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Some existence results for a nonlinear fractional differential equation on partially ordered Banach spaces

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Abstract

By using fixed point results on cones, we study the existence and uniqueness of positive solutions for some nonlinear fractional differential equations via given boundary value problems. Examples are presented in order to illustrate the obtained results.

1 Introduction

The field of fractional differential equations has been subjected to an intensive development of the theory and the applications (see, for example, [1–6] and the references therein). It should be noted that most of papers and books on fractional calculus are devoted to the solvability of linear initial fractional differential equations on terms of special functions. There are some papers dealing with the existence of solutions of nonlinear initial value problems of fractional differential equations by using the techniques of nonlinear analysis such as fixed point results, the Leray-Schauder theorem, stability, *etc.* (see, for example, [7–19] and the references therein). In fact, fractional differential equations arise in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena and aerodynamics (see, for example, [20–23] and the references therein). The main advantage of using the fractional nonlinear differential equations is related to the fact that we can describe the dynamics of complex non-local systems with memory. In this line of taught, the equations involving various fractional orders are important from both theoretical and applied view points. We need the following notions.

Definition 1.1 ([1, 4]) For a continuous function $f : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order α is defined by

$$ {}^{c}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,ds, $$

where $n-1 < \alpha < n$, $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Definition 1.2 ([1, 4]) The Riemann-Liouville fractional derivative of order α for a continuous function f is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n-1}}\,ds \quad \bigl(n = [\alpha]+1\bigr),$$

where the right-hand side is pointwise defined on $(0,\infty)$.

Definition 1.3 ([1, 4]) Let $[a,b]$ be an interval in $\mathbb{R}$ and $\alpha > 0$. The Riemann-Liouville fractional order integral of a function $f \in L^1([a,b],\mathbb{R})$ is defined by

$$I_a^\alpha f(t) = \frac{1}{\gamma(\alpha)}\int_a^t \frac{f(s)}{(t-s)^{1-\alpha}}\,ds$$

whenever the integral exists.

Suppose that E is a Banach space which is partially ordered by a cone $P \subseteq E$, that is, $x \le y$ if and only if $y - x \in P$. We denote the zero element of E by θ. A cone P is called normal if there exists a constant $N > 0$ such that $\theta \le x \le y$ implies $\|x\| \le N\|y\|$ (see [24]). Also, we define the order interval $[x_1,x_2] = \{x \in E | x_1 \le x \le x_2\}$ for all $x_1, x_2 \in E$ [24]. We say that an operator $A : E \to E$ is increasing whenever $x \le y$ implies $Ax \le Ay$. Also, $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \le y \le \mu x$ (see [24]). Finally, put $P_h = \{x \in E | x \sim h\}$ for all $h > \theta$. It is easy to see that $P_h \subseteq P$ is convex and $\lambda P_h = P_h$ for all $\lambda > 0$. We recall the following in our results. Let E be a real Banach space and let P be a cone in E. Let (a,b) be an interval and let τ and φ be two positive-valued functions such that $\varphi(t) \ge \tau(t)$ for all $t \in (a,b)$ and $\tau : (a,b) \to (0,1)$ is a surjection. We say that an operator $A : P \to P$ is τ-φ-concave whenever $A(\tau(t)x) \ge \varphi(t)Ax$ for all $t \in (a,b)$ and $x \in P$ [13]. We say that A is φ-concave whenever $\tau(t) = t$ for all t [13]. We recall the following result.

Theorem 1.1 ([13]) *Let E be a Banach space, let P be a normal cone in E, and let $A : P \to P$ be an increasing and τ-φ-concave operator. Suppose that there exists $\theta \ne h \in P$ such that $Ah \in P_h$. Then there are $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \le u_0 \le v_0$ and $u_0 \le Au_0 \le Av_0 \le v_0$, the operator A has a unique fixed point $x^* \in [u_0,v_0]$, and for $x_0 \in P_h$ and the sequence $\{x_n\}$ with $x_n = Ax_{n-1}$, we have $\|x_n - x^*\| \to 0$.*

2 Main results

We study the existence and uniqueness of a solution for the fractional differential equation

$$D^\alpha u(t) + f\bigl(t,u(t)\bigr) = 0$$

on partially ordered Banach spaces with two types of boundary conditions and two types of fractional derivatives, Riemann-Liouville and Caputo.

