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Asymptotic properties of solutions of third-order nonlinear neutral dynamic equations

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Abstract

The purpose of this article is to give oscillation criteria for the third-order neutral dynamic equation $(r_2(t)[(r_1(t)[y(t)+p(t)y(\tau(t))]^{\triangle})^{\triangle}]^{\gamma})^{\triangle}+f(t,y(\delta(t)))=0$, where $\gamma \geq 1$ is a ratio of odd positive integers with $r_1(t)$, $r_2(t)$, and $p(t)$ are positive real-valued rd-continuous functions defined on $\mathbb{T}$. We give new results for the third-order neutral dynamic equations and an example to illustrate the importance of our results.

Keywords: oscillation; dynamic equations; time scales; neutral equations

1 Introduction

In the present article, we are concerned with oscillations of the third-order nonlinear neutral dynamic equation

$$\left(r_2(t)\left[\left(r_1(t)\left[y(t)+p(t)y\left(\tau(t)\right)\right]^{\triangle}\right)^{\triangle}\right]^{\gamma}\right)^{\triangle}+f\left(t,y\left(\delta(t)\right)\right)=0 \tag{1}$$

on a time scale $\mathbb{T}$. Throughout this paper it is assumed that $\gamma \geq 1$ is a ratio of odd positive integers, $\tau(t):\mathbb{T}\to\mathbb{T}$ and $\delta(t):\mathbb{T}\to\mathbb{R}$ are rd-continuous functions such that $\tau(t)\leq t$, $\delta(t)\leq t$, $\lim_{t\to\infty}\delta(t)=\lim_{t\to\infty}\tau(t)=\infty$ and $\delta^{\triangle}(t)>0$ is rd-continuous, $r_1(t)$, $r_2(t)$ and $p(t)$ are positive real valued rd-continuous functions defined on $\mathbb{T}$, $0\leq p(t)\leq p<1$ is increasing. We define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}}=[t_0,\infty)\cap\mathbb{T}$. Furthermore, $f:\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is a continuous function such that $uf(t,u)>0$ for all $u\neq 0$ and there exists a rd-continuous positive function $q(t)$ defined on $\mathbb{T}$ such that $|f(t,u)|\geq q(t)|u^{\gamma}|$.

We use throughout this paper the following notations for convenience and for shortening the equations:

$$\begin{aligned} &x(t)=y(t)+p(t)y\left(\tau(t)\right), \qquad x^{[1]}=\left(r_1x^{\triangle}\right)^{\triangle}, \\ &x^{[2]}=r_2\left(x^{[1]}\right)^{\gamma}, \qquad x^{[3]}=\left(x^{[2]}\right)^{\triangle}. \end{aligned} \tag{2}$$

A nontrivial function $y(t)$ is said to be a solution of (1) if $x\in C^1_{\mathrm{rd}}[t_y,\infty)$, $r_1x^{\triangle}\in C^1_{\mathrm{rd}}[t_y,\infty)$ and $x^{[2]}\in C^1_{\mathrm{rd}}[t_y,\infty)$ for $t_y\geq t_0$ and $y(t)$ satisfies equation (1) for $t_y\geq t_0$. A solution of (1) which is nontrivial for all large t is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory.

Recently, there has been many important research activity on the oscillatory behavior of dynamic equations. For example, on second-order dynamic equations, Saker [1], and

Agarwal *et al.* [2], Saker [3], Hassan [4] and Candan [5, 6] considered the following nonlinear dynamic equations:

$$\big(r(t)\big(\big(y(t)+p(t)y(t-\tau)\big)^{\triangle}\big)^{\gamma}\big)^{\triangle}+f\big(t,y(t-\delta)\big)=0,$$

$$\big(r(t)\big(\big(y(t)+p(t)y\big(\tau(t)\big)\big)^{\triangle}\big)^{\gamma}\big)^{\triangle}+f\big(t,y\big(\delta(t)\big)\big)=0,$$

$$\big(r(t)\big(x^{\triangle}(t)\big)^{\gamma}\big)^{\triangle}+p(t)x^{\gamma}(t)=0,$$

and

$$\big(r(t)\big(\big(y(t)+p(t)y\big(\tau(t)\big)\big)^{\triangle}\big)^{\gamma}\big)^{\triangle}+\int_c^d f\big(t,y\big(\theta(t,\xi)\big)\big)\triangle\xi=0,$$

respectively, and they gave sufficient conditions which guarantee that every solution of the equation oscillates. Moreover, there are also some papers on third-order dynamic equations. For instance, Erbe *et al.* [7] considered the third-order nonlinear dynamic equation

$$\big(c(t)\big(a(t)x^{\triangle}(t)\big)^{\triangle}\big)^{\triangle}+q(t)f\big(x(t)\big)=0.$$

