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Existence, uniqueness and stability of positive solutions to a general sublinear elliptic systems

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Abstract

In this paper, we make use of a new stability result and bifurcation theory to study the existence and uniqueness of positive solutions to semilinear elliptic systems with some general sublinear conditions. Moreover, we obtain the precise global bifurcation diagrams of the system in a single monotone solution curve.
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Keywords: semilinear elliptic systems; positive solution; stability; existence; uniqueness

1 Introduction

We consider positive solutions of a semilinear elliptic system with n equations ($n \geq 2$)

$$
\begin{cases}
\Delta u_1 + \lambda f_1(u_1, u_2, \ldots, u_n) = 0, & x \in \Omega, \\
\Delta u_2 + \lambda f_2(u_1, u_2, \ldots, u_n) = 0, & x \in \Omega, \\
\cdots \\
\Delta u_n + \lambda f_n(u_1, u_2, \ldots, u_n) = 0, & x \in \Omega, \\
u_1(x) = \cdots = u_n(x) = 0, & x \in \partial\Omega,
\end{cases}
\tag{1.1}
$$

where $\lambda > 0$ is a positive parameter, Ω is a bounded smooth domain in $\mathbf{R}^N$, and f_i ($i = 1, 2, \ldots, n$) are smooth real-valued functions defined on $\mathbf{R}_+^n = \{(u_1, u_2, \ldots, u_n) \in \mathbf{R}^n : u_i > 0, 1 \leq i \leq n\}$ satisfying the following.

Cooperativeness Define the Jacobian of the vector field $(f_1, \ldots, f_n)$ as

$$
H(u_1, \ldots, u_n) = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_n} \end{pmatrix} = \begin{pmatrix} f'_{11} & \cdots & f'_{1n} \\ \vdots & \ddots & \vdots \\ f'_{n1} & \cdots & f'_{nn} \end{pmatrix}.
\tag{1.2}
$$

Then $\partial f_i / \partial u_j \geq 0$ ($i \neq j$) for $(u_1, \ldots, u_n) \in \mathbf{R}_+^n$.

The purpose of this paper is to study the existence, uniqueness and stability of positive solutions of such cooperative system (1.1) under certain conditions of f_i.

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The existence, uniqueness and stability of positive solutions to sublinear semilinear elliptic systems with two equations have been recently studied in [1–4]. The sublinear condition plays an important role. In this paper, we continue the effort in [3] to prove the stability of a positive solution to (1.1) under some reasonable sublinear conditions, and the stability implies the uniqueness of the positive solution. We also prove corresponding existence results using bifurcation theory and the continuation method. This is motivated by the existence study of exact multiplicity (and uniqueness) of positive solutions to the scalar semilinear elliptic equation:

$$
\begin{cases}
\Delta u + \lambda f(u) = 0, & x \in \Omega, \\
u(x) = 0, & x \in \partial\Omega,
\end{cases}
\tag{1.3}
$$

starting from Korman *et al.* [5, 6]. In Ouyang and Shi [7, 8], their result classified the different global exact multiplicity of (1.3) for more general nonlinearity f. There are more results on the existence and uniqueness of solution to the semilinear cyclic elliptic system

$$
\begin{cases}
\Delta u + \lambda f(v) = 0, & x \in \Omega, \\
\Delta v + \lambda g(u) = 0, & x \in \Omega, \\
u(x) = v(x) = 0, & x \in \partial\Omega.
\end{cases}
\tag{1.4}
$$

The notation of sublinearity and superlinearity of the nonlinear vector field or the ones in higher dimension was considered in Sirakov [9]. Our definition of sublinear nonlinearity is quite different, and ours is similar to the one in Ouyang and Shi [8] for the scalar case. Dalmasso [10] obtained the existence and uniqueness result for a more special sublinear system, and it was extended by Shi and Shivaji [4]. The uniqueness of a positive solution for large λ was proved in Hai [11, 12], Hai and Shivaji [13]. If Ω is a finite ball or the whole space, then the positive solutions of systems are radically symmetric and decreasing in radical direction by the result of Troy [14]. Hence the system can be converted into a system of ODEs. Several authors have taken that approach for the existence of the solutions, see Serrin and Zou [15, 16], and much success has been achieved for Lane-Emden systems. Using the scaling invariant in the Lane-Emden system, the uniqueness of the radial positive solution was shown in Dalmasso [10], Korman and Shi [17]. Cui *et al.* [18, 19] considered cyclic systems with three equations, and the uniqueness and existence of positive solutions were obtained. For the Lane-Emden systems with n equations, Maniwa [20] obtained the uniqueness and existence of positive solutions to systems under the sublinear conditions.

We organize the rest of this paper in the following way. In Section 2, we recall the maximum principle and prove the main stability result. In Section 3, we use the stability result and bifurcation theory to prove the existence and uniqueness of a positive solution. We also obtain the precise global bifurcation diagrams of the system (the bifurcation diagram is a single monotone solution curve in all cases) and give some examples. In Section 4, we consider the similar question for merely Hölder continuous nonlinearities, and we use monotone methods for existence. We use $W^{2,p}(\Omega)$ and $W^{2,p}_{\text{loc}}(\Omega)$ for the standard Sobolev space, $C(\overline{\Omega})$ for the space of continuous functions defined on $\overline{\Omega}$, and $C_0(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : u(x) = 0, x \in \partial\Omega\}$. We use $N(L)$ and $R(L)$ to denote the null space and the range space of a linear operator L.

