 \\ &\quad + 2\alpha_k(1-\alpha_k)\langle g(x^k) - x^*, S(t^k) - x^*\rangle \\ &\le \alpha_k^2||g(x^k) - x^*||^2 + (1-\alpha_k)^2||x^k - x^*||^2 \\ &\quad + 2\alpha_k(1-\alpha_k)\langle g(x^k) - x^*, S(t^k) - x^*\rangle \\ &= \alpha_k^2||g(x^k) - x^*||^2 + (1-\alpha_k)^2||x^k - x^*||^2 \\ &\quad + 2\alpha_k(1-\alpha_k)\langle g(x^k) - g(x^*), S(t^k) - x^*\rangle \\ &\quad + 2\alpha_k(1-\alpha_k)\langle g(x^*) - x^*, S(t^k) - x^*\rangle \\ &\le \alpha_k^2||g(x^k) - x^*||^2 + (1-\alpha_k)^2||x^k - x^*||^2 \\ &\quad + 2\delta\alpha_k(1-\alpha_k)||x^k - x^*||||(t^k) - x^*|| \\ &\quad + 2\alpha_k(1-\alpha_k)\langle g(x^*) - x^*, S(t^k) - x^*\rangle \\ &\le \alpha_k^2||g(x^k) - x^*||^2 + ((1-\alpha_k)^2 + 2\delta\alpha_k(1-\alpha_k))||x^k - x^*||^2 \\ &\quad + 2\alpha_k(1-\alpha_k)\langle g(x^*) - x^*, S(t^k) - x^*\rangle \\ &\le (1 - \alpha_k + 2\delta\alpha_k)||x^k - x^*||^2 + \alpha_k^2||g(x^k) - x^*||^2 \\ &\quad + 2\alpha_k(1-\alpha_k)\max\{0, \langle g(x^*) - x^*, S(t^k) - x^*\rangle\} \\ &= (1 - A_k)||x^k - x^*||^2 + B_k,\end{aligned}$$

where A_k and B_k are defined by

$$\begin{cases} A_k = \alpha_k(1 - 2\delta), \\ B_k = \alpha_k^2||g(x^k) - x^*||^2 + 2\alpha_k(1-\alpha_k)\max\{0,\ \langle g(x^*) - x^*,\ S(t^k) - x^*\rangle\}. \end{cases}$$

Since $\lim_{k\to\infty}\alpha_k = 0, \sum_{k=1}^{\infty}\alpha_k = \infty$, Step 4, we have $\limsup_{k\to\infty}\langle x^* - g(x^*), S(t^k) - x^*\rangle \ge 0$ and hence

$$B_k = o(A_k), \lim_{k\to\infty} A_k = 0, \sum_{k=1}^{\infty} A_k = \infty.$$

By Lemma 2.1, we obtain that the sequence $\{x^k\}$ converges strongly to x^*. It follows from Step 1 that the sequences $\{y^k\}$ and $\{t^k\}$ also converge strongly to the same solution $x^* = Pr_{Fix(S)\cap Sol(f,C)}g(x^*)$.

□

4 Applications

Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and F be a function from C into $\mathcal{H}$. In this section, we consider the variational inequality problem which is presented as follows:

$$\text{Find } x^* \in C \text{ such that } \langle F(x^*), x - x^*\rangle \ge 0 \text{ for all } x \in C. \qquad VI(F,\ C)$$

Let $f : C \times C \to \mathcal{R}$ be defined by $f(x, y) = \langle F(x), y - x\rangle$. Then Problem $EP(f, C)$ can be written in $VI(F, C)$. The set of solutions of $VI(F, C)$ is denoted by $Sol(F, C)$. Recall that the function F is called *strongly monotone* on C with ß >0 if

$$\langle F(x) - F(y),\ x - y\rangle \ge \beta||x - y||^2,\ \forall x,\ y \in C;$$

monotone on C if

$$\langle F(x) - F(y),\ x - y\rangle \geq 0,\ \forall x,\ y \in C;$$

pseudomonotone on C if

$$\langle F(y),\ x - y\rangle \geq 0 \Rightarrow \langle F(x),\ x - y\rangle \geq 0,\ \forall x,\ y \in C;$$

