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The Clark dual and multiple periodic solutions of delay differential equations

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Abstract

We study the multiplicity of periodic solutions of a class of non-autonomous delay differential equations. By making full use of the Clark dual, the dual variational functional is considered. Some sufficient conditions are obtained to guarantee the existence of multiple periodic solutions.

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1 Introduction

The existence and multiplicity of periodic solutions of delay differential equations have been investigated since 1962. Various methods have been used to study such a problem [1-10]. Among those methods, critical point theory is a very important tool. By combining with Kaplan-Yorke method, it can be used indirectly to study the existence of periodic solutions of delay differential equation [11-16]. By building the variational frame for some special systems, it can be used directly to study the existence of delay differential systems [17,18]. However, the variational functionals in the above two cases are strongly indefinite. They are hard to be dealt with.

In this article, we study multiple periodic solutions of the following non-autonomous delay differential equations

$$x'(t) = -f(t, x(t - \frac{\pi}{2})), \tag{1}$$

where $x(t) \in \mathbb{R}^n, f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. We assume that

(f1) $f(t, x)$ is odd with respect to x and $\pi/2$-periodic with respect to t, i.e.,

$$f(t, -x) = -f(t, x), \quad f(t + \frac{\pi}{2}, x) = f(t, x), \quad \forall(t, x) \in \mathbb{R} \times \mathbb{R}^n;$$

(f2) There exists a continuous differentiable function $F(t, x)$, which is strictly convex with respect to x uniformly in t, such that $f(t, x)$ is the gradient of $F(t, x)$ with respect to x;

(f3)

$$f(t, x) = A_0 x + g(t, x), \quad g(t, x) = o(|x|) \quad \text{as} \quad |x| \to 0 \quad \text{uniformly in } t;$$

$$f(t, x) = A_\infty x + h(t, x), \quad h(t, x) = o(|x|) \quad \text{as} \quad |x| \to \infty \quad \text{uniformly in } t;$$

where A_0, A_∞ are positive definite constant matrices.

By making use of the Clarke dual, we study the dual variational functional associated with (1), which is an indefinite functional. Since the dimension of its negative space is finite, we define it as the Morse index of the dual variational functional. This Morse index is significant. Then Z_2-index theory can be used and some sufficient conditions are obtained to guarantee the existence of multiple periodic solutions of (1).

The rest of this article is organized as follows: in Section 2, some preliminary results will be stated; in Section 3, linear system is discussed and the Morse index of the variational functional associated with linear system is defined; in Section 4, our main results will be stated and proved.

2 Preliminaries

Denote by $\mathbb{N}$, $\mathbb{N}^*$, $\mathbb{Z}$, $\mathbb{R}$ the sets of all positive integers, nonnegative integers, integers, real numbers, respectively. We define $S^1 := \mathbb{R}/(2\pi\mathbb{Z})$.

For a matrix A, denote by $\sigma(A)$ the set of eigenvalue of A. The identity matrix of order n is denoted by I_n and for simplicity by I.

Let $x, y \in L^2(S^1, \mathbb{R}^n)$. For every $z \in C^\infty(S^1, \mathbb{R}^n)$, if

$$\int_0^{2\pi} (x(t), z'(t)) dt = - \int_0^{2\pi} (y(t), z(t)) dt,$$

then y is called a weak derivative of x, denoted by $\dot{x}$.

The space $H^1 = W^{1,2}(S^1, \mathbb{R}^n)$ consists of 2π-periodic vector-valued functions with dimension n, which possess square integrable derivative of order 1. We can choose the usual norm and inner product in H^1 as follows:

$$\| x \|^2 = \int_0^{2\pi} [|x(t)|^2 + |\dot{x}(t)|^2] dt,$$

$$<x, y> = \int_0^{2\pi} [(x(t), y(t)) + (\dot{x}(t), \dot{y}(t))] dt, \quad \forall x, y \in H^1,$$

where $|\cdot|$, $(\cdot, \cdot)$ denote the usual norm and inner product in $\mathbb{R}^n$, respectively. Then H^1 is a Hilbert space.