2 Stability and linearized equations

In this section, we study the stability result about a positive solution. Let $U = (u_1, \ldots, u_n)$ be a solution of (1.1). We shall denote the partial derivative of $f_i(U)$ with respect to u_j by $f'_{ij}(U)$ or f'_{ij}. The stability of U is determined by the eigenvalue equation

$$
\begin{cases}
\Delta \xi_1 + \lambda f'_{11} \xi_1 + \lambda f'_{12} \xi_2 + \cdots + \lambda f'_{1n} \xi_n = -\mu \xi_1, & x \in \Omega, \\
\Delta \xi_2 + \lambda f'_{21} \xi_1 + \lambda f'_{22} \xi_2 + \cdots + \lambda f'_{2n} \xi_n = -\mu \xi_2, & x \in \Omega, \\
\cdots \\
\Delta \xi_n + \lambda f'_{n1} \xi_1 + \lambda f'_{n2} \xi_2 + \cdots + \lambda f'_{nn} \xi_n = -\mu \xi_n, & x \in \Omega, \\
\xi_1(x) = \xi_2(x) = \cdots = \xi_n(x) = 0, & x \in \partial \Omega,
\end{cases}
\tag{2.1}
$$

which can be written as

$$
L\mathbf{u} = H\mathbf{u} + \mu \mathbf{u},
\tag{2.2}
$$

where

$$
\mathbf{u} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \qquad
L\mathbf{u} = \begin{pmatrix} -\Delta \xi_1 \\ \vdots \\ -\Delta \xi_n \end{pmatrix}, \quad \text{and} \quad
H = \begin{pmatrix} f'_{11} & \cdots & f'_{1n} \\ \vdots & \ddots & \vdots \\ f'_{n1} & \cdots & f'_{nn} \end{pmatrix}.
\tag{2.3}
$$

Definition 2.1 [16] An $n \times n$ matrix A is reducible if for some permutation matrix Q,

$$
QAQ^T = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix},
$$

where B and C are square matrices and Q^T is the transpose of Q. Otherwise, A is irreducible.

Throughout this paper, H is assumed to be irreducible, since if not the case, the linearized system (2.1) can be reduced to two subsystems with one being not coupled with the other. If we assume that H is cooperative and irreducible, then the maximum principles hold for such systems. Before stating our results, we recall some known results as required.

Lemma 2.2 *Let* $X = [W^{2,p}_{\text{loc}}(\Omega) \cap C_0(\overline{\Omega})]^n$, *and let* $Y = [L^p(\Omega)]^n$ *for* $p > n$. *Suppose that* L, H *are given as in* (2.3), *and* $H = (f'_{ij})$ *is irreducible and satisfies* $f'_{ij} \geq 0$ $(i \neq j)$ *for* $(u_1, \ldots, u_n) \in \mathbf{R}^n_+$. *Then we have the following.*

1. $\mu_1 = \inf\{\mu \in \text{spt}(L - H)\}$ *is a real eigenvalue of* $L - H$, *where* $\text{spt}(L - H)$ *is the spectrum of* $L - H$.
2. *For* $\mu = \mu_1$, *there exists a unique* (*up a constant multiple*) *eigenfunction* $\mathbf{u}_1 \in X$, *and* $\mathbf{u}_1 > 0$ *in* Ω.
3. *For* $\mu < \mu_1$, *the equation* $L\mathbf{u} = H\mathbf{u} + \mu\mathbf{u} + \mathbf{f}$ *is uniquely solvable for any* $\mathbf{f} \in Y$, *and* $\mathbf{u} > 0$ *as long as* $\mathbf{f} \geq 0$.
4. (*Maximum principle*) *For* $\mu \leq \mu_1$, *suppose that* $\mathbf{u} \in [W^{2,p}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})]^n$ *satisfies* $L\mathbf{u} \geq H\mathbf{u} + \mu\mathbf{u}$ *in* Ω, $\mathbf{u} \geq 0$ *on* $\partial \Omega$, *then* $\mathbf{u} \geq 0$ *in* Ω.

5. *If there exists* $\mathbf{u} \in [W_{\text{loc}}^{2,p}(\Omega) \cap C(\overline{\Omega})]^n$ *satisfying* $L\mathbf{u} \geq H\mathbf{u}$ *in* Ω, $\mathbf{u} \geq 0$ *on* $\partial\Omega$, *and either* $\mathbf{u} \not\equiv 0$ *on* $\partial\Omega$ *or* $L\mathbf{u} \not\equiv H\mathbf{u}$ *in* Ω, *then* $\mu_1 > 0$.

For the result and proofs, see Sweers [21], Proposition 3.1 and Theorem 1.1. Moreover, from a standard compactness argument, there are countably many eigenvalues $\{\mu_i\}$ of $L - H$, and $|\mu_i - \mu_1| \to \infty$ as $i \to \infty$. We notice that μ_i is not necessarily real-valued. We call a solution U *stable* if $\mu_1 > 0$; and otherwise, it is *unstable* ($\mu_1 \leq 0$).

For our purpose, in this section, we also need to consider the adjoint operator of $L - H$. Let the transpose matrix of the Jacobian be

$$H^T = \begin{pmatrix} f'_{11} & \cdots & f'_{n1} \\ \vdots & \ddots & \vdots \\ f'_{1n} & \cdots & f'_{nn} \end{pmatrix}. \tag{2.4}$$

Then, evidently, the results in Lemma 2.2 also hold for the eigenvalue problem

$$L\mathbf{u}^* = H^T\mathbf{u}^* + \mu\mathbf{u}^*, \tag{2.5}$$

which is

$$\begin{cases} \Delta\xi_1^* + \lambda f'_{11}\xi_1^* + \lambda f'_{21}\xi_2^* + \cdots + \lambda f'_{n1}\xi_n^* = -\mu\xi_1^*, & x \in \Omega, \\ \Delta\xi_2^* + \lambda f'_{12}\xi_1^* + \lambda f'_{22}\xi_2^* + \cdots + \lambda f'_{n2}\xi_n^* = -\mu\xi_2^*, & x \in \Omega, \\ \cdots \\ \Delta\xi_n^* + \lambda f'_{1n}\xi_1^* + \lambda f'_{2n}\xi_2^* + \cdots + \lambda f'_{nn}\xi_n^* = -\mu\xi_n^*, & x \in \Omega, \\ \xi_1^*(x) = \xi_2^*(x) = \cdots = \xi_n^*(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.6}$$

where $\mathbf{u}^* = (\xi_1^*, \xi_2^*, \ldots, \xi_n^*)^T$. It is easy to verify that $L - H^T$ is the adjoint operator of $L - H$, while both are considered as operators defined on subspaces of $[L^2(\Omega)]^n$. By using the well-known functional analytic techniques (see [21, 22]), one can show the following.