Later, Erbe *et al.* [8] considered the third-order nonlinear dynamic equation

$$x^{\triangle\triangle\triangle}(t)+p(t)x(t)=0$$

by giving Hille and Nehari type criteria. Then, Hassan [9] studied the third-order nonlinear dynamic equation

$$\big(a(t)\big(\big(r(t)x^{\triangle}(t)\big)^{\triangle}\big)^{\gamma}\big)^{\triangle}+f\big(t,x\big(\tau(t)\big)\big)=0.$$

Lastly, Wang and Xu [10] studied asymptotic properties of a certain third-order dynamic equation,

$$\big(r_2(t)\big(\big(r_1(t)x^{\triangle}(t)\big)^{\triangle}\big)^{\gamma}\big)^{\triangle}+q(t)f\big(x(t)\big)=0.$$

As we see from all the above, our equation, a neutral dynamic equation, is more general than other third-order dynamic equations and therefore it is very important. For some other important articles on oscillations of second-order nonlinear neutral delay dynamic equation on time scales and oscillations of third-order neutral differential equations, we refer the reader to the papers [11, 12], and [13], respectively. We give [14, 15] as references for books on the time scale calculus.

2 Main results

Lemma 1 *Assume that y is an eventually positive solution of* (1) *and*

$$\int_{t_0}^{\infty}\frac{\triangle t}{r_1(t)}=\infty,\qquad \int_{t_0}^{\infty}\left(\frac{1}{r_2(t)}\right)^{\frac{1}{\gamma}}\triangle t=\infty. \tag{3}$$

Then, there is a $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that either

$$\text{(i)}\quad x(t)>0,\qquad x^{\triangle}(t)>0,\qquad x^{[1]}(t)>0,\quad t\in[t_1,\infty)_{\mathbb{T}},$$

or

$$\text{(ii)}\quad x(t) > 0, \qquad x^{\triangle}(t) < 0, \qquad x^{[1]}(t) > 0, \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

Proof Assume that $y(t) > 0$ for $t \geq t_0$ and therefore $y(\tau(t)) > 0$ and $y(\delta(t)) > 0$ for $t \geq t_1 > t_0$. Consequently, $x(t) > 0$, eventually. Using (2) in (1) and the fact that $|f(t,u)| \geq q(t)|u^{\gamma}|$, we obtain

$$x^{[3]}(t) + q(t)\big(y\big(\delta(t)\big)\big)^{\gamma} \leq 0, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{4}$$

Hence, we conclude that $x^{[2]}(t)$ is a strictly decreasing function on $[t_1, \infty)_{\mathbb{T}}$. We claim that $x^{[2]}(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. If not, then there exists a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $x^{[2]}(t) < 0$ on $[t_2, \infty)_{\mathbb{T}}$. Then, there exist a negative constant c and $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that

$$x^{[2]}(t) \leq c < 0, \quad t \in [t_3, \infty)_{\mathbb{T}}$$

and it follows that

$$x^{[1]}(t) \leq \left(\frac{c}{r_2(t)}\right)^{\frac{1}{\gamma}}. \tag{5}$$

Integrating (5) from t_3 to t and using (3), we obtain

$$r_1(t)x^{\triangle}(t) \leq r_1(t_3)x^{\triangle}(t_3) + c^{\frac{1}{\gamma}} \int_{t_3}^{t} \left(\frac{1}{r_2(s)}\right)^{\frac{1}{\gamma}} \triangle s,$$

which implies that $r_1(t)x^{\triangle}(t) \to -\infty$ as $t \to \infty$. Therefore, there exists a $t_4 \in [t_3, \infty)_{\mathbb{T}}$ such that

$$r_1(t)x^{\triangle}(t) \leq r_1(t_4)x^{\triangle}(t_4) < 0, \quad t \in [t_4, \infty)_{\mathbb{T}}. \tag{6}$$