2.1 Existence results for the fractional differential equation with the Riemann-Liouville fractional derivative

First, we study the existence and uniqueness of a positive solution for the fractional differential equation

$$D^\alpha u(t) + f\bigl(t,u(t)\bigr) = 0 \quad \bigl(0 < \varepsilon < T, T \ge 1, t \in [\varepsilon,T], 0 < \alpha < 1\bigr), \tag{2.1}$$

$$u(\eta) = u(T) \quad \bigl(\eta \in (\varepsilon, t)\bigr), \tag{2.2}$$

where D^α is the Riemann-Liouville fractional derivative of order α. Let $E = C[\varepsilon, T]$. Consider the Banach space of continuous functions on $[\varepsilon, T]$ with the sup norm and set $P = \{y \in C[\varepsilon, T] : \min_{t\in[\varepsilon,T]} y(t) \geq 0\}$. Then P is a normal cone.

Lemma 2.1 *Let $0 < \varepsilon < T$, $T \geq 1$, $t \in [\varepsilon, T]$, $\eta \in (\varepsilon, t)$ and $0 < \alpha < 1$. Then the problem $D^\alpha u(t) + f(t, u(t)) = 0$ with the boundary value condition $u(\eta) = u(T)$ has a solution u_0 if and only if u_0 is a solution of the fractional integral equation*

$$u(t) = \int_\varepsilon^T G(t,s) f\bigl(s, u(s)\bigr)\, ds,$$

where

$$G(t,s) = \begin{cases} \frac{t^{\alpha-1}(\eta-s)^{\alpha-1} - t^{\alpha-1}(T-s)^{\alpha-1}}{(\eta^{\alpha-1}-T^{\alpha-1})\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & \varepsilon \leq s \leq \eta \leq t \leq T, \\ \frac{-t^{\alpha-1}(T-s)^{\alpha-1}}{(\eta^{\alpha-1}-T^{\alpha-1})\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & \varepsilon \leq \eta \leq s \leq t \leq T, \\ \frac{-t^{\alpha-1}(T-s)^{\alpha-1}}{(\eta^{\alpha-1}-T^{\alpha-1})\Gamma(\alpha)}, & \varepsilon \leq \eta \leq t \leq s \leq T. \end{cases}$$

Proof From $D^\alpha u(t) + f(t, u(t)) = 0$ and the boundary condition, it is easy to see that $u(t) - c_1 t^{\alpha-1} = -I_\varepsilon^\alpha f(t, u(t))$. By the definition of a fractional integral, we get

$$u(t) = c_1 t^{\alpha-1} - \int_\varepsilon^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, u(s)\bigr)\, ds.$$

Thus, $u(\eta) = c_1 \eta^{\alpha-1} - \int_\varepsilon^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))\, ds$ and

$$u(T) = c_1 T^{\alpha-1} - \int_\varepsilon^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, u(s)\bigr)\, ds.$$

Since $u(\eta) = u(T)$, we obtain

$$c_1 = \frac{1}{\eta^{\alpha-1} - T^{\alpha-1}} \int_\varepsilon^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, u(s)\bigr)\, ds - \frac{1}{\eta^{\alpha-1} - T^{\alpha-1}} \int_\varepsilon^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, u(s)\bigr)\, ds.$$

Hence,

$$\begin{aligned} u(t) = {} & \frac{t^{\alpha-1}}{\eta^{\alpha-1} - T^{\alpha-1}} \int_\varepsilon^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, u(s)\bigr)\, ds - \frac{t^{\alpha-1}}{\eta^{\alpha-1} - T^{\alpha-1}} \int_\varepsilon^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, u(s)\bigr)\, ds \\ & - \int_\varepsilon^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, u(s)\bigr)\, ds = \int_\varepsilon^T G(t,s) f\bigl(s, u(s)\bigr)\, ds. \end{aligned}$$

This completes the proof. □

Now, we are ready to state and prove our first main result.