Lemma 2.3 *Let X, Y, L and H be the same as in Lemma* 2.2. *Then the principal eigenvalue μ_1 of $L - H$ is also a real eigenvalue of $L - H^T$, $\mu_1 = \inf\{\mu \in \text{spt}(L - H^T)\}$, and for $\mu = \mu_1$, there exists a unique eigenfunction $\mathbf{u}_1^* \in [W_{\text{loc}}^{2,p}(\Omega) \cap C_0(\overline{\Omega})]^n$ of $L - H^T$ (up a constant multiple), and $\mathbf{u}_1^* > 0$ in Ω.*

Cui *et al.* [3] obtained the stability result of a positive solution for the system with two equations. We give the following stability result about a positive solution of (1.1).

Theorem 2.4 *Suppose that $U = (u_1, \ldots, u_n)$ is a positive solution of* (1.1), *f_i is cooperative and $H = (f'_{ij})$ is irreducible, then U is stable if f_i satisfies one of the following conditions: for any $U \in \mathbf{R}_+^n$,*

(H$_1$)

$$f_i(U) > \sum_{j=1}^n f'_{ij}(U)u_j, \quad i = 1, \ldots, n; \quad or \tag{2.7}$$

(H_2)

$$f_i(U) > \sum_{j=1}^{n} f'_{ji}(U)u_j, \quad i = 1, \ldots, n. \tag{2.8}$$

Proof Let $U = (u_1, \ldots, u_n)$ be a positive solution of (1.1), and let $(\mu_1, \mathbf{u}^*)$ be the corresponding principal eigen-pair of (2.6) such that $\xi_i^* > 0$ in Ω for any $1 \le i \le n$.

We assume that f_i satisfies (H_1). Multiplying the system (1.1) by $\mathbf{u}^*$, the system (2.6) by U, integrating over Ω and subtracting, we obtain

$$\mu_1 \int_\Omega \sum_{i=1}^{n} u_i \xi_i^* \, dx = \lambda \int_\Omega \sum_{i=1}^{n} f_i \xi_i^* \, dx - \lambda \int_\Omega \sum_{i=1}^{n} \xi_i^* \sum_{j=1}^{n} f'_{ij} u_j \, dx$$

$$= \lambda \int_\Omega \sum_{i=1}^{n} \xi_i^* \left(f_i - \sum_{j=1}^{n} f'_{ij} u_j \right) dx. \tag{2.9}$$

Hence $\mu_1 > 0$ if (H_1) is satisfied.

Similar to the proof above, let $(\mu_1, \mathbf{u})$ be the corresponding principal eigen-pair of (2.1) such that $\xi_i > 0$ in Ω for any $i = 1, \ldots, n$. We assume that f_i satisfies (H_2). Multiplying the system (1.1) by $\mathbf{u}$, the system (2.1) by U, integrating over Ω and subtracting, we can get

$$\mu_1 \int_\Omega \sum_{i=1}^{n} u_i \xi_i \, dx = \lambda \int_\Omega \sum_{i=1}^{n} f_i \xi_i \, dx - \lambda \int_\Omega \sum_{i=1}^{n} \xi_i \sum_{j=1}^{n} f'_{ji} u_j \, dx$$

$$= \lambda \int_\Omega \sum_{i=1}^{n} \xi_i \left(f_i - \sum_{j=1}^{n} f'_{ji} u_j \right) dx. \tag{2.10}$$

Hence $\mu_1 > 0$ if (H_2) is satisfied. □

On the other hand, the same proof also implies the following instability result under the opposite condition of (H_1) and (H_2).

Theorem 2.5 *Suppose that $U = (u_1, \ldots, u_n)$ is a positive solution of* (1.1), *f_i is cooperative and $H^T = (f'_{ji})$ is irreducible, then U is unstable if f_i satisfies one of the following conditions: for any $U \in \mathbf{R}_+^n$,*

(H_1')

$$f_i(U) < \sum_{j=1}^{n} f'_{ij}(U)u_j, \quad i = 1, \ldots, n; \quad or \tag{2.11}$$

(H_2')

$$f_i(U) < \sum_{j=1}^{n} f'_{ji}(U)u_j, \quad i = 1, \ldots, n. \tag{2.12}$$

Remark 2.6

1. In [8], for positive solutions of the scalar equation

$$\Delta u + \lambda h(u) = 0, \quad x \in \Omega, \qquad u(x) = 0, \quad x \in \partial\Omega,$$

   the function $h(u)$ is called a sublinear function if $h(u) > uh'(u)$, and it is superlinear if $h(u) < uh'(u)$. It was proved in Proposition 3.14 of [8] that a positive solution u is stable if h is sublinear, and u is unstable if h is superlinear. Now our conclusions, Theorems 2.4 and 2.5, are generalizations of the corresponding results. The condition (H$_i$) for $i = 1, 2$ is the generalization of sublinearity (or superlinearity) to n-variable vector fields.

2. The conditions (H$_1$) and (H$_2$) can be written in a vector form $F(U) > H(U)U$ and $F(U) > H^T(U)U$, where $F(U) = (f_1(U), \dots, f_n(U))^T$, $U = (u_1, \dots, u_n)^T$, and H is the original Jacobian matrix of the vector field $(f_1(U), \dots, f_n(U))$, and H^T is the transpose matrix of the Jacobian H. The condition (H$_1$) is clearly more natural as the conditions for f_i are separate. Hence the sublinearity can be defined for a single n-variable function. The condition (H$_2$) is defined for the whole vector field $(f_1(U), \dots, f_n(U))$.

3. If a solution U is stable, then it is necessarily a non-degenerate solution. That is, any eigenvalue μ_i of (2.1) has a positive real part. But when a solution is proved to be unstable, it can be a degenerate one with zero or pure imaginary eigenvalues.