Lipschitz continuous on C with constants $L > 0$ if

$$||F(x) - F(y)|| \leq L||x - y||,\ \forall x,\ y \in C.$$

Since

$$\begin{aligned} y^k &= \text{argmin}\{\lambda_k f(x^k,\ y) + \frac{1}{2}||y - x^k||^2\ :\ y \in C\} \\ &= \text{argmin}\,\{\lambda_k\langle F(x^k),\ y - x^k\rangle + \frac{1}{2}||y - x^k||^2 : y \in C\} \\ &= Pr_C(x^k - \lambda_k F(x^k)), \end{aligned}$$

(2.4), (2.5) and Theorem 3.1, we obtain that the following convergence theorem for finding a common element of the set of fixed points of a nonexpansive mapping S and the solution set of problem $VI(F, C)$.

Theorem 4.1 *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, F be a function from C to $\mathcal{H}$such that F is monotone and L-Lipschitz continuous on C, $g : C \to C$ is contractive with constant $\delta \in (0,\ \frac{1}{2})$, $S: C \to C$ be nonexpansive and positive sequences $\{a_k\}$ and $\{\lambda_k\}$ satisfy the following restrictions*

$$\begin{cases} \sum_{k=0}^{\infty} |\alpha_{k+1} - \alpha_k| < \infty, \\ \lim_{k\to\infty} \alpha_k = 0, \\ \sum_{k=0}^{\infty} \alpha_k = \infty, \\ \sum_{k=0}^{\infty} \sqrt{|\lambda_{k+1} - \lambda_k|} < \infty, \\ \{\lambda_k\} \subset [a,\ b] \text{ for some } a,\ b \in (0,\ \frac{1}{L}). \end{cases}$$

Then sequences $\{x^k\}$, $\{y^k\}$ and $\{t^k\}$ generated by

$$\begin{cases} y^k = Pr_C(x^k - \lambda_k F(x^k)), \\ t^k = Pr_C(x^k - \lambda_k F(y^k)), \\ x^{k+1} = \alpha_k g(x^k) + (1 - \alpha_k)S(t^k), \end{cases}$$

converge strongly to the same point $x^* \in Pr_{Fix(S)\cap Sol(F,C)}g(x^*)$.

Thus, this scheme and its convergence become results proposed by Nadezhkina and Takahashi in [23]. As direct consequences of Theorem 3.1, we obtain the following corollary.

Corollary 4.2 *Suppose that Assumptions A_1-A_3 are satisfied, $Sol(f, C) \neq \emptyset$, $x^0 \in C$ and two positive sequences $\{\lambda_k\}$, $\{a_k\}$ satisfy the following restrictions*:

$$\begin{cases} \sum_{k=0}^{\infty} |\alpha_{k+1} - \alpha_k| < \infty, \\ \lim_{k\to\infty} \alpha_k = 0, \\ \sum_{k=0}^{\infty} \alpha_k = \infty, \\ \sum_{k=0}^{\infty} \sqrt{|\lambda_{k+1} - \lambda_k|} < \infty, \\ \{\lambda_k\} \subset [a, b] \text{ for some } a, b \in (0, \frac{1}{L}), \text{ where } L = \max\{2c_1, 2c_2\}. \end{cases}$$

Then, the sequences $\{x^k\}$, $\{y^k\}$ *and* $\{t^k\}$ *generated by*

$$\begin{cases} y^k = \text{argmin}\, \{\lambda_k f(x^k, y) + \frac{1}{2}||y - x^k||^2 \ : \ y \in C\}, \\ t^k = \text{argmin}\, \{\lambda_k f(y^k, y) + \frac{1}{2}||y - x^k||^2 \ : \ y \in C\} \\ x^{k+1} = \alpha_k g(x^k) + (1 - \alpha_k) t^k, \end{cases}$$

where $g : C \to C$ *is a* δ*-contraction with* $0 < \delta < \frac{1}{2}$, *converge strongly to the same point* $x^* = Pr_{Sol(f,C)} g(x^*)$.

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Authors' contributions
The main idea of this paper is proposed by P.N. Anh. The revision is made by DDT. PNA and DDT prepared the manuscript initially and performed all the steps of proof in this research. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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