Theorem 2.2 *Let $0 < \varepsilon < T$ be given and let τ and φ be two functions on (ε, T) such that $\varphi(t) \geq \tau(t)$ for all $t \in (\varepsilon, T)$. Suppose that $\tau : (\varepsilon, T) \to (0,1)$ is a surjection and $f(t, u(t)) \in C([\varepsilon, T] \times [0, \infty])$ is increasing in u for each fixed t, $f(t, u(t)) \leq 0$ and $f(t, \tau(\lambda) u(t)) \geq$*

$\varphi(\lambda)f(t,u(t))$ for all $t,\lambda \in (\varepsilon,T)$ and $u \in P$. Assume that there exist $M_1 > 0$, $M_2 > 0$ and $\theta \neq h \in P$ such that

$$M_1 h(t) \le \int_\varepsilon^T G(t,s)f\bigl(s,h(s)\bigr)\,ds \le M_2 h(t)$$

for all $t \in [\varepsilon,T]$, where $G(t,s)$ is the green function defined in Lemma 2.1. *Then the problem* (2.1) *with the boundary value condition* (2.2) *has a unique positive solution $u^* \in P_h$. Moreover, for the sequence $u_{n+1} = \int_\varepsilon^T G(t,s)f(s,u_n(s))\,ds$, we have $\|u_n - u^*\| \to 0$ for all $u_0 \in P_h$.*

Proof By using Lemma 2.1, the problem is equivalent to the integral equation

$$u(t) = \int_\varepsilon^T G(t,s)f\bigl(s,u(s)\bigr)\,ds,$$

where

$$G(t,s) = \begin{cases} \frac{t^{\alpha-1}(\eta-s)^{\alpha-1}-t^{\alpha-1}(T-s)^{\alpha-1}}{(\eta^{\alpha-1}-T^{\alpha-1})\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & \varepsilon \le s \le \eta \le t \le T, \\ \frac{-t^{\alpha-1}(T-s)^{\alpha-1}}{(\eta^{\alpha-1}-T^{\alpha-1})\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & \varepsilon \le \eta \le s \le t \le T, \\ \frac{-t^{\alpha-1}(T-s)^{\alpha-1}}{(\eta^{\alpha-1}-T^{\alpha-1})\Gamma(\alpha)}, & \varepsilon \le \eta \le t \le s \le T. \end{cases}$$

Define the operator $A : P \to E$ by $Au(t) = \int_\varepsilon^T G(t,s)f(s,u(s))\,ds$. Then u is a solution for the problem if and only if $u = Au$. It is easy to check that the operator A is increasing on P. On the other hand,

$$\begin{aligned} A\bigl(\tau(\lambda)u\bigr)(t) &= \int_\varepsilon^T G(t,s)f\bigl(s,\tau(\lambda)u(s)\bigr)\,ds \\ &\ge \varphi(\lambda)\int_\varepsilon^T G(t,s)f\bigl(s,u(s)\bigr)\,ds = \varphi(\lambda)Au(t) \end{aligned}$$

for all $\lambda \in [\varepsilon,T]$ and $u \in P$. Thus, the operator A is τ-φ-concave. Since

$$M_1 h(t) \le Ah(t) = \int_\varepsilon^T G(t,s)f\bigl(s,h(s)\bigr)\,ds \le M_2 h(t)$$

for all $t \in [\varepsilon,T]$, we get $Ah \in P_h$. Now, by using Theorem 1.1, the operator A has a unique positive solution $u^* \in P_h$. This completes the proof. □

Here, we give the following example to illustrate Theorem 2.2.

Example 2.1 Let $0 < \varepsilon < 1$ be given. Consider the periodic boundary value problem

$$\begin{aligned} &D^{\frac{1}{3}}u(t) + \bigl\{g(t) + \bigl[u(t)\bigr]^\alpha\bigr\} = 0 \quad \bigl(t \in [\varepsilon,1]\bigr), \\ &u(\eta) = u(1), \end{aligned}$$

where $\eta \in (\varepsilon, t)$, g is continuous on $[\varepsilon, 1]$ and $\min_{t\in[\varepsilon,1]} g(t) > 0$. Put