3 Existence and uniqueness

In this section, we consider the uniqueness and existence of positive solutions for the following problem:

$$\begin{cases} \Delta u_1 + \lambda(g_{11}(u_1) + g_{12}(u_2) + \cdots + g_{1n}(u_n)) = 0, & x \in \Omega, \\ \Delta u_2 + \lambda(g_{21}(u_1) + g_{22}(u_2) + \cdots + g_{2n}(u_n)) = 0, & x \in \Omega, \\ \cdots \\ \Delta u_n + \lambda(g_{n1}(u_1) + g_{n2}(u_2) + \cdots + g_{nn}(u_n)) = 0, & x \in \Omega, \\ u_1(x) = u_2(x) = \cdots = u_n(x) = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

Suppose that each $g_{ij}(x)$ $(i, j = 1, 2, \dots, n)$ is a smooth real function defined on $\mathbf{R}_+$ satisfying

(A$_1$) $g_{ij}(0) \geq 0$;
(A$_2$) $g'_{ij}(x) \geq 0$, $(g_{ij}(x)/x)' \leq 0$, for all $x \geq 0$.

The Perron-Frobenius theorem plays a critical role in our main result.

Lemma 3.1 (Perron-Frobenius theorem: strong form [23, Theorem 5.3.1]) *Let $n \times n$ matrix A be a nonnegative irreducible matrix. Then $\rho(A)$ is a simple eigenvalue of A, associated to a positive eigenvector, where $\rho(A)$ denotes the spectral radius of A. Moreover, $\rho(A) > 0$.*

Here let (λ_1, φ_1) be the principal eigen-pair of

$$-\Delta\varphi = \lambda\varphi, \quad x \in \Omega, \qquad \varphi(x) = 0, \quad x \in \partial\Omega, \tag{3.2}$$

such that $\varphi_1(x) > 0$ in Ω and $\|\varphi_1\|_\infty = 1$. We have the following existence and uniqueness result for this sublinear problem.

Theorem 3.2 *Assume that each of $g_{ij}(x)$ satisfies* (A$_1$), (A$_2$) *and*

(A$_3$) $\lim_{x\to\infty} \frac{g_{ij}(x)}{x} = 0$.

1. *If at least one of $g_{ij}(0)$ $(i = 1, 2, \ldots, n)$ is positive and matrix $G = (g_{ij}(0))$ is irreducible, then* (3.1) *has a unique positive solution $U(\lambda) = (u_1(\lambda), u_2(\lambda), \ldots, u_n(\lambda))$ for all $\lambda > 0$;*
2. *If $g_{ij}(0) = 0$, $g'_{ij}(0) \geq 0$ for each $i, j = 1, 2, \ldots, n$ and matrix $G' = (g'_{ij}(0))$ is irreducible, then for some $\lambda_* > 0$,* (3.1) *has no positive solution when $\lambda \leq \lambda_*$, and* (3.1) *has a unique positive solution $U(\lambda) = (u_1(\lambda), u_2(\lambda), \ldots, u_n(\lambda))$ for $\lambda > \lambda_*$.*

Moreover, $\{(\lambda, u_1(\lambda), u_2(\lambda), \ldots, u_n(\lambda)) : \lambda > \lambda_*\}$ (in the first case, we assume $\lambda_* = 0$) is a smooth curve so that $u_i(\lambda)$ is strictly increasing in λ, and $u_i(\lambda) \to 0$ as $\lambda \to \lambda_*^+$.

Proof Our proof follows that of Theorem 6.1 in [4]. Firstly, we extend g_{ij} to be defined on **R** and they are continuously differentiable on **R**. From (A$_2$), $g_{ij}(x) > g'_{ij}(x)x$ implies that $\sum_{j=1}^n g_{ij}(u_j) > \sum_{j=1}^n g'_{ij}(u_j)u_j$, so $f_i(U) = f_i(u_1, \ldots, u_n) = \sum_{j=1}^n g_{ij}(u_j)$ satisfies (H$_1$). Hence from Theorem 2.4, any positive solution of (3.1) is stable.

Let us define

$$
F(\lambda, U) = \begin{pmatrix} \Delta u_1 + \lambda[g_{11}(u_1) + g_{12}(u_2) + \cdots + g_{1n}(u_n)] \\ \Delta u_2 + \lambda[g_{21}(u_1) + g_{22}(u_2) + \cdots + g_{2n}(u_n)] \\ \cdots \\ \Delta u_n + \lambda[g_{n1}(u_1) + g_{n2}(u_2) + \cdots + g_{nn}(u_n)] \end{pmatrix},
\tag{3.3}
$$

where $\lambda \in \mathbf{R}$ and $u_1, u_2, \ldots, u_n \in C_0^{2,\alpha}(\overline{\Omega})$. Here g_{ij} are at least C^1, then $F : \mathbf{R} \times X \to Y$ is continuously differentiable, where $X = [C_0^{2,\alpha}(\overline{\Omega})]^n$ and $Y = [C^\alpha(\overline{\Omega})]^n$. For weak solutions $U = (u_1, \ldots, u_n)$, one can also consider $X = [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)]^n$ and $Y = [L^p(\Omega)]^n$ where $p > 1$ is properly chosen.

It is easy to see that $(\lambda, U) = (0, 0, \ldots, 0)$ is a solution of (3.1). We apply the implicit function theorem at $(\lambda, U) = (0, 0, \ldots, 0)$. The Fréchet derivative of F is given by

$$
F_U(\lambda, U) \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{pmatrix} = \begin{pmatrix} \Delta \phi_1 + \lambda[g'_{11}(u_1)\phi_1 + g'_{12}(u_2)\phi_2 + \cdots + g'_{1n}(u_n)\phi_n] \\ \Delta \phi_2 + \lambda[g'_{21}(u_1)\phi_1 + g'_{22}(u_2)\phi_2 + \cdots + g'_{2n}(u_n)\phi_n] \\ \cdots \\ \Delta \phi_n + \lambda[g'_{n1}(u_1)\phi_1 + g'_{n2}(u_2)\phi_2 + \cdots + g'_{nn}(u_n)\phi_n] \end{pmatrix}.
\tag{3.4}
$$