Define the shift operator $K : H^1 \to H^1$ by $Kx(\cdot) = x(\cdot + \pi/2)$, for all $x \in H^1$. Clearly, K is a bounded linear operator from H^1 to H^1. Set

$$E = \{x \in H^1 \mid K^2 x = -x\}.$$

Then E is a closed subspace of H^1. If $x \in E$, its Fourier expansion is

$$x(t) = \sum_{k=1}^{\infty} [a_k \cos(2k-1)t + b_k \sin(2k-1)t], \tag{2}$$

where a_k, $b_k \in \mathbb{R}^n$. In particular, E does not contain $\mathbb{R}^n$ as its subspace. In addition, $\int_0^{2\pi} x(t)dt = 0$ for all $x \in E$.

The dual variational functional corresponding to (1) defined on H^1 is

$$J(y) = \int_0^{2\pi} [\frac{1}{2}(\dot{y}(t + \frac{\pi}{2}), y(t)) + F^*(t, \dot{y}(t))]dt. \tag{3}$$

By Hypothesis (f3), $F(t, x)/|x| \to +\infty$ as $|x| \to \infty$ uniformly in t. Since $F(t, \cdot)$ is strictly convex, Proposition 2.4 of [19] implies that $F^*(t, \cdot) \in C^1(\mathbb{R}^n, \mathbb{R})$. Since f satisfies (f3), it follows the discussion in Chapter 7 of [19] that

$$f^*(t, y) = B_\infty y + o(|y|) \qquad \text{as} \qquad |y| \to \infty \quad \text{uniformly in } t, \tag{4}$$

$$f^*(t, y) = B_0 y + o(|y|) \qquad \text{as} \qquad |y| \to 0 \quad \text{uniformly in } t, \tag{5}$$

where $B_\infty = A_\infty^{-1}$ and $B_0 = A_0^{-1}$.

Lemma 2.1. *If $y \in E$ is a critical point of J, then the function x defined by $x(t) = f^*(t, \dot{y}(t))$ is a 2π-periodic solution of (1).*

Proof. Since $f(\cdot, x)$ is $\pi/2$-periodic, it follows that $F(\cdot, x)$ and then $F^*(\cdot, \dot{y})$ are $\pi/2$-periodic. So is $f^*(\cdot, \dot{y})$. (4) implies that there exist positive constants a_1, a_2 such that $|F^*(t, y)| \le a_1 + a_2|y|^s$ for some $s > 2$ and all $t \in \mathbb{R}$, $y \in H^1$. We define ϕ by the formulas

$$\varphi(y) = \int_0^{2\pi} F^*(t, \dot{y}(t))dt.$$

It follows Proposition B. 37 of [20] that $\phi \in C^1(H^1, \mathbb{R})$ and

$$< \varphi'(y), z > = \int_0^{2\pi} (f^*(t, \dot{y}(t)), \dot{z}(t))dt, \qquad \forall y, z \in H^1.$$

We claim: $\phi'(y) \in E$ if $y \in E$.

To prove the above claim, let $z \in H^1$ and $y \in E$,

$$< K^2 \varphi'(y), z > = < \varphi'(y), K^{-2}z > = \int_0^{2\pi} (f^*(t, \dot{y}(t)), \dot{z}(t - \pi))dt$$

$$= \int_0^{2\pi} (f^*(t + \pi, \dot{y}(t + \pi)), \dot{z}(t))dt = \int_0^{2\pi} (f^*(t, -\dot{y}(t)), \dot{z}(t))dt$$

$$= \int_0^{2\pi} (-f^*(t, \dot{y}(t)), \dot{z}(t))dt = < -\varphi'(y), z > .$$

The arbitrary of z implies that $\phi'(y) \in E$.