$$G(t,s) = \begin{cases} \frac{t^{-2/3}(\eta-s)^{-2/3}-t^{-2/3}(1-s)^{-2/3}}{(\eta^{-2/3}-1^{-2/3})\Gamma(1/3)} - \frac{(t-s)^{-2/3}}{\Gamma(1/3)}, & \varepsilon \le s \le \eta \le t \le 1, \\ \frac{-t^{-2/3}(1-s)^{-2/3}}{(\eta^{-2/3}-1^{-2/3})\Gamma(1/3)} - \frac{(t-s)^{-2/3}}{\Gamma(1/3)}, & \varepsilon \le \eta \le s \le t \le 1, \\ \frac{-t^{-2/3}(1-s)^{-2/3}}{(\eta^{-2/3}-1^{-2/3})\Gamma(1/3)}, & \varepsilon \le \eta \le t \le s \le 1. \end{cases}$$

Then $\int_\varepsilon^1 G(t,s)\,ds = \frac{t^{-2/3}(\eta-\varepsilon)^{1/3}-t^{-2/3}(1-\varepsilon)^{1/3}-(t-\varepsilon)^{1/3}(\eta^{-2/3}-1)}{\Gamma(4/3)(\eta^{-2/3}-1)}$. Now, define $\tau(t) = t$, $\varphi(t) = t^{1/3}$, $\gamma_1 = \min_{t\in[\varepsilon,1]} g(t)$, $\gamma_2 = \max_{t\in[\varepsilon,1]} g(t)$ and also $f(t,u) = g(t) + u^{1/3}$ for all t. Then $\tau : (0,1) \to (0,1)$ is a surjection and $\varphi(t) > \tau(t)$ for all $t \in (\varepsilon, 1)$. For each $u \ge 0$, we have

$$\begin{aligned} f\big(t, \tau(\lambda)u(t)\big) &= f\big(t, \lambda u(t)\big) = g(t) + \lambda^{1/3}\big[u(t)\big]^{1/3} \\ &\ge \lambda^{1/3}\big(g(t) + \big[u(t)\big]^{1/3}\big) = \varphi(\lambda) f\big(t, u(t)\big). \end{aligned}$$

Now, put $h \equiv 1$, $M_1 = (\gamma_1 + 1)\min_{t\in[\varepsilon,1],\eta\in[\varepsilon,1]} \frac{-t^{-2/3}(1-\varepsilon)^{1/3}-(t-\varepsilon)^{1/3}(\eta^{-2/3}-1)}{\Gamma(4/3)(\eta^{-2/3}-1)}$ and $M_2 = (\gamma_2 + 1)\max_{\eta\in[\varepsilon,1]} \frac{\varepsilon^{-2/3}\eta^{1/3}}{\Gamma(4/3)(\eta^{-2/3}-1)}$. Then we get

$$\begin{aligned} &\int_\varepsilon^1 G(t,s)\big\{g(s) + \big[h(s)\big]^{1/3}\big\}\,ds \\ &\quad\le \int_\varepsilon^1 G(t,s)(\gamma_2 + 1)\,ds \\ &\quad\le (\gamma_2 + 1)\max_{t\in[\varepsilon,1]} \int_\varepsilon^1 G(t,s)\,ds \le (\gamma_2 + 1)\left(\max_{\eta\in[\varepsilon,1]} \frac{\varepsilon^{-2/3}\eta^{1/3}}{\Gamma(4/3)(\eta^{-2/3}-1)}\right) = M_2 h \end{aligned}$$

and

$$\begin{aligned} &\int_\varepsilon^1 G(t,s)\big\{g(s) + \big[h(s)\big]^{1/3}\big\}\,ds \\ &\quad\ge (\gamma_1 + 1)\min_{t\in[\varepsilon,1]} \int_\varepsilon^1 G(t,s)\,ds \\ &\quad\ge (\gamma_1 + 1)\min_{t\in[\varepsilon,1],\eta\in[\varepsilon,1]} \frac{-t^{-2/3}(1-\varepsilon)^{1/3} - (t-\varepsilon)^{1/3}(\eta^{-2/3}-1)}{\Gamma(4/3)(\eta^{-2/3}-1)} = M_1 h. \end{aligned}$$

Thus, by using Theorem 2.2, the problem has a unique solution in $P_h = P_1$.