Then $F_U(0, 0, \ldots, 0)(\phi_1, \ldots, \phi_n)^{\mathrm{T}} = (\Delta\phi_1, \ldots, \Delta\phi_n)^{\mathrm{T}}$ is an isomorphism from X to Y, and the implicit function theorem implies that $F(\lambda, U) = 0$ has a unique solution $(\lambda, U(\lambda))$ for $\lambda \in (0, \delta)$ for some small $\delta > 0$, and $(u'_1(0), \ldots, u'_n(0))$ is the unique solution of

$$
\begin{cases} \Delta\phi_1 + g_{11}(0) + g_{12}(0) + \cdots + g_{1n}(0) = 0, & x \in \Omega, \\ \Delta\phi_2 + g_{21}(0) + g_{22}(0) + \cdots + g_{2n}(0) = 0, & x \in \Omega, \\ \cdots \\ \Delta\phi_n + g_{n1}(0) + g_{n2}(0) + \cdots + g_{nn}(0) = 0, & x \in \Omega, \\ \phi_1(x) = \cdots = \phi_n(x) = 0, & x \in \partial\Omega. \end{cases}
\tag{3.5}
$$

Therefore,

$$\begin{cases} u_1'(0) = (g_{11}(0) + g_{12}(0) + \cdots + g_{1n}(0))e \\ u_2'(0) = (g_{21}(0) + g_{22}(0) + \cdots + g_{2n}(0))e \\ \cdots \\ u_n'(0) = (g_{n1}(0) + g_{n2}(0) + \cdots + g_{nn}(0))e, \end{cases}$$

where e is the unique positive solution of

$$\Delta e + 1 = 0, \quad x \in \Omega, \qquad e(x) = 0, \quad x \in \partial\Omega. \tag{3.6}$$

If $\sum_{j=1}^{n} g_{ij}(0) > 0$ for any $i = 1, 2, \ldots, n$, then $U(\lambda) = (u_1(\lambda), \ldots, u_n(\lambda))$ is positive for $\lambda \in (0, \delta)$. If there exists i such that $g_{i1}(0) + g_{i2}(0) + \cdots + g_{in}(0) > 0$, then $u_i(\lambda) > 0$ for $\lambda \in (0, \delta)$. For $k \neq i$, $\Delta u_k(\lambda) = -\lambda[g_{k1}(\lambda) + \cdots + g_{kn}(\lambda)]$ and g_{ij} is positive, hence $u_k(\lambda) > 0$ as well.

Next we assume that $g_{ij}(0) = 0$ and $g_{ij}'(0) > 0$ $(i, j = 1, \ldots, n)$. The linearized operator is

$$\begin{aligned}
F_U(\lambda, 0, \ldots, 0) \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_n \end{pmatrix} &= \begin{pmatrix} \Delta\Phi_1 + \lambda[g_{11}'(0)\Phi_1 + g_{12}'(0)\Phi_2 + \cdots + g_{1n}'(0)\Phi_n] \\ \Delta\Phi_2 + \lambda[g_{21}'(0)\Phi_1 + g_{22}'(0)\Phi_2 + \cdots + g_{2n}'(0)\Phi_n] \\ \cdots \\ \Delta\Phi_n + \lambda[g_{n1}'(0)\Phi_1 + g_{n2}'(0)\Phi_2 + \cdots + g_{nn}'(0)\Phi_n] \end{pmatrix} \\
&= \begin{pmatrix} \Delta\Phi_1 \\ \Delta\Phi_2 \\ \vdots \\ \Delta\Phi_n \end{pmatrix} + \lambda \begin{pmatrix} g_{11}'(0) & g_{12}'(0) & \cdots & g_{1n}'(0) \\ g_{21}'(0) & g_{22}'(0) & \cdots & g_{2n}'(0) \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1}'(0) & g_{n2}'(0) & \cdots & g_{nn}'(0) \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_n \end{pmatrix} \\
&= \begin{pmatrix} \Delta\Phi_1 \\ \Delta\Phi_2 \\ \vdots \\ \Delta\Phi_n \end{pmatrix} + \lambda J \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_n \end{pmatrix}, \tag{3.7}
\end{aligned}$$

where

$$J = \begin{pmatrix} g_{11}'(0) & \cdots & g_{1n}'(0) \\ \vdots & \ddots & \vdots \\ g_{n1}'(0) & \cdots & g_{nn}'(0) \end{pmatrix}.$$

Since $g_{ij}'(0) > 0$, all entries of matrix J are positive. Therefore, by using Lemma 3.1, there exist a positive principal eigenvalue χ_J and the corresponding eigenvector $(k_1, k_2, \ldots, k_n)^T$ of J for some $k_i > 0$ such that $(k_1\varphi_1, k_2\varphi_1, \ldots, k_n\varphi_1)^T$ is a positive eigenvector of $F_{(u_1, \ldots, u_n)}(\lambda_*, 0, \ldots, 0)$, where $\lambda_* = \lambda_1/\chi_J$. Similarly, J^T has the same principal eigenvalue χ_J, and the corresponding eigenvector $(k_1^*\varphi_1, k_2^*\varphi_1, \ldots, k_n^*\varphi_1)^T$, where k_i^* $(i = 1, \ldots, n)$ is a positive constant.

Hence when $\lambda = \lambda_* = \lambda_1/\chi_J$, $F_U(\lambda, 0, \ldots, 0)$ is not invertible and $\lambda = \lambda_*$ is a potential bifurcation point. More precisely, the null space $N(F_U(\lambda_*, 0, \ldots, 0)) = \text{span}\{(k_1\varphi_1, \ldots, k_n\varphi_n)\}$ is one-dimensional.

Suppose that $(h_1, \ldots, h_n)^{\mathrm{T}} \in R(F_U(\lambda_*, 0, \ldots, 0))$, then there exist $(\psi_1, \ldots, \psi_n) \in X$ such that

$$
\begin{aligned}
&F_U(\lambda_*, 0, \ldots, 0) \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} \\
&= \begin{pmatrix} \Delta \psi_1 \\ \vdots \\ \Delta \psi_n \end{pmatrix} + \lambda_* \begin{pmatrix} g'_{11}(0) & \cdots & g'_{1n}(0) \\ \cdots & \ddots & \cdots \\ g'_{n1}(0) & \cdots & g'_{nn}(0) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}.
\end{aligned}
\tag{3.8}
$$

Consider the adjoint eigenvalue equation

$$
\begin{pmatrix} \Delta \psi_1^* \\ \vdots \\ \Delta \psi_n^* \end{pmatrix} + \lambda_* J^{\mathrm{T}} \begin{pmatrix} \psi_1^* \\ \vdots \\ \psi_n^* \end{pmatrix} = \begin{pmatrix} \Delta \psi_1^* \\ \vdots \\ \Delta \psi_n^* \end{pmatrix} + \lambda_* \begin{pmatrix} g'_{11}(0) & \cdots & g'_{n1}(0) \\ \cdots & \ddots & \cdots \\ g'_{1n}(0) & \cdots & g'_{nn}(0) \end{pmatrix} \begin{pmatrix} \psi_1^* \\ \vdots \\ \psi_n^* \end{pmatrix} = 0,
\tag{3.9}
$$

where $(\psi_1^*, \ldots, \psi_n^*)^{\mathrm{T}} = (k_1^* \varphi_1, \ldots, k_n^* \varphi_n)^{\mathrm{T}}$.