Since $\phi \in C^1(H^1, \mathbb{R})$, it is easy to check that $J \in C^1(H^1, \mathbb{R})$ and

$$< J'(y), h > = \int_0^{2\pi} (\frac{1}{2}y(t - \frac{\pi}{2}) - \frac{1}{2}y(t + \frac{\pi}{2}) + f^*(t, \dot{y}(t)), \dot{h}(t))dt, \quad h \in H^1.$$

Assume that $y \in E$ is a critical point of J. For any $h \in H^1$, $h = h_1 + h_2$, where $h_1 \in E$, $h_2 \in E^\perp$. Then

$$< J'(y), h > = < J'(y), h_1 > + < J'(y), h_2 > .$$

Since $\phi'(y) \in E$, it is easy to check that $J'(y) \in E$ and $<J'(y), h_2> = 0$. Since y is a critical point of J on E, then $<J'(y), h_1> = 0$. Thus y is a critical point of J on H^1. Applying the fundamental Lemma (cf. [19]), there exists c_1 such that

$$\gamma\left(t - \frac{\pi}{2}\right) + f^*(t, \dot{y}(t)) = c_1, \qquad \text{a.e. on } [0, 2\pi].$$

Setting

$$x(t) = f^*(t, \dot{y}(t)) = -\gamma\left(t - \frac{\pi}{2}\right) + c_1,$$

we obtain $x \in H^1$, $\dot{x}(t - \pi/2) = \dot{y}(t)$ and by duality

$$\dot{y}(t) = f(t, x(t)).$$

Thus,

$$\dot{x}\left(t - \frac{\pi}{2}\right) = f(t, x(t)),$$

i.e.,

$$\dot{x}(t) = -f\left(t, x\left(t - \frac{\pi}{2}\right)\right) \qquad \text{a.e. on } [0, 2\pi].$$

Moreover, $x(0) = x(2\pi)$ since $x \in H^1$. It follows a regular discussion that $x(t) = f^*(t, \dot{y}(t))$ is a periodic solution of (1). □

Let X be a Hilbert space, $\Phi \in C^1(X, \mathbb{R})$, i.e., Φ is a continuously Fréchet differentiable functional defined on X. If X_1 is a closed subspace of X, denote by $X_1^\perp$ the orthogonal complement of X_1 in X. Fix a prime integer $p > 1$. Define a map $\mu : X \to X$ such that $||\mu x|| = ||x||$ for any $x \in X$ and $\mu^p = id_X$, where id_X is the identity map on X. Then μ is a linear isometric action of Z_p on X, where Z_p is the cyclic group with order p.

A subset $A \subset X$ is called μ-invariant if $\mu(A) \subset A$. A continuous map $h : A \to E$ is called μ-equivariant if $h(\mu x) = \mu h(x)$ for any $x \in A$. A continuous functional $H : X \to \mathbb{R}$ is called μ-invariant if $H(\mu x) = H(x)$ for any $x \in X$.

Φ is said to be satisfying (PS)-condition if any sequence $\{x_j\} \subset X$ for which $\{\Phi(x_j)\}$ is bounded and $\Phi'(x_j) \to 0$ as $j \to \infty$, possesses a convergent subsequence. A sequence $\{x_j\}$ is called (PS)-sequence if $\{\Phi(x_j)\}$ is bounded and $\Phi'(x_j) \to 0$ as $j \to \infty$.

Lemma 2.2 [21]. *Let $\Phi \in C^1(X, \mathbb{R})$ be a μ-invariant functional satisfying the (PS)-condition. Let Y and Z be closed μ-invariant subspaces of X with codimY and dimZ finite and*

$$codimY < dimZ.$$

Assume that the following conditions are satisfied:
(F1) $Fix_\mu \subset Y$, $Z \cap Fix_\mu = \{0\}$;
(F2) $\inf_{x \in Y} \Phi(x) > -\infty$;
(F3) there exist constants $r > 0$ and $c < 0$ such that $\Phi(x) \leq c$ whenever $x \in Z$ and $||x|| = r$;
(F4) if $x \in Fix_\mu$ and $\Phi'(x) = 0$, then $\Phi(x) \geq 0$.
Then there exists at least dimZ - codimY distinct Z_p-orbits of critical points of Φ outside of Fix_μ with critical value less than or equal to c.

3 Morse index

Let A be a positive definite constant matrix. We consider the periodic boundary value problem

$$\begin{cases} x'(t) + Ax(t - \frac{\pi}{2}) = 0 \\ x(0) = x(2\pi) \end{cases}. \tag{6}$$

Since $F(t, x) = 1/2(Ax, x)$, it is easy to verify that its Legendre transform $F^*(t, y)$ is of the form $F^*(t, y) = 1/2(By, y)$, where $B = A^{-1}$. The dual action of (6) is defined on H^1 by

$$\chi_A(\gamma) = \int_0^{2\pi} \frac{1}{2}[(\dot{\gamma}(t + \frac{\pi}{2}), \gamma(t)) + (B\dot{\gamma}(t), \dot{\gamma}(t))]dt. \tag{7}$$

A is positively definite, so is B. Thus there exists $\delta_1 > 0$ such that $(By, y) \geq \delta_1|y|^2$ for all $y \in \mathbb{R}^n$. Wirtinger's inequality implies that the symmetric bilinear form given by

$$((\gamma, z))_1 = \int_0^{2\pi} (B\dot{\gamma}(t), \dot{z}(t))dt \tag{8}$$

define an inner product on E. The corresponding norm $|| \cdot ||_1$ is such that

$$\| \gamma \|_1^2 \geq \delta_1 \| \dot{\gamma} \|_{L^2}^2. \tag{9}$$

Proposition 3.1. *The norm $|| \cdot ||_1$ is equivariant to the stand norm $|| \cdot ||$ of E.*

Proof. Since B is positive, Wirtinger's inequality and (9) imply that there exists a positive constant δ_2 such that

$$\| x \|_1^2 = \int_0^{2\pi} (B\dot{x}(t), \dot{x}(t))dt \geq \delta_1 \| \dot{x} \|_{L^2}^2 \geq \delta_1 \delta_2 \| x \|^2.$$

On the other hand, since B is a positive definite matrix, there exists a constant $M > 0$ such that

$$\| x \|^2 \geq \| \dot{x} \|_{L^2}^2 \geq \frac{1}{M} \int_0^{2\pi} (B\dot{x}(t), \dot{x}(t))dt = \frac{1}{M} \| x \|_1^2.$$

Thus those two norms are equivariant to each other, which completes our proof. □

Let us define the linear operator L on E by setting

$$((L\gamma, z))_1 = \int_0^{2\pi} \left(\gamma(t + \frac{\pi}{2}), \dot{z}(t)\right) dt. \tag{10}$$

One can easily check that L is a compact self-adjoint operator. (7) can be rewritten as

$$2\chi_A(\gamma) = \int_0^{2\pi} \left[\left(\dot{\gamma}\left(t + \frac{\pi}{2}\right), \gamma(t)\right) + (B\dot{\gamma}(t), \dot{\gamma}(t))\right] dt = (((I - L)\gamma, \gamma))_1. \tag{11}$$

It follows from the spectral theory that E can be decomposed as the orthogonal sum of $Ker(I - L)$, E^+ and E^- with $I - L$ positive definite (resp. negative definite) on E^+ (resp. E^-). Since L is compact, it has at most finite many eigenvalues (counting the multiplicity) greater than one. Thus the index $\dim E^- < \infty$.

Definition 3.1. *The index $i(A)$ is the Morse index of χ_A, i.e. the supermum of the dimension of the subspace of E on which χ_A is negative definite.*

On the other hand, there exists a positive constant δ_3 such that

$$(((I - L)\gamma, \gamma))_1 \geq \delta_3 \parallel \gamma \parallel_1^2, \forall \gamma \in E^+ \quad \text{and} \quad (((I - L)\gamma, \gamma))_1 \leq -\delta_3 \parallel \gamma \parallel_1^2, \quad \forall \gamma \in E^-.$$

Setting $\delta = \delta_1 \delta_3 > 0$, we reduce from (9) and (11) the estimates

$$\chi_A(\gamma) \geq \frac{\delta}{2} \parallel \dot{\gamma} \parallel_{L^2}^2, \quad \forall \gamma \in E^+ \quad \text{and} \quad \chi_A(\gamma) \leq -\frac{\delta}{2} \parallel \dot{\gamma} \parallel_{L^2}^2, \quad \forall \gamma \in E^-. \tag{12}$$

Proposition 3.2. *The dimension of $Ker(I - L)$ is equal to the number of linearly independent solutions of (6).*

Proof. If $y \in E$, by a Fourier argument, $y \in Ker(I - L)$ if and only if

$$B\dot{y}(t) = \gamma\left(t + \frac{\pi}{2}\right) + c_2, \tag{13}$$

for some $c_2 \in \mathbb{R}^n$ and a.e. $t \in [0, 2\pi]$. Since $B = A^{-1}$ is invertible, y is continuous differentiable. Thus (13) hold for all $t \in [0, 2\pi]$, and $B\dot{y} \in E \oplus \mathbb{R}^n$. Set

$$x(t) = (\varphi\gamma)(t) = B\dot{y}(t). \tag{14}$$

In view of (13), (14) and the Clark dual, we have

$$\dot{x}(t) = \frac{d}{dt}(B\dot{y}(t)) = \dot{\gamma}\left(t + \frac{\pi}{2}\right) = Ax\left(t + \frac{\pi}{2}\right).$$

Thus x is a solution of (6). □

Conversely, assume now that $x \in E \oplus \mathbb{R}^n$ is a solution of (6). Then

$$\int_0^{2\pi} Ax\left(t - \frac{\pi}{2}\right) dt = 0,$$

and hence there exists an unique $y \in E$ such that $x = \phi y$, where ϕ is defined by (14). Thus

$$-\dot{x}(t) = Ax\left(t - \frac{\pi}{2}\right) = \dot{\gamma}\left(t - \frac{\pi}{2}\right)$$

for a.e. $t \in [0, 2\pi]$. Integrating the above equality, we have $x(t) = y(t + \pi/2) + c_3$ for some $c_3 \in \mathbb{R}^n$ and all $t \in [0, 2\pi]$. Consequently,

$$A[\gamma\left(t + \frac{\pi}{2}\right) + c_3] = \dot{\gamma}(t)$$

for a.e. $t \in [0, 2\pi]$, which is equivariant to (13) and shows that $y \in Ker(I - L)$. Thus ϕ is an isomorphism between $Ker(I - L)$ and the space of solutions of (6).

Consider the following eigenvalue problem

$$(I - L)\gamma = \lambda\gamma, \quad \forall \gamma \in E.$$

For any $z \in E$, we have $(((I - L)y, z))_1 = > \lambda((y, z))_1$. Computing directly, we have

$$\int_0^{2\pi} \left(\sum_{k=1}^{\infty} [(2k - 1)B + (-1)^k I][-a_k \cos(2k - 1)t + b_k \sin(2k - 1)t], \dot{z}(t)\right) dt$$

$$= \lambda \int_0^{2\pi} \left(\sum_{k=1}^{\infty} (2k - 1)B[-a_k \cos(2k - 1)t + b_k \sin(2k - 1)t], \dot{z}(t)\right) dt.$$

Denote by e_i, $i = 1, 2, ..., n$ the basic of $\mathbb{R}^n$. Choosing

$$z(t) = \begin{Bmatrix} \sin(2k-1)t \\ \cos(2k-1)t \end{Bmatrix} e_i, \quad k \in \mathbb{N}, i = 1, 2, \ldots, n,$$

it follows that

$$\begin{cases} ((2k-1)B + (-1)^k I)a_k = \lambda(2k-1)Ba_k, \\ ((2k-1)B + (-1)^k I)b_k = \lambda(2k-1)Bb_k. \end{cases}$$

Since $B = A^{-1}$, then

$$\begin{cases} \lambda a_k = (\dfrac{(-1)^k}{2k-1}A + I)a_k, \\ \lambda b_k = (\dfrac{(-1)^k}{2k-1}A + I)b_k. \end{cases}$$

Thus

$$\dim \operatorname{Ker}(I - L) = 2\sum_{k=1}^{\infty} \dim \operatorname{Ker}\left(\frac{(-1)^k}{2k-1}A + I\right).$$

Proposition 3.3. *If* $\sigma(A) \cap \{4l + 1, l \in \mathbb{N}^*\} = \varnothing$, *then* $I - L$ *is invertible and* $\operatorname{codim}E^+ = i(A)$.

Proof. The above analysis implies that $\operatorname{Ker}(I - L) = \{0\}$ if $\sigma(A) \cap \{4l + 1, l \in \mathbb{N}^*\} = \varnothing$. Thus $I - L$ is invertible. If we decompose E as $E = E^+ \oplus E^-$, then $\operatorname{codim}E^+ = i(A)$. □

4 Main results and their proofs

Now we consider the multiplicity of periodic solutions of (1). The main result reads as follows:

Theorem 4.1. *Assume that (f1)-(f3) are satisfied. Moreover,*

(A1) $\sigma(A_\infty) \cap \{4l + 1, l \in \mathbb{N}^*\} = \varnothing$;

(A2) $i(A_0) > i(A_\infty)$.

Then (1) has at least $i(A_0) - i(A_\infty)$ *pairs of nontrivial* 2π-*periodic solutions.*

By (f3), $f(t, 0) = 0$ uniformly in t. Since $F(t, \cdot)$ is strictly convex, it follows that 0 is the unique equilibrium point of (1). Without loss of generality, we can assume that $F(t, 0) = 0$ uniformly in t. Since $f(t, 0) = 0$, then $F^*(t, 0) = 0$ uniformly in t.

Lemma 4.1. *The functional J satisfies (PS)-condition.*

Proof. Denote by $((\cdot, \cdot))_{1,\infty}$ the inner product, defined by (8) replacing B by B_∞. Let L_∞ be linear self-adjoint operator, defined by (10), under the inner product $((\cdot, \cdot))_{1,\infty}$. We define the operator N over E by setting

$$((N\gamma, z))_{1,\infty} = \int_0^{2\pi} (f^*(t, \dot{\gamma}) - B_\infty\dot{\gamma}, \dot{z})dt, \quad \forall \gamma, z \in E.$$

Let $\{\gamma_j\}_{j=1}^{\infty}$ be a (PS)-sequence. Define $f_j := y_j - L_\infty y_j + Ny_j$, $j \in \mathbb{N}$. Since

$$< J'(\gamma), z >= \int_0^{2\pi} \left(\gamma\left(t - \frac{\pi}{2}\right) + f^*(t, \dot{\gamma}), \dot{z}\right)dt = ((\gamma - L_\infty\gamma + N\gamma, z))_{1,\infty},$$

then $f_j \to 0$ in E as $j \to \infty$. Then there exists $R > 0$ such that for all $j \in \mathbb{N}$, $|| f_j || \leq R$. (A1) implies that $M = I - L_\infty$ is invertible. Thus it follows from (4) that there exists a positive constant c_4 such that, for all $y \in E$

$$ || N\gamma ||_{1,\infty} \leq \frac{1}{2} || M^{-1} ||_{1,\infty}^{-1} || \gamma ||_{1,\infty} + c_4 $$

where $|| \cdot ||_{1,\infty}$ denotes by the norm corresponding to $((\cdot, \cdot))_{1,\infty}$. Since $|| f_j || \leq R$, we have

$$ || \gamma_j ||_{1,\infty} \leq || M^{-1} ||_{1,\infty}(|| N\gamma_j ||_{1,\infty} + || f_j ||_{1,\infty}) \leq \frac{1}{2} || \gamma_j ||_{1,\infty} + || M^{-1} ||_{1,\infty}(c_4 + R), \qquad j \in \mathbb{N}. $$

The above inequality implies that $\{y_j\}$ is bounded. A standard argument as Lemma 4.5 in [19] shows that $\{y_j\}$ has a convergent sequence. □

Lemma 4.2. *The functional J is bounded from below on a closed invariant subspace Y of E of codimension $i(A_\infty)$.*

Proof. Consider the linear delay differential system

$$ x'(t) = -A_\infty x \left(t - \frac{\pi}{2} \right). $$

Its dual variational functional is

$$ \chi_{A_\infty}(\gamma) = \int_0^{2\pi} \frac{1}{2} \left[\left(\dot{\gamma} \left(t + \frac{\pi}{2} \right), \gamma(t) \right) + (B_\infty \dot{\gamma}(t), \dot{\gamma}(t)) \right] dt, $$

where $((\cdot, \cdot))_{1,\infty}$, L_∞ are defined as Lemma 4.1.

Because of (A1) and Proposition 3.3, the operator $I - L_\infty$ is invertible. We decompose E as $E = E_\infty^+ \oplus E_\infty^-$, where E_∞^+(resp. E_∞^-) is the positive (resp. negative) definite eigenspace of $I - L_\infty$. Then $\operatorname{codim} E_\infty^+ = \dim E_\infty^- = i(A_\infty)$. Let $Y := E_\infty^+$. Inequality (12) implies that there exists a positive constant δ_4 such that, for each $y \in Y$,

$$ \chi_{A_\infty}(\gamma) = \int_0^{2\pi} \frac{1}{2} \left[\dot{\gamma} \left(t + \frac{\pi}{2} \right), \gamma(t) \right) + (B_\infty \dot{\gamma}, \dot{\gamma}) \right] dt \geq \frac{\delta_4}{2} || \dot{\gamma} ||_{L^2}^2. $$

It follows (4) that there exists a positive constant c_5 such that

$$ | f^*(t, \gamma) - B_\infty \gamma | \leq \frac{\delta_4}{2} | \gamma | + c_5 $$

for each $y \in \mathbb{R}^n$. Hence, by the mean value theorem,

$$ | F^*(t, \gamma) - \frac{1}{2}(B_\infty \gamma, \gamma) | \leq \int_0^1 | (f^*(t, \sigma\gamma) - \sigma B_\infty \gamma, \gamma) | \, d\sigma $$

$$ \leq \int_0^1 \left[\frac{\delta_4}{2} \sigma | \gamma |^2 + c_5 | \gamma | \right] d\sigma = \frac{\delta_4}{4} | \gamma |^2 + c_5 | \gamma |. $$

Consequently, we have, for $y \in Y$,

$$ J(\gamma) = \chi_{A_\infty}(\gamma) + \int_0^{2\pi} [F^*(t, \dot{\gamma}) - \frac{1}{2}(B_\infty \dot{\gamma}, \dot{\gamma})] dt $$

$$ \geq \frac{\delta_4}{2} || \dot{\gamma} ||_{L^2}^2 - \int_0^{2\pi} \left[\frac{\delta_4}{4} | \dot{\gamma} |^2 + c_5 | \dot{\gamma} | \right] dt $$

$$ = \frac{\delta_4}{4} || \dot{\gamma} ||_{L^2}^2 - c_5 || \dot{\gamma} ||_{L^1} \geq \frac{\delta_4}{4} || \dot{\gamma} ||_{L^2}^2 - c_5 \sqrt{2\pi} || \dot{\gamma} ||_{L^2} $$

and J is bounded from below on Y.

Lemma 4.3. *There exists an invariant subspace Z of E with dimension $i(A_0)$ and some $r > 0$ such that $J(y) < 0$ whenever $y \in Z$ and $||y||_1 = r$.*

Proof. Consider the linear delay differential system

$$x'(t) = -A_0 x\left(t - \frac{\pi}{2}\right).$$

Its dual variational functional is

$$\chi_{A_0}(\gamma) = \int_0^{2\pi} \frac{1}{2}\left[\left(\dot{\gamma}\left(t + \frac{\pi}{2}\right), \gamma(t)\right) + (B_0\dot{\gamma}(t), \dot{\gamma}(t))\right] dt,$$

where $((\cdot, \cdot))_{1,0}$, L_0 are defined by (8) and (10) with B replaced by B_0, respectively.

We decompose E as $E = E_0^+ \oplus E_0^0 \oplus E_0^-$, where $E_0^0 = \ker(I - L_0)$, $E_0^+(\text{resp. } E_0^-)$ is the positive (resp. negative) definite eigenspace of $I - L_0$. Let $Z := E_0^-$. Then Z is a finite dimensional space. Inequality (12) implies that there exists $\delta_5 > 0$ such that, for each $y \in Z$,

$$\chi_{A_0}(\gamma) = \int_0^{2\pi} \frac{1}{2}\left[\left(\dot{\gamma}\left(t + \frac{\pi}{2}\right), \gamma(t)\right) + (B_0\dot{\gamma}, \dot{\gamma})\right] dt \le -\frac{\delta_5}{2} \| \dot{\gamma} \|_{L^2}^2$$

whenever $y \in Z$. By (5), there exists $\rho > 0$ such that

$$|f^*(t, \gamma) - B_0\gamma| \le \frac{\delta_5}{2} |\gamma|,$$

for $y \in \mathbb{R}^n$ with $|y| \le \rho$. Hence, by the mean value theorem, we have

$$
\begin{aligned}
\left| F^*(t, \gamma) - \frac{1}{2}(B_0\gamma, \gamma) \right| &\le \int_0^1 |(f^*(t, \sigma\gamma) - \sigma B_0\gamma, \gamma)| \, d\sigma \\
&\le \int_0^1 \frac{\delta_5}{2} \sigma |\gamma|^2 d\sigma = \frac{\delta_5}{4} |\gamma|^2
\end{aligned}
\tag{15}
$$

whenever $|y| \le \rho$. Consequently, if $y \in Z$ and $0 < |y| < \rho$, we get

$$
\begin{aligned}
J(\gamma) &= \chi_{A_0}(\gamma) + \int_0^{2\pi} \left[F^*(t, \dot{\gamma}) - \frac{1}{2}(B_0\dot{\gamma}, \dot{\gamma})\right] dt \\
&\le -\frac{\delta_5}{2} \| \dot{\gamma} \|_{L^2}^2 + \int_0^{2\pi} \frac{\delta_5}{4} |\dot{\gamma}|^2 dt = -\frac{\delta_5}{4} \| \dot{\gamma} \|_{L^2}^2 \le 0.
\end{aligned}
$$

If $J(y) = 0$, then $\| \dot{\gamma} \|_{L^2} = 0$, which implies that $|\dot{\gamma}| = 0$ for a.e. $t \in [0, 2\pi]$. Thus $y \in \mathbb{R}^n \cap Z = \{0\}$, which contradicts with $0 < |y| < \rho$. Thus $J(y) < 0$.

Proof of Theorem 4.1. Now, we apply Lemma 2.2 to prove Theorem 4.1.

Define the action $\mu : E \to E$ by $\mu x = -x$. Then μ is a generator of group Z_2 and $\text{Fix}_\mu = \{0\}$. Obviously, (F1) and (F4) hold. It is easy to check that J is μ-invariant. Lemma 4.1 implies that J satisfies (PS)-condition.

Let Y, Z define as in Lemma 4.2 and 4.3. Then Y, Z are μ-invariant subspaces and (A2) implies that

$$\dim Z = i(A_0) > i(A_\infty) = \text{codim} Y.$$

It follows from Lemmas 4.2 and 4.3 that (F2) and (F3) hold. Applying Lemma 2.2, J has at least $i(A_0) - i(A_\infty)$ pairs of distinct critical points with critical values less than or

equal to c. Since $J(0) = 0$ and $E \cap \mathbb{R}^n = \{0\}$, if follows that all those $i(A_0) - i(A_\infty)$ pairs of distinct critical points are nonconstant.

If y is a critical point of J, then $x := f^*(t, \dot{y})$ is a 2π-periodic solution of (1). Clearly, $y \neq 0$ implies that $x \neq 0$. Let x_1, x_2 are two periodic solutions satisfying $x_1 = -x_2$. Setting $\dot{y}_1 = f(t, x_1)$ and $\dot{y}_2 = f(t, x_2)$, then

$$\dot{y}_1 = f(t, x_1) = f(t, -x_2) = -f(t, x_2) = -\dot{y}_2.$$

Integrating he above equality, we have $y_1 = -y_2 + c_6$ for some $c_6 \in \mathbb{R}^n$. Since $y_1, y_2 \in E$, it follows that $c_6 = 0$. Then y_1, y_2 belongs the same Z_2-orbit. Thus (1) has $i(A_0) - i(A_\infty)$ pairs of nontrivial periodic solutions.

Corollary 4.1. *Under the hypotheses of Theorem 4.1, (1) has at least* $2[i(A_0) - i(A_\infty)]$ *nontrivial 2π-periodic solutions.*

Since every Z_2-orbit has two elements and those two elements are different from each other, this corollary is obvious.

Authors' contributions
The manuscript was drafted by HX and it is based on his PhD thesis. JY and ZG were the supervisors of the thesis and gave detailed comments on the manuscript. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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