2.2 Existence results for the fractional differential equation with the Caputo fractional derivative

Here, we study the existence and uniqueness of a positive solution for the fractional differential equation

$$^cD^\alpha u(t) + f\big(t, u(t)\big) = 0 \quad \big(t \in [0,T], T \ge 1, 1 < \alpha < 2\big), \tag{2.3}$$

$$u(0) = \beta_1 u(\eta), \qquad u(T) = \beta_2 u(\eta) \quad \big(\eta \in (0,t), 0 < \beta_1 < \beta_2 < 1\big), \tag{2.4}$$

where $^cD^\alpha$ is the Caputo fractional derivative of order α. Let $E = C[0,T]$ be the Banach space of continuous functions on $[0,T]$ with the sup norm and

$$P = \Big\{y \in C[0,T] : \min_{t\in[0,T]} y(t) \ge 0\Big\}.$$

It is known that P is a normal cone. Similar to the proof of Lemma 2.1, we can prove the following result.

Lemma 2.3 *Let $1 < \alpha < 2$, $T \geq 1$, $t \in [0,T]$, $\eta \in (0,t)$ and $0 < \beta_1 < \beta_2 < 1$. Then the problem ${}^cD^\alpha u(t) + f(t,u(t)) = 0$ with the boundary value conditions $u(0) = \beta_1 u(\eta)$ and $u(T) = \beta_2 u(\eta)$ has a solution u_0 if and only if u_0 is a solution of the fractional integral equation $u(t) = \int_0^T G(t,s)f(s,u(s))\,ds$, where*

$$G(t,s) = \begin{cases} \frac{[\beta_1 T + t(\beta_2-\beta_1)](\eta-s)^{\alpha-1} + t(T-s)^{\alpha-1} - T(t-s)^{\alpha-1}}{T\Gamma(\alpha)}, & 0 \leq s \leq \eta \leq t \leq T, \\ \frac{t(T-s)^{\alpha-1} - T(t-s)^{\alpha-1}}{T\Gamma(\alpha)}, & 0 \leq \eta \leq s \leq t \leq T, \\ \frac{t(T-s)^{\alpha-1}}{T\Gamma(\alpha)}, & 0 \leq \eta \leq t \leq s \leq T. \end{cases}$$

Theorem 2.4 *Let $T \geq 1$ be given and let τ and φ be two positive-valued functions on $(0,T)$ such that $\varphi(t) \geq \tau(t)$ for all $t \in (0,T)$. Suppose that $\tau : (0,T) \to (0,1)$ is a surjection and $f(t,u(t)) \in C([\varepsilon,T] \times [0,\infty])$ is increasing in u for each fixed t, $f(t,u(t)) = 0$ whenever $0 < \eta < s < t < T$ and $f(t,u(t)) \geq 0$ otherwise, and also $f(t,\tau(\lambda)u(t)) \geq \varphi(\lambda)f(t,u(t))$ for all $t,\lambda \in (0,T)$ and $u \in P$. Assume that there exist $M_1 > 0$, $M_2 > 0$ and $\theta \neq h \in P$ such that*

$$M_1 h(t) \leq \int_0^T G(t,s)f\bigl(s,h(s)\bigr)\,ds \leq M_2 h(t)$$

for all $t \in [0,T]$, where $G(t,s)$ is the green function defined in Lemma 2.3. Then the problem (2.3) with the boundary value conditions (2.4) has a unique positive solution $u^* \in P_h$. Moreover, for the sequence $u_{n+1} = \int_\varepsilon^T G(t,s)f(s,u_n(s))\,ds$, we have $\|u_n - u^*\| \to 0$ for all $u_0 \in P_h$.

Proof It is sufficient to define the operator $A : P \to E$ by

$$Au(t) = \int_0^T G(t,s)f\bigl(s,u(s)\bigr)\,ds.$$

Now, by using a similar proof of Theorem 2.2, one can show that $Au(t) \geq 0$ for all $u \in P$ and $t \in [0,T]$, and also the operator A is τ-φ-concave. By using Theorem 1.1, the operator A has a unique positive solution $u^* \in P_h$. This completes the proof by using Lemma 2.3. □

Below we present an example to illustrate Theorem 2.4.