Multiplying the system (3.8) by $(\psi_1^*, \ldots, \psi_n^*)$, the system (3.9) by $(\psi_1, \ldots, \psi_n)$, integrating over Ω and subtracting, we get

$$
\int_\Omega \left(h_1 \psi_1^* + \cdots + h_n \psi_n^* \right) dx = \int_\Omega \left(k_1^* h_1 \varphi_1 + \cdots + k_n^* h_n \varphi_1 \right) dx = 0.
\tag{3.10}
$$

Hence $(h_1, \ldots, h_n)^{\mathrm{T}} \in R(F_U(\lambda_*, 0, \ldots, 0))$ if and only if (3.9) holds, which implies that the codimension of $R(F_U(\lambda_*, 0, \ldots, 0))$ is one.

Next we verify that $F_{\lambda U}(\lambda_*, 0, \ldots, 0)[k_1 \varphi_1, \ldots, k_n \varphi_1]^{\mathrm{T}} \notin R(F_U(\lambda_*, 0, \ldots, 0))$. Indeed,

$$
F_{\lambda U}(\lambda_*, 0, \ldots, 0) \begin{pmatrix} k_1 \varphi_1 \\ \vdots \\ k_n \varphi_1 \end{pmatrix} = J \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} \varphi_1 = \chi_J \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix}.
\tag{3.11}
$$

But this contradicts with

$$
0 = \chi_J \int_\Omega \left(k_1 k_1^* + \cdots + k_n k_n^* \right) \varphi_1^2 \, dx > 0.
\tag{3.12}
$$

Hence

$$
F_{\lambda U}(\lambda_*, 0, \ldots, 0)[k_1 \varphi_1, \ldots, k_n \varphi_1]^{\mathrm{T}} \notin R\big(F_U(\lambda_*, 0, \ldots, 0) \big).
$$

By using a bifurcation from a simple eigenvalue theorem of Crandall-Rabinowitz [24], we conclude that $(\lambda_*, 0, \ldots, 0)$ is a bifurcation point. The nontrivial solutions of $F(\lambda, u_1, \ldots, u_n) = (0, \ldots, 0)$ near $(\lambda_*, 0, \ldots, 0)$ are in the form of $\{(\lambda(s), u_1(s), \ldots, u_n(s))\}$ for $s \in (-\delta, \delta)$, where $u_i(s) = k_i s \varphi_1 + o(s)$ $(i = 1, \ldots, n)$. From the stability of positive solutions, each positive solution is non-degenerate.

Next we claim that (1.1) has no positive solution when $\lambda \leq \lambda_*$. Let $(u_1, \ldots, u_n)$ is a positive solution of (1.1) and $(k_1^* \varphi_1, \ldots, k_n^* \varphi_1)$ satisfy

$$
\begin{pmatrix} \Delta k_1^* \varphi_1 \\ \vdots \\ \Delta k_n^* \varphi_1 \end{pmatrix} + \lambda_* \begin{pmatrix} g_{11}'(0) & \cdots & g_{n1}'(0) \\ \cdots & \ddots & \cdots \\ g_{1n}'(0) & \cdots & g_{nn}'(0) \end{pmatrix} \begin{pmatrix} k_1^* \varphi_1 \\ \vdots \\ k_n^* \varphi_1 \end{pmatrix} = 0. \tag{3.13}
$$

Multiplying the system (3.1) by $(k_1^* \varphi_1, k_2^* \varphi_1, \ldots, k_n^* \varphi_n)$, the system (3.13) by $(u_1, \ldots, u_n)$, integrating over Ω and subtracting, by (A$_2$) and $\sum_{j=1}^n g_{ij}(0) = 0$, we obtain

$$
\lambda_* \int_\Omega \sum_{i=1}^n \left[\sum_{j=1}^n k_i^* g_{ij}'(0) u_j \varphi_1 \right] dx = \lambda \int_\Omega \sum_{j=1}^n \left[\sum_{i=1}^n k_i^* g_{ij}(u_j) \varphi_1 \right] dx
$$

$$
< \lambda \int_\Omega \sum_{j=1}^n \left[\sum_{i=1}^n k_i^* g_{ij}'(0) u_j \varphi_1 \right] dx. \tag{3.14}
$$

Hence (3.1) has no positive solution when $\lambda \leq \lambda_*$. And the solution $\{(\lambda(s), u_1(s), \ldots, u_n(s))\}$ can also be parameterized as $(\lambda, u_1(\lambda), \ldots, u_n(\lambda))$ for $\lambda \in (\lambda_*, \lambda_* + \delta)$. With the implicit function theorem, we can extend this curve to the largest λ^*.

Let $\Gamma = \{(\lambda, u_1(\lambda), \ldots, u_n(\lambda)) : \lambda_* < \lambda < \lambda^*\}$. We show that $(u_1(\lambda), \ldots, u_n(\lambda))$ is strictly increasing in λ for $\lambda \in (\lambda_*, \lambda^*)$. In fact, $(\partial u_1(\lambda)/\partial\lambda, \ldots, \partial u_n(\lambda)/\partial\lambda)$ satisfies the equation

$$
F_U(\lambda, U) \begin{pmatrix} \frac{\partial u_1(\lambda)}{\partial\lambda} \\ \vdots \\ \frac{\partial u_n(\lambda)}{\partial\lambda} \end{pmatrix}
$$

$$
= \begin{pmatrix} \Delta \frac{\partial u_1(\lambda)}{\partial\lambda} \\ \vdots \\ \Delta \frac{\partial u_n(\lambda)}{\partial\lambda} \end{pmatrix} + \lambda \begin{pmatrix} g_{11}'(u_1(\lambda)) & \cdots & g_{1n}'(u_n(\lambda)) \\ \vdots & \ddots & \vdots \\ g_{n1}'(u_1(\lambda)) & \cdots & g_{nn}'(u_n(\lambda)) \end{pmatrix} \begin{pmatrix} \frac{\partial u_1(\lambda)}{\partial\lambda} \\ \vdots \\ \frac{\partial u_n(\lambda)}{\partial\lambda} \end{pmatrix}
$$

$$
= - \begin{pmatrix} g_{11}(u_1(\lambda)) + \cdots + g_{1n}(u_n(\lambda)) \\ \vdots \\ g_{n1}(u_1(\lambda)) + \cdots + g_{nn}(u_n(\lambda)) \end{pmatrix}. \tag{3.15}
$$

Hence $(\partial u_1(\lambda)/\partial\lambda, \ldots, \partial u_n(\lambda)/\partial\lambda)$ from the maximum principle (Lemma 2.2 part 3) and the fact that $\mu_1((u_1(\lambda), \ldots, u_n(\lambda))) > 0$ from the stability of positive solutions. We claim that $\lambda^* = \infty$. Suppose not, then $\lambda^* < \infty$ and $\|(u_1(\lambda), \ldots, u_n(\lambda))\|_X < \infty$. Then one can show that the curve Γ can be extended to $\lambda = \lambda^*$ from some standard elliptic estimates, then from the implicit function theorem, Γ can be extended beyond $\lambda = \lambda^*$, which is a contradiction; if $\lambda^* < \infty$ and $\|(u_1(\lambda), \ldots, u_n(\lambda))\|_X = \infty$, a contradiction can be derived with the solution curve which cannot blow up at finite λ^* (see similar arguments for the scalar equation in [25]). Hence we must have $\lambda^* = \infty$.

Finally, we claim the uniqueness. If there is another positive solution for some $\lambda > \lambda_*$, then the arguments above show that this solution also belongs to a solution curve defined for $\lambda \in (\lambda_*, \infty)$, and the solutions on the curve are increasing in λ, but the nonexistence of positive solutions for $\lambda < \lambda_*$ and the local bifurcation at $\lambda = \lambda_*$ exclude the possibility of

another solution curve. Hence the positive solution is unique for all $\lambda > \lambda_*$. This completes the proof. □

Example 3.3 We consider the following cyclic system:

$$\begin{cases} \Delta u_1 + \lambda f_1(u_2) = 0, & x \in \Omega, \\ \Delta u_2 + \lambda f_2(u_3) = 0, & x \in \Omega, \\ \cdots \\ \Delta u_n + \lambda f_n(u_1) = 0, & x \in \Omega, \\ u_1(x) = \cdots = u_n(x) = 0, & x \in \partial\Omega, \end{cases} \tag{3.16}$$

where f_i are smooth real functions defined on $\mathbf{R}_+$ satisfying (A_1), (A_2) and (A_3). Then Theorem 3.2 implies the existence and uniqueness of a positive solution of problem (3.16), and the bifurcation diagram is a single monotone solution curve. The n-dimensional cyclic positone and semipositone system was considered in [26] and [27]. They got the existence and multiplicity of positive solutions result for some combined sublinear condition by the method of sub-super solutions.

Example 3.4 Consider the general Lane-Emden system

$$\begin{cases} \Delta u_1 + \lambda b_1(a_{11} + u_1)^{p_{11}} \cdots (a_{1n} + u_n)^{p_{1n}} = 0, & x \in \Omega, \\ \cdots \\ \Delta u_i + \lambda b_i(a_{i1} + u_1)^{p_{i1}} \cdots (a_{in} + u_n)^{p_{in}} = 0, & x \in \Omega, \\ \cdots \\ \Delta u_i + \lambda b_n(a_{n1} + u_1)^{p_{n1}} \cdots (a_{nn} + u_n)^{p_{nn}} = 0, & x \in \Omega, \\ u_1(x) = \cdots = u_n(x) = 0, & x \in \partial\Omega, \end{cases} \tag{3.17}$$

where $\lambda > 0$ is a positive parameter, b_i, a_{ij} $(1 \le i, j \le n)$ are positive constants, p_{ij} $(1 \le i, j \le n)$ are nonnegative constants satisfying $\sum_{i=1}^{n} p_{ij} < 1$ and $\Omega \subset \mathbf{R}^N$ denotes a bounded domain of class $C^{2,\alpha}$, $\alpha \in (0,1)$. For the case $N = 2$, (3.17) has been studied by many authors. Especially, Dalmasso [10] proved the uniqueness and existence of positive solutions to (3.17) for the case $N = 2$, $p_{11} = p_{22} = 0$, $0 < p_{12}p_{21} < 1$.

In this section, we show the uniqueness and existence of positive solutions for (3.17) by using super-subsolution methods and the stability of positive solutions by using Theorem 2.4.

Let $e(x)$ be the unique positive solution of

$$\Delta e + 1 = 0, \quad x \in \Omega, \qquad e(x) = 0, \quad x \in \partial\Omega.$$

We construct a super-solution $(\overline{u}_1, \ldots, \overline{u}_n) = M(e(x), \ldots, e(x))$. There exists a suitable positive constant M such that

$$\Delta(Me) + \lambda b_i(a_{i1} + Me)^{p_{i1}} \cdots (a_{in} + Me)^{p_{in}}$$

$$\le -M + \lambda b_i(\overline{a}_i + Me)^{p_{i1} + \cdots + p_{in}} \le 0,$$

where $\overline{a}_i = \max\{a_{i1},\ldots,a_{in}\}$. We construct a sub-solution in the form of $(\underline{u}_1,\ldots,\underline{u}_n) = (\varepsilon\varphi_1,\ldots,\varepsilon\varphi_1)$, where ε will be specified later. Recall that φ_1 is the positive principal eigenfunction with $\|\varphi_1\|_\infty = 1$. Now, for the ith equation of (3.17), we have

$$\Delta(\varepsilon\varphi_1) + \lambda b_i(a_{i1} + \varepsilon\varphi_1)^{p_{i1}} \cdots (a_{in} + \varepsilon\varphi_1)^{p_{in}} \geq -\varepsilon\lambda_1\varphi_1 + \lambda b_i(\underline{a}_i + \varepsilon\varphi_1)^{p_{i1}+\cdots+p_{in}}$$

$$\geq -\varepsilon\lambda_1\varphi_1 + \lambda b_i(\varepsilon\varphi_1)^{p_{i1}+\cdots+p_{in}},$$

where $\underline{a}_i = \min\{a_{i1},\ldots,a_{in}\}$. Hence $(\underline{u}_1,\ldots,\underline{u}_n)$ is a subsolution of (3.17), if we choose ε smaller, and it satisfies $\varepsilon < (\lambda b_i/\lambda_1)^{\frac{1}{1-(p_{i1}+\cdots+p_{in})}}\|\varphi_1\|_\infty^{-1}$. Hence if we choose ε smaller so that $(\underline{u}_1,\ldots,\underline{u}_n) \leq (\overline{u}_1,\ldots,\overline{u}_n)$, then $(\underline{u},\underline{v})$ is a subsolution of (4.1). Therefore, there exists a positive solution $(u_1,\ldots,u_n)$ of (3.17) between the supersolution and the subsolution.

Next, we show the solution is stable. Letting $f_i(U) = b_i \prod_{j=1}^n (a_{ij} + u_j)^{p_{ij}}$, by simple calculation, we get

$$f'_{ij}(U) = b_i p_{ij} \prod_{l=1,l\neq j}^n (a_{il} + u_l)^{p_{il}} (a_{ij} + u_{ij})^{p_{ij}-1}$$

and

$$f_i(U) - \sum_{j=1}^n f'_{ij}(U)u_j = b_i \prod_{j=1}^n (a_{ij} + u_j)^{p_{ij}} - \sum_{j=1}^n b_i p_{ij} \prod_{l=1,l\neq j}^n (a_{il} + u_l)^{p_{il}} (a_{ij} + u_{ij})^{p_{ij}-1}$$

$$= b_i \prod_{l=1}^n (a_{il} + u_l)^{p_{il}} \left[1 - \sum_{j=1}^n \frac{p_{ij}}{a_{ij} + u_j} u_j \right]$$

$$> b_i \prod_{l=1}^n (a_{il} + u_l)^{p_{il}} \left[1 - \sum_{j=1}^n p_{ij} \right] > 0.$$

So, f_i satisfies (H$_1$), hence from Theorem 2.4, any positive solution of (3.17) is stable.

4 Application: Hölder continuous case

In this section, we consider that $f_i \in C^\alpha(\overline{\mathbf{R}}_+^n) \cap C^1(\mathbf{R}_+^n)$ for $\alpha \in (0,1)$.

Example 4.1 Consider

$$\begin{cases} \Delta u + \lambda(au - u^2 + \sqrt{uv}) = 0, & x \in \Omega, \\ \Delta v + \lambda(bv - v^2 + \sqrt{uv}) = 0, & x \in \Omega, \\ u(x) = v(x) = 0, & x \in \partial\Omega, \end{cases} \tag{4.1}$$

where $\lambda > 0$ is a positive parameter, $a, b \geq 0$.

We use the monotone method to prove the existence of a solution. For the supersolution, we choose $(\overline{u},\overline{v}) = (a + b + 1, a + b + 1)$. Then

$$\Delta\overline{u} + \lambda\left(a\overline{u} - \overline{u}^2 + \sqrt{\overline{u}\overline{v}}\right) \leq \lambda\left[a(a + b + 1) - (a + b + 1)^2 + (a + b + 1)\right]$$

$$\leq \lambda(a + b + 1)\left[(a + 1) - (a + b + 1)\right] \leq 0.$$

Similarly, we have $0 < v < (a + b + 1)$. Then it is clear that $(\overline{u}, \overline{v})$ is a supersolution of (4.1). We construct a sub-solution in the form of $(\underline{u}, \underline{v}) = (\varepsilon\varphi_1, \varepsilon\varphi_1)$, where ε will be specified later. Recall that φ_1 is the positive principal eigenfunction with $\|\varphi_1\|_\infty = 1$. Now, for the equation of u or the equation of v, we have

$$\Delta(\varepsilon\varphi_1) + \lambda\left(a\varepsilon\varphi_1 - (\varepsilon\varphi_1)^2 + \varepsilon\varphi_1\right) = -\varepsilon\lambda_1\varphi_1 + \lambda\left(a\varepsilon\varphi_1 - (\varepsilon\varphi_1)^2 + \varepsilon\varphi_1\right)$$
$$= \varepsilon\varphi_1\left[\lambda(a + 1) - \lambda_1 - \lambda\varepsilon\varphi_1\right].$$

Hence if we choose ε smaller and it satisfies $\varepsilon < \min\{\frac{\lambda(a+1)-\lambda_1}{\lambda\|\varphi_1\|_\infty}, \frac{\lambda(b+1)-\lambda_1}{\lambda\|\varphi_1\|_\infty}\}$, so that $(\underline{u}, \underline{v}) \leq (\overline{u}, \overline{v})$, then $(\underline{u}, \underline{v})$ is a subsolution of (4.1). Therefore, there exists a positive solution (u, v) of (4.1) between the supersolution and the subsolution when $\lambda > \max\{\lambda_1/(a+1), \lambda_1/(b+1)\}$. We remark that the stability defined in Section 2 can still be established for a nonlinear function f_i' to become ∞ near $\partial\Omega$ by using Remark 3.1 in [28]. Thus the positive solution (u, v) of (4.1) is stable.

Remark 4.2 Since (4.1) is a cooperative model from ecology with logistic growth rate and sublinear interaction term, we can get the stable result. When the interaction terms are uv (Lotka-Volterra type) and $\sqrt{uv}$ as proposed here, and they do not satisfy the conditions of Theorem 3.2, thus, the solution may not be unique.

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