Dividing both sides of (6) by $r_1(t)$ and integrating from t_4 to t, we obtain

$$x(t) - x(t_4) \leq r_1(t_4)x^{\triangle}(t_4) \int_{t_4}^{t} \left(\frac{1}{r_1(s)}\right) \triangle s.$$

Hence, we see from (3) that $x(t) \to -\infty$ as $t \to \infty$, which contradicts the fact that $x(t) > 0$, and therefore $x^{[2]}(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. As a result of $x^{[1]}(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$ it follows that $r_1(t)x^{\triangle}(t) < 0$ on $[t_1, \infty)_{\mathbb{T}}$ or $r_1(t)x^{\triangle}(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$, which completes the proof. □

Lemma 2 *Let y be an eventually positive solution of* (1). *Assume that Case* (i) *of Lemma* 1 *holds. Then, there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that*

$$x^{\triangle}(t) \geq \frac{r_2(t, t_1)}{r_1(t)} \big[x^{[2]}(t)\big]^{\frac{1}{\gamma}}, \quad t \in [t_1, \infty)_{\mathbb{T}}, \tag{7}$$

where $r_2(t,t_1) = \int_{t_1}^{t} \frac{\triangle s}{(r_2(s))^{\frac{1}{\gamma}}}$ *and*

$$x(t) \geq r_1(t,t_1)\big[x^{[2]}(t)\big]^{\frac{1}{\gamma}}, \quad t \in [t_1,\infty)_{\mathbb{T}},$$

where $r_1(t,t_1) = \int_{t_1}^{t} \frac{r_2(s,t_1)}{r_1(s)} \triangle s$.

Proof Since $x^{[2]}(t)$ is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$, we have

$$\begin{aligned} r_1(t)x^{\triangle}(t) &\geq r_1(t)x^{\triangle}(t) - r_1(t_1)x^{\triangle}(t_1) \\ &= \int_{t_1}^{t} \frac{[x^{[2]}(s)]^{\frac{1}{\gamma}}}{(r_2(s))^{\frac{1}{\gamma}}} \triangle s, \end{aligned}$$

it follows that

$$x^{\triangle}(t) \geq \frac{[x^{[2]}(t)]^{\frac{1}{\gamma}}}{r_1(t)} \int_{t_1}^{t} \frac{\triangle s}{(r_2(s))^{\frac{1}{\gamma}}}$$

or

$$x^{\triangle}(t) \geq \frac{r_2(t,t_1)}{r_1(t)}\big[x^{[2]}(t)\big]^{\frac{1}{\gamma}}, \quad t \in [t_1,\infty)_{\mathbb{T}}. \tag{8}$$

Similarly, integrating (8) from t_1 to t, we obtain

$$x(t) \geq \big[x^{[2]}(t)\big]^{\frac{1}{\gamma}} \int_{t_1}^{t} \frac{r_2(s,t_1)}{r_1(s)} \triangle s$$

or

$$x(t) \geq r_1(t,t_1)\big[x^{[2]}(t)\big]^{\frac{1}{\gamma}}, \quad t \in [t_1,\infty)_{\mathbb{T}}.$$

This completes the proof. □

Lemma 3 *Let* y *be an eventually positive solution of* (1). *Assume that Case* (ii) *of Lemma* 1 *holds. If*

$$\int_{t_0}^{\infty} \frac{1}{r_1(t)} \int_{t}^{\infty} \left[\frac{1}{r_2(s)} \int_{s}^{\infty} q(u)\triangle u\right]^{1/\gamma} \triangle s \triangle t = \infty, \tag{9}$$

then $\lim_{t\to\infty} y(t) = 0$.

Proof Since Case (ii) of Lemma 1 is satisfied,

$$\lim_{t\to\infty} x(t) = l \geq 0.$$

We claim that $\lim_{t\to\infty} x(t) = 0$. Assume that $l > 0$. Then for any $\epsilon > 0$, we have $l < x(t) < l + \epsilon$ for sufficiently large $t \geq t_1$. Choose $0 < \epsilon < \frac{l(1-p)}{p}$. On the other hand, since

$$x(t) = y(t) + p(t)y\big(\tau(t)\big),$$

we have

$$\begin{aligned} y(t) &\geq x(t) - px\bigl(\tau(t)\bigr) \\ &> l - p(l+\epsilon) \\ &= k(l+\epsilon) \\ &> kx(t), \quad t \geq t_2 \geq t_1, \end{aligned}$$

where $k = \frac{l-p(l+\epsilon)}{l+\epsilon} > 0$. Then,

$$\bigl(y\bigl(\delta(t)\bigr)\bigr)^{\gamma} \geq k^{\gamma}\bigl(x\bigl(\delta(t)\bigr)\bigr)^{\gamma}, \quad t \geq t_3 \geq t_2. \tag{10}$$

Substituting (10) into (4), we obtain

$$x^{[3]}(t) \leq -q(t)k^{\gamma}\bigl(x\bigl(\delta(t)\bigr)\bigr)^{\gamma}, \quad t \geq t_3. \tag{11}$$

Integrating (11) from t to ∞, we get

$$x^{[2]}(t) \geq k^{\gamma}\int_t^{\infty} q(s)\bigl(x\bigl(\delta(s)\bigr)\bigr)^{\gamma}\triangle s, \quad t \geq t_3$$

or using $x(\delta(t)) > l$,

$$x^{[1]}(t) \geq kl\left[\frac{1}{r_2(t)}\int_t^{\infty} q(s)\triangle s\right]^{1/\gamma}, \quad t \geq t_3. \tag{12}$$

Integrating (12) from t to ∞ and dividing both sides by $r_1(t)$, we have

$$-x^{\triangle}(t) \geq \frac{kl}{r_1(t)}\int_t^{\infty}\left[\frac{1}{r_2(u)}\int_u^{\infty} q(s)\triangle s\right]^{1/\gamma}\triangle u, \quad t \geq t_3. \tag{13}$$

Integrating (13) from t_3 to ∞, we obtain

$$x(t_3) \geq kl\int_{t_3}^{\infty}\frac{1}{r_1(t)}\int_t^{\infty}\left[\frac{1}{r_2(s)}\int_s^{\infty} q(u)\triangle u\right]^{1/\gamma}\triangle s\triangle t,$$

which contradicts (9) and therefore $l = 0$. By making use of $0 \leq y(t) \leq x(t)$, we conclude that $\lim_{t\to\infty} y(t) = 0$. □

Theorem 2.1 *Assume that* $\delta(\sigma(t)) = \sigma(\delta(t))$. *Furthermore, suppose that* (3), (9), *and*

$$\int_{t_0}^{\infty} Q(s)\triangle s = \infty, \tag{14}$$

where $Q(s) = q(s)(1-p)^{\gamma}$, *hold. Then, every solution* $y(t)$ *of* (1) *is either oscillatory on* $[t_0,\infty)_{\mathbb{T}}$ *or* $\lim_{t\to\infty} y(t) = 0$.

Proof Assume that (1) has a nonoscillatory solution; without loss of generality we may suppose that $y(t) > 0$ for $t \geq t_0$ and therefore $y(\tau(t)) > 0$ and $y(\delta(t)) > 0$ for $t \geq t_1 > t_0$. In the case when $y(t)$ is negative the proof is similar. As we see from Lemma 1 we have two cases to consider. First we assume that $x(t)$ satisfies Case (i) in Lemma 1. Then, by using (2) we see that

$$y(t) \geq x(t) - p(t)x\big(\tau(t)\big) \geq (1-p)x(t), \quad t \geq t_2 \geq t_1$$

or

$$\big(y\big(\delta(t)\big)\big)^{\gamma} \geq (1-p)^{\gamma}\big(x\big(\delta(t)\big)\big)^{\gamma}, \quad t \geq t_3 \geq t_2. \tag{15}$$

Substituting (15) into (4), we obtain

$$x^{[3]}(t) \leq -q(t)(1-p)^{\gamma}\big(x\big(\delta(t)\big)\big)^{\gamma} = -Q(t)\big(x\big(\delta(t)\big)\big)^{\gamma}, \quad t \geq t_3. \tag{16}$$

Furthermore, using Pötzche's chain rule, we find

$$\begin{aligned}
&\big(\big(x\big(\delta(t)\big)\big)^{\gamma}\big)^{\triangle} \\
&\quad = \gamma \int_0^1 \big[h\big(x\big(\delta(t)\big)\big)^{\sigma} + (1-h)x\big(\delta(t)\big)\big]^{\gamma-1}\big(x\big(\delta(t)\big)\big)^{\triangle}\,dh \\
&\quad \geq \gamma \int_0^1 \big[hx\big(\delta(t)\big) + (1-h)x\big(\delta(t)\big)\big]^{\gamma-1}\big(x\big(\delta(t)\big)\big)^{\triangle}\,dh \\
&\quad = \gamma\big(x\big(\delta(t)\big)\big)^{\gamma-1}x^{\triangle}\big(\delta(t)\big)\delta^{\triangle}(t) > 0.
\end{aligned} \tag{17}$$

Define the function

$$z(t) = \frac{x^{[2]}(t)}{(x(\delta(t)))^{\gamma}}, \quad t \geq t_3. \tag{18}$$

It is easy to see that $z(t) > 0$. Taking the derivative of $z(t)$, we see that

$$\begin{aligned}
z^{\triangle}(t) &= \frac{x^{[3]}(t)}{(x(\delta(t)))^{\gamma}} + \big(x^{[2]}(t)\big)^{\sigma}\left(\frac{1}{(x(\delta(t)))^{\gamma}}\right)^{\triangle} \\
&= \frac{x^{[3]}(t)}{(x(\delta(t)))^{\gamma}} - \big(x^{[2]}(t)\big)^{\sigma}\frac{((x(\delta(t)))^{\gamma})^{\triangle}}{(x(\delta^{\sigma}(t)))^{\gamma}(x(\delta(t)))^{\gamma}}.
\end{aligned} \tag{19}$$

Substituting (16) into (19) and using (17), respectively, we have

$$\begin{aligned}
z^{\triangle}(t) &\leq -Q(t) - \big(x^{[2]}(t)\big)^{\sigma}\frac{((x(\delta(t)))^{\gamma})^{\triangle}}{(x(\delta^{\sigma}(t)))^{\gamma}(x(\delta(t)))^{\gamma}} \\
&\leq -Q(t) - \big(x^{[2]}(t)\big)^{\sigma}\frac{\gamma(x(\delta(t)))^{\gamma-1}x^{\triangle}(\delta(t))\delta^{\triangle}(t)}{(x(\delta^{\sigma}(t)))^{\gamma}(x(\delta(t)))^{\gamma}} \\
&= -Q(t) - \gamma\big(x^{[2]}(t)\big)^{\sigma}\frac{x^{\triangle}(\delta(t))\delta^{\triangle}(t)}{(x(\delta^{\sigma}(t)))^{\gamma}x(\delta(t))} \\
&\leq -Q(t) - \gamma\big(x^{[2]}(t)\big)^{\sigma}\frac{x^{\triangle}(\delta(t))\delta^{\triangle}(t)}{(x(\delta^{\sigma}(t)))^{\gamma+1}}.
\end{aligned} \tag{20}$$

Using (7) in (20) and the fact that $x^{[2]}(t)$ is strictly decreasing, we obtain from (18)

$$\begin{aligned} z^{\triangle}(t) &\leq -Q(t) - \gamma \frac{\delta^{\triangle}(t) r_2(\delta(t), t_1)}{r_1(\delta(t))} \frac{(x^{[2]}(\delta(t)))^{\frac{1}{\gamma}}}{x(\delta^{\sigma}(t))} \frac{(x^{[2]}(t))^{\sigma}}{(x(\delta^{\sigma}(t)))^{\gamma}} \\ &\leq -Q(t) - \gamma \frac{\delta^{\triangle}(t) r_2(\delta(t), t_1)}{r_1(\delta(t))} \big(z^{\sigma}(t)\big)^{\frac{\gamma+1}{\gamma}}. \end{aligned} \tag{21}$$

Finally, integrating (21) from t_3 to t, we get

$$z(t) - z(t_3) \leq \int_{t_3}^{t} \left[-Q(s) - \gamma \frac{r_2(\delta(s), t_1)\delta^{\triangle}(s)}{r_1(\delta(s))} \big(z^{\sigma}(s)\big)^{\frac{\gamma+1}{\gamma}} \right] \triangle s \leq -\int_{t_3}^{t} Q(s) \triangle s$$

and consequently

$$\int_{t_3}^{t} Q(s) \triangle s \leq z(t_3),$$

which contradicts (14). When Case (ii) holds, we can conclude from Lemma 3 that $\lim_{t\to\infty} y(t) = 0$. □

Theorem 2.2 *Suppose that* (3), (9) *hold and* $\delta(\sigma(t)) = \sigma(\delta(t))$. *Furthermore, assume that there exists a positive rd-continuous* $\triangle$*-differentiable function* $\alpha(t)$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\alpha(s) Q(s) - \left(\frac{(\alpha^{\triangle}(s))_+}{\gamma+1} \right)^{\gamma+1} \left(\frac{r_1(\delta(s))}{\alpha(s) r_2(\delta(s), t_1)\delta^{\triangle}(s)} \right)^{\gamma} \right] \triangle s = \infty, \tag{22}$$

where $(\alpha^{\triangle}(s))_+ = \max\{0, \alpha^{\triangle}(s)\}$ *and* $Q(s) = q(s)(1-p)^{\gamma}$. *Then, every solution* $y(t)$ *of* (1) *is either oscillatory on* $[t_0, \infty)_{\mathbb{T}}$ *or* $\lim_{t\to\infty} y(t) = 0$.

Proof Suppose to the contrary that $y(t)$ is nonoscillatory solution of (1). We assume that $y(t) > 0$ for $t \geq t_0$, then $y(\tau(t)) > 0$ and $y(\delta(t)) > 0$ for $t \geq t_1 > t_0$. We first consider that $x(t)$ satisfies Case (i) in Lemma 1. We proceed as in the proof of Theorem 2.1, and we obtain (16). Let us define the function

$$z(t) = \alpha(t) \frac{x^{[2]}(t)}{(x(\delta(t)))^{\gamma}}, \quad t \geq t_3. \tag{23}$$

It is clear that $z(t) > 0$. Taking the derivative of $z(t)$, we see that

$$\begin{aligned} z^{\triangle}(t) &= \big(x^{[2]}(t)\big)^{\sigma} \left(\frac{\alpha(t)}{(x(\delta(t)))^{\gamma}} \right)^{\triangle} + \frac{\alpha(t)}{(x(\delta(t)))^{\gamma}} x^{[3]}(t) \\ &= \frac{\alpha(t) x^{[3]}(t)}{(x(\delta(t)))^{\gamma}} + \big(x^{[2]}(t)\big)^{\sigma} \left(\frac{(x(\delta(t)))^{\gamma} \alpha^{\triangle}(t) - \alpha(t)((x(\delta(t)))^{\gamma})^{\triangle}}{(x(\delta(t)))^{\gamma} (x(\delta^{\sigma}(t)))^{\gamma}} \right). \end{aligned} \tag{24}$$

Now using (16) in (24), we obtain

$$z^{\triangle}(t) \leq -\alpha(t) Q(t) + \frac{\alpha^{\triangle}(t) z^{\sigma}(t)}{\alpha^{\sigma}(t)} - \frac{\alpha(t)(x^{[2]}(t))^{\sigma}((x(\delta(t)))^{\gamma})^{\triangle}}{(x(\delta(t)))^{\gamma}(x(\delta^{\sigma}(t)))^{\gamma}}. \tag{25}$$

Substituting (17) into (25), we obtain

$$z^{\triangle}(t) \le -\alpha(t)Q(t) + \frac{\alpha^{\triangle}(t)z^{\sigma}(t)}{\alpha^{\sigma}(t)} - \gamma\frac{\alpha(t)(x^{[2]}(t))^{\sigma}x^{\triangle}(\delta(t))\delta^{\triangle}(t)}{x(\delta(t))(x(\delta^{\sigma}(t)))^{\gamma}}. \tag{26}$$

By using (7) into (26), we obtain

$$z^{\triangle}(t) \le -\alpha(t)Q(t) + \frac{(\alpha^{\triangle}(t))_{+}z^{\sigma}(t)}{\alpha^{\sigma}(t)} - \gamma\frac{\alpha(t)r_2(\delta(t),t_1)\delta^{\triangle}(t)}{r_1(\delta(t))}\left(\frac{z^{\sigma}(t)}{\alpha^{\sigma}(t)}\right)^{\lambda}, \tag{27}$$

where $\lambda = \frac{\gamma+1}{\gamma}$. Let

$$A^{\lambda} = \gamma\frac{\alpha(t)r_2(\delta(t),t_1)\delta^{\triangle}(t)}{r_1(\delta(t))}\left(\frac{z^{\sigma}(t)}{\alpha^{\sigma}(t)}\right)^{\lambda}$$

and

$$B^{\lambda-1} = \frac{(\alpha^{\triangle}(t))_{+}}{\lambda}\left(\frac{r_1(\delta(t))}{\gamma\alpha(t)r_2(\delta(t),t_1)\delta^{\triangle}(t)}\right)^{1/\lambda}.$$

By making use of the inequality

$$\lambda AB^{\lambda-1} - A^{\lambda} \le (\lambda-1)B^{\lambda}, \quad \lambda > 1, A, B \ge 0 \tag{28}$$

in (27), we have

$$z^{\triangle}(t) \le -\alpha(t)Q(t) + \left(\frac{(\alpha^{\triangle}(t))_{+}}{\gamma+1}\right)^{\gamma+1}\left(\frac{r_1(\delta(t))}{\alpha(t)r_2(\delta(t),t_1)\delta^{\triangle}(t)}\right)^{\gamma}. \tag{29}$$

Integrating both sides of (29) from t_3 to t then yields

$$\int_{t_3}^{t}\left(\alpha(s)Q(s) - \left(\frac{(\alpha^{\triangle}(s))_{+}}{\gamma+1}\right)^{\gamma+1}\left(\frac{r_1(\delta(s))}{\alpha(s)r_2(\delta(s),t_1)\delta^{\triangle}(s)}\right)^{\gamma}\right)\triangle s \le z(t_3) - z(t) \le z(t_3),$$

which contradicts (22).

When Case (ii) holds, we can conclude from Lemma 3 that $\lim_{t\to\infty} y(t) = 0$. □

Let $\mathbb{D}_0 \equiv \{(t,s) \in \mathbb{T}^2 : t > s \ge t_0\}$ and $\mathbb{D} \equiv \{(t,s) \in \mathbb{T}^2 : t \ge s \ge t_0\}$. The function $H \in C_{\mathrm{rd}}(\mathbb{D},\mathbb{R})$ is said to belong to class $\Re$ if $H(t,t) = 0$, $t \ge t_0$, $H(t,s) > 0$ on $\mathbb{D}_0$ and H has a continuous $\triangle$-partial derivative $H^{\triangle_s}(t,s)$ on $\mathbb{D}_0$ with respect to the second variable.

Theorem 2.3 *Assume that* (3) *and* (9) *hold and* $\delta(\sigma(t)) = \sigma(\delta(t))$. *Furthermore*, $\alpha(t)$ *is defined as in Theorem* 2.2 *and* $H \in \Re$ *such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\Bigg[H(t,s)\alpha(s)Q(s) - \left(\frac{\alpha^{\sigma}(s)C(t,s)}{\gamma+1}\right)^{\gamma+1}\left(\frac{r_1(\delta(s))}{H(t,s)\alpha(s)r_2(\delta(s),t_1)\delta^{\triangle}(s)}\right)^{\gamma}\Bigg]\triangle s = \infty, \tag{30}$$

where $C(t,s) = \max\{0, H^{\triangle_s}(t,s) + \frac{H(t,s)(\alpha^{\triangle}(s))_+}{\alpha^{\sigma}(s)}\}$. *Then every solution* $y(t)$ *of* (1) *is either oscillatory on* $[t_0,\infty)_{\mathbb{T}}$ *or* $\lim_{t\to\infty} y(t) = 0$.

Proof Assume that $y(t)$ is a nonoscillatory solution of (1). Define $z(t)$ as in (23). We proceed as in the proof of Theorem 2.2 to obtain (27). Multiplying both sides of (27) by $H(t,s)$, integrating with respect to s from t_3 to t, we get

$$\int_{t_3}^{t} H(t,s)\alpha(s)Q(s)\triangle s \le -\int_{t_3}^{t} H(t,s)z^{\triangle}(s)\triangle s + \int_{t_3}^{t} \frac{H(t,s)(\alpha^{\triangle}(s))_+ z^{\sigma}(s)}{\alpha^{\sigma}(s)}\triangle s - \int_{t_3}^{t} \frac{\gamma H(t,s)\alpha(s)r_2(\delta(s),t_1)\delta^{\triangle}(s)}{r_1(\delta(s))}\left(\frac{z^{\sigma}(s)}{\alpha^{\sigma}(s)}\right)^{\lambda}\triangle s, \tag{31}$$

where $\lambda = \frac{\gamma+1}{\gamma}$. Integrating by parts yields by (31)

$$\begin{aligned}
&\int_{t_3}^{t} H(t,s)\alpha(s)Q(s)\triangle s \\
&\quad\le H(t,t_3)z(t_3) + \int_{t_3}^{t} H^{\triangle_s}(t,s)z^{\sigma}(s)\triangle s \\
&\qquad + \int_{t_3}^{t} \frac{H(t,s)(\alpha^{\triangle}(s))_+ z^{\sigma}(s)}{\alpha^{\sigma}(s)}\triangle s - \int_{t_3}^{t} \frac{\gamma H(t,s)\alpha(s)r_2(\delta(s),t_1)\delta^{\triangle}(s)}{r_1(\delta(s))}\left(\frac{z^{\sigma}(s)}{\alpha^{\sigma}(s)}\right)^{\lambda}\triangle s \\
&\quad\le H(t,t_3)z(t_3) + \int_{t_3}^{t} C(t,s)z^{\sigma}(s)\triangle s \\
&\qquad - \int_{t_3}^{t} \frac{\gamma H(t,s)\alpha(s)r_2(\delta(s),t_1)\delta^{\triangle}(s)}{r_1(\delta(s))}\left(\frac{z^{\sigma}(s)}{\alpha^{\sigma}(s)}\right)^{\lambda}\triangle s.
\end{aligned} \tag{32}$$

Let

$$A^{\lambda} = \frac{\gamma H(t,s)\alpha(s)r_2(\delta(s),t_1)\delta^{\triangle}(s)}{r_1(\delta(s))}\left(\frac{z^{\sigma}(s)}{\alpha^{\sigma}(s)}\right)^{\lambda}$$

and

$$B^{\lambda-1} = \frac{C(t,s)\alpha^{\sigma}(s)}{\lambda}\left(\frac{r_1(\delta(t))}{\gamma H(t,s)\alpha(t)r_2(\delta(t),t_1)\delta^{\triangle}(t)}\right)^{1/\lambda}.$$

Then, using the inequality (28) in (32), we have

$$\int_{t_3}^{t} H(t,s)\alpha(s)Q(s)\triangle s \le H(t,t_3)z(t_3) + \int_{t_3}^{t}\left(\frac{\alpha^{\sigma}(s)C(t,s)}{\gamma+1}\right)^{\gamma+1}\left(\frac{r_1(\delta(s))}{H(t,s)\alpha(s)r_2(\delta(s),t_1)\delta^{\triangle}(s)}\right)^{\gamma}\triangle s$$

or

$$\frac{1}{H(t,t_3)}\int_{t_3}^{t}\left[H(t,s)\alpha(s)Q(s) - \left(\frac{\alpha^{\sigma}(s)C(t,s)}{\gamma+1}\right)^{\gamma+1}\left(\frac{r_1(\delta(s))}{H(t,s)\alpha(s)r_2(\delta(s),t_1)\delta^{\triangle}(s)}\right)^{\gamma}\right]\triangle s \le z(t_3),$$

which contradicts (30) and completes the proof.

When Case (ii) holds, we can conclude from Lemma 3 that $\lim_{t\to\infty} y(t) = 0$. □

Example 2.4 Consider the following third-order neutral nonlinear dynamic equation:

$$\left(t^3\left[\left(\left(t\left[y(t)+\frac{1}{2}y\left(\frac{t}{2}\right)\right]^{\Delta}\right)^{\Delta}\right]^3\right)^{\Delta}+\frac{3}{t}y^3\left(\frac{t}{2}\right)=0,\quad t\in[t_0,\infty)_{\mathbb{T}}, t_0>0, \tag{33}$$

where $\gamma = 3$, $r_1(t) = t$, $r_1(t) = r_2(t) = t^3$ $p(t) = \frac{1}{2}$, $\tau(t) = \delta(t) = \frac{t}{2}$, and $q(t) = 3t^{-1}$. We can verify that all conditions of Theorem 2.1 are satisfied, therefore every solution of (33) is oscillatory or $\lim_{t\to\infty} y(t) = 0$. In fact, $y(t) = t^{-1}$ is a solution of (33).

Competing interests
The author declares that they have no competing interests.

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