Example 2.2 Let $\alpha = \frac{3}{2}$. Consider the periodic boundary value problem

$$\begin{aligned} &{}^cD^\alpha u(t) + g(t) + \bigl[u(t)\bigr]^\alpha = 0 \quad \bigl(t \in [0,1]\bigr), \\ &u(0) = \frac{1}{3}u\left(\frac{1}{2}\right) \qquad u(1) = \frac{1}{2}u\left(\frac{1}{2}\right), \end{aligned}$$

where g is a continuous function on $[0,1]$ with $\min_{t\in[0,1]} g(t) > 0$. Put $\beta_2 = \eta = 1/2$, $\beta_1 = 1/3$ and

$$G(t,s) = \begin{cases} \frac{[\frac{1}{3} + \frac{1}{6}t](\frac{1}{2}-s)^{1/2} + t(1-s)^{1/2} - (t-s)^{1/2}}{\Gamma(3/2)}, & 0 \leq s \leq \eta \leq t \leq 1, \\ \frac{t(1-s)^{1/2} - (t-s)^{1/2}}{\Gamma(3/2)}, & 0 \leq \eta \leq s \leq t \leq 1, \\ \frac{t(1-s)^{1/2}}{\Gamma(3/2)}, & 0 \leq \eta \leq t \leq s \leq 1. \end{cases}$$

Then $\int_0^1 G(t,s)\,ds = \frac{[\frac{1}{3}+\frac{1}{6}t](\frac{1}{2})^{3/2}+t-t^{3/2}}{\Gamma(5/2)}$. Now, define $\tau(t) = t$, $\varphi(t) = t^\alpha$, $\gamma_1 = \min_{t\in[0,1]} g(t)$, $\gamma_2 = \max_{t\in[0,1]} g(t)$ and $f(t,u) = g(t) + u^\alpha$. Then it is easy to see that $\tau : (0,1) \to (0,1)$ is a surjection map and $\varphi(t) > \tau(t)$ for $t \in (0,1)$. Also, we have

$$\begin{aligned} f\bigl(t,\tau(\lambda)u(t)\bigr) &= f\bigl(t,\lambda u(t)\bigr) = g(t) + \lambda^\alpha\bigl[u(t)\bigr]^\alpha \\ &\geq \lambda^\alpha\bigl(g(t) + \bigl[u(t)\bigr]^\alpha\bigr) = \varphi(\lambda) f\bigl(t,u(t)\bigr) \end{aligned}$$

for all $u \geq 0$. Now, put $h \equiv 1$, $M_1 = (\gamma_1 + 1)\min_{t\in[0,1]} \frac{-\frac{1}{3}t(\frac{1}{2})^{3/2}-t^{3/2}}{\Gamma(5/2)}$ and also $M_2 = (\gamma_2 + 1)\frac{\frac{5}{6}(\frac{1}{2})^{3/2}+1}{\Gamma(5/2)}$. Then we have

$$\begin{aligned} &\int_0^1 G(t,s)\bigl\{g(s) + \bigl[h(s)\bigr]^{3/2}\bigr\}\,ds \\ &\quad\leq \int_0^1 G(t,s)(\gamma_2+1)\,ds \\ &\quad\leq (\gamma_2+1)\max_{t\in[0,1]}\int_0^1 G(t,s)\,ds \leq (\gamma_2+1)\frac{\frac{5}{6}(\frac{1}{2})^{3/2}+1}{\Gamma(5/2)} = M_2 h \end{aligned}$$

and

$$\begin{aligned} \int_0^1 G(t,s)\bigl\{g(s) + \bigl[h(s)\bigr]^{3/2}\bigr\}\,ds &\geq (\gamma_1+1)\min_{t\in[0,1]}\int_0^1 G(t,s)\,ds \\ &\geq (\gamma_1+1)\min_{t\in[0,1]}\frac{-\frac{1}{3}t(\frac{1}{2})^{3/2}-t^{3/2}}{\Gamma(5/2)} = M_1 h. \end{aligned}$$

Thus, by using Theorem 2.4, the problem has a unique solution in $P_h = P_1